Vol. 2 No. 2 ACTA MATHEMATICAE APPLICATAE SINICA December, 1985

STRONG CONSISTENCY OF RECURSIVE IDENTIFICATION BY NO USE OF PERSISTENT EXCITATION CONDITION*

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Abstract

For the discrete-time stochastic system without monitoring, the strong consistency of the estimate given by the stochastic gradient algorithm is considered when the persistent excitation condition is possibly not fulfilled. In addition, the convergence rate is given for a specific class of system noises, while in the adaptive tracking case the convergence rate for the parameter estimates as well as the tracking error are obtained when the reference signal is disturbed by a "dither".

§ 1. Introduction

For stochastic linear control systems considerable attention has been paid to parameter estimation given by various recursive algorithm, among which the stochastic gradient algorithm is probably the simplest one. Goodwin, Ramadge and Caines [1] have proved the global convergence of the system and the asymptotic optimality of adaptive tracking when the stochastic gradient algorithm is applied. In this case, using a method that combines the probabilistic method with the ordinary differential equation treatment (Ljung[3], Kushner and Clark[4]), Chen and Caines[2] have shown results similar to those obtained by Chen[5] for the modified least squares algorithm, namely, the sufficient conditions for strong consistency of the estimate for systems without monitoring, and consistency of parameter estimates and asymptotic suboptimality for systems with adaptive tracking. Recently, for a class of noises including the martingale difference sequence and other dependent random sequences as special cases, Chen and Guo[6] have obtained necessary and sufficient conditions for strong consistency of the estimate given by the stochastic gradient algorithm.

However, for this algorithm there are still some questions left open. For example, can the estimate given by the algorithm remain consistent if the usual persistent excitation condition is not satisfied? And what is the convergence rate in the case of convergence? Further, for adaptive tracking how does the tracking error depend on time? These problems are discussed in the present paper.

Consider the multi-input and multi-output system

$$y_n + A_1 y_{n-1} + \cdots + A_p y_{n-p} = B_1 u_{n-1} + \cdots + B_q u_{n-q} + \varepsilon_n \tag{1}$$

where y_n and u_n denote the m-dimensional output and the l-dimensional input respectively, A_i, B_j ($i=1, \cdots, p$; $j=1, \cdots, q$) are the unknown matrices to be identified.

* Received October 22, 1984.

ε_n is the m-dimensional system noise driven by a martingale difference sequence $\{w_n\}$, that is

$$E(w_n|\mathscr{F}_{n-1})=0, \quad \forall n\geqslant 1, \tag{2}$$

$$\varepsilon_n=w_n+C_1w_{n-1}+\cdots+C_rw_{n-r} \tag{3}$$

where $\{\mathscr{F}_n\}$ is a family of nondecreasing σ-algebras defined on the probability space $(\Omega, \mathscr{F}, P)$ and C_k $(k=1, \cdots, r)$ are unknown matrices.

Let z be the shift-back operator and set

$$\theta^\tau=[-A_1\cdots-A_pB_1\cdots B_qC_1\cdots C_r], \tag{4}$$

$$C(z)=I+C_1z+\cdots C_rz^r. \tag{5}$$

Denote by θ_n the estimate for θ at time n and let it be given by the stochastic gradient algorithm:

$$\theta_{n+1}=\theta_n+\frac{\varphi_n}{r_n}(y^\tau_{n+1}-\varphi^\tau_n\theta_n) \tag{6}$$

where

$$\varphi^\tau_n=[y^\tau_n\cdots y^\tau_{n-p+1}, u^\tau_n\cdots u^\tau_{n-q+1}, y^\tau_n-\varphi^\tau_{n-1}\theta_{n-1}, \cdots, y^\tau_{n-r+1}-\varphi^\tau_{n-r}\theta_{n-r}], \tag{7}$$

$$r_n=1+\sum_{i=1}^n\|\varphi_i\|^2, \quad r_0=1. \tag{8}$$

The initial values θ_0 and φ_0 can be arbitrarily chosen.

Assume that φ_n is $\mathscr{F}_n$-measurable and that w_n is such that

$$E(\|w_n\|^2|\mathscr{F}_{n-1})\leqslant c_0r^\varepsilon_{n-1} \tag{9}$$

with constants $c_0>0$, $\varepsilon\in[0, 1)$. It is worth noting that condition (9) is weaker than the uniform boundedness condition $E(\|w_n\|^2|\mathscr{F}_n)\leqslant\sigma^2$ since $r_n\geqslant 1$ by (8).

Let matrix $\Phi(n, i)$ be recursively defined by

$$\Phi(n+1, i)=\left(I-\frac{\varphi_n\varphi^\tau_n}{r_n}\right)\cdot\Phi(n, i), \quad \Phi(i, i)=I. \tag{10}$$

We shall see in the sequel that the properties of $\Phi(n, i)$ are of great importance for the convergence of the parameter estimates.

§ 2. Strong Consistency of Parameter Estimates

For strong consistency of parameter estimates the so called persistent excitation Condition a) or b) is commonly used (e.g., Ljung[7], Moore[8] and Chen[9]):

a) $\frac{1}{n}\sum_{i=1}^n\varphi_i\varphi^\tau_i\xrightarrow[n\to\infty]{}R>0$ a.s.

b) $r_n\to\infty$ and

$$\lambda^n_{\max}/\lambda^n_{\min}\leqslant\gamma<\infty, \quad \forall n\geqslant 0, \quad \text{a.s.}$$

where $\lambda^n_{\max}$ and $\lambda^n_{\min}$ denote respectively the maximum and minimum eigenvalues of the matrix $\sum_{i=1}^n\varphi_i\varphi^\tau_i+\frac{1}{d}I$, d is the dimension of φ_n and γ may depend on ω.

It is easy to see that Condition a) implies Condition b), which means that the matrix $\sum_{i=1}^n\varphi_i\varphi^\tau_i$ will never be ill-conditioned. We now show that, if it is ill-conditioned, θ_n given by (6) still can be consistent.

Theorem 1. Provided that $C(z)-\frac{1}{2}I$ is strictly positive real, $r_n\to\infty$, $\overline{\lim_{n\to\infty}} r_n/r_{n-1}<\infty$ and that there exist constants N_0 and M (which are allowed to depend on ω) such that

$$\lambda_{\max}^n/\lambda_{\min}^n\leqslant M(\log r_n)^{1/4} \quad \text{a.s.} \quad \forall n\geqslant N_0$$

then $\theta_n\to\theta$, as $n\to\infty$. a.s.

We first prove some lemmas.

Let

$$t_n=\sum_{i=2}^{n-1}\frac{\|\varphi_i\|^2}{r_i(\log r_{i-1})^{1/4}}, \tag{11}$$

$$m(t)=\max[n:\ t_n\leqslant t], \quad \forall t\geqslant 0. \tag{12}$$

Lemma 1. Under the conditions of Theorem 1 there are positive constants α, β, N (which may depend on ω) such that

$$\sum_{i=m(N+(k-1)\alpha)}^{m(N+k\alpha)-1}\frac{\varphi_i\varphi_i^\tau}{r_i}\geqslant\beta\cdot I \quad \text{a.s.} \tag{13}$$

Proof. We first show that

$$m(t)\xrightarrow[t\to\infty]{}\infty. \tag{14}$$

It is known from $\overline{\lim_{n\to\infty}} r_n/r_{n-1}<\infty$ that there exists a constant $l\in(0,\ \infty)$ such that

$$r_n/r_{n-1}\leqslant l, \quad \forall n\geqslant 1. \tag{15}$$

Then

$$\begin{aligned}t_n&=\sum_{i=2}^{n-1}\frac{\|\varphi_i\|^2}{r_i(\log r_{i-1})^{1/4}}\geqslant\frac{1}{l}\sum_{i=2}^{n-1}\frac{\|\varphi_i\|^2}{r_{i-1}(\log r_{i-1})^{1/4}}\\&=\frac{1}{l}\sum_{i=2}^{n-1}\int_{r_{i-1}}^{r_i}\frac{dt}{r_{i-1}(\log r_{i-1})^{1/4}}\\&\geqslant\frac{1}{l}\sum_{i=2}^{n-1}\int_{r_{i-1}}^{r_i}\frac{dt}{t(\log t)^{1/4}}=\frac{1}{l}\int_{r_1}^{r_{n-1}}\frac{dt}{t(\log t)^{1/4}}\\&=\frac{4}{3l}[\log^{3/4}r_{n-1}-\log^{3/4}r_1].\end{aligned} \tag{16}$$

By (16), from $r_n\to\infty$ it follows that $t_n\to\infty$ and hence $m(t)\to\infty$.

From (14) we know that there exists N such that $m(N)\geqslant N_0$ and

$$(\log r_i)^{1/4}\geqslant 1, \quad (\log r_i)^{1/4}/r_i\leqslant\frac{1}{2M}, \quad \forall i\geqslant m(N). \tag{17}$$

For any $k\geqslant 1$, by summation by parts we have

$$\begin{aligned}\sum_{i=m(N+(k-1)\alpha)}^{m(N+k\alpha)-1}\frac{\varphi_i\varphi_i^\tau}{r_i}&\geqslant\sum_{i=m(N+(k-1)\alpha)}^{m(N+k\alpha)}\frac{\varphi_i\varphi_i^\tau}{r_i}-I\\&=\sum_{i=m(N+(k-1)\alpha)}^{m(N+k\alpha)}\frac{1}{r_i}\Big(\sum_{j=1}^{i}\varphi_j\varphi_j^\tau-\sum_{j=1}^{i-1}\varphi_j\varphi_j^\tau\Big)-I\\&=\frac{1}{r_{m(N+k\alpha)}}\sum_{j=1}^{m(N+k\alpha)}\varphi_j\varphi_j^\tau-\frac{1}{r_{m(N+(k-1)\alpha)}}\sum_{j=1}^{m(N+(k-1)\alpha)-1}\varphi_j\varphi_j^\tau\\&\quad+\sum_{i=m(N+(k-1)\alpha)+1}^{m(N+k\alpha)}\sum_{j=1}^{i-1}\varphi_j\varphi_j^\tau\Big(\frac{1}{r_{i-1}}-\frac{1}{r_i}\Big)-I\end{aligned}$$

$$\geqslant \sum_{i=m(N+(k-1)\alpha)+1}^{m(N+k\alpha)} \sum_{j=1}^{i-1} \varphi_j\varphi_j^\tau \frac{\|\varphi_i\|^2}{r_{i-1}\cdot r_i} - 2I$$

$$\geqslant \sum_{i=m(N+(k-1)\alpha)+1}^{m(N+k\alpha)} \left(\lambda_{\min}^{i-1} - \frac{1}{d}\right)\cdot I \frac{\|\varphi_i\|^2}{r_{i-1}\cdot r_i} - 2I$$

$$\geqslant \sum_{i=m(N+(k-1)\alpha)+1}^{m(N+k\alpha)} \left(\frac{\lambda_{\max}^{i-1}}{M(\log r_{i-1})^{1/4}} - \frac{1}{d}\right)\cdot I \frac{\|\varphi_i\|^2}{r_{i-1}\cdot r_i} - 2I$$

$$\geqslant \sum_{i=m(N+(k-1)\alpha)+1}^{m(N+k\alpha)} \frac{1}{d}\left(\frac{r_{i-1}}{M(\log r_{i-1})^{1/4}} - 1\right) I \frac{\|\varphi_i\|^2}{r_{i-1}\cdot r_i} - 2I$$

$$= \frac{1}{d} \sum_{i=m(N+(k-1)\alpha)+1}^{m(N+k\alpha)} \left(\frac{1}{M} - \frac{(\log r_{i-1})^{1/4}}{r_{i-1}}\right)\cdot \frac{\|\varphi_i\|^2}{r_i(\log r_{i-1})^{1/4}} - 2I$$

$$\geqslant \frac{1}{2Md}(t_{m(N+k\alpha)+1} - t_{m(N+(k-1)\alpha)+1}) I - 2I$$

$$\geqslant \frac{1}{2Md}[N+k\alpha - (N+(k-1)\alpha+1)] I - 2I$$

$$= \left[\frac{1}{2Md}(\alpha-1) - 2\right] I.$$

Then we take

$$\alpha > 4Md+1, \quad \beta = \frac{1}{2Md}(\alpha-1) - 2$$

and Lemma 1 holds.

Lemma 2. Under the conditions of Theorem 1, for any $k \geqslant 1$ the following estimate holds

$$\|\Phi(m(N+k\alpha), m(N+(k-1)\alpha))\| \leqslant \sqrt{1 - \frac{\beta^2}{c_0 k}}$$

where c_0 is some constant independent of k and β is given by Lemma 1.

Proof. Denote by ρ_k the maximum eigenvalue of matrix $\Phi^\tau(m(N+k\alpha), m(N+(k-1)\alpha))\cdot\Phi(m(N+k\alpha), m(N+(k-1)\alpha))$ and $x_m(N+(k-1)\alpha)$ the unit eigenvector corresponding to ρ_k.

By the definition of matrix norm we have

$$\|\Phi(m(N+k\alpha), m(N+(k-1)\alpha))\| = \sqrt{\rho_k}. \tag{18}$$

For $i \in [m(N+(k-1)\alpha), m(N+k\alpha)-1]$, define x_i recursively by

$$x_{i+1} = \left(I - \frac{\varphi_i\varphi_i^\tau}{r_i}\right)x_i. \tag{19}$$

It is easy to see from (10) that

$$x_{m(N+k\alpha)} = \Phi(m(N+k\alpha), m(N+(k-1)\alpha))\cdot x_{m(N+(k-1)\alpha)}$$

and then

$$\begin{aligned} x_{m(N+k\alpha)}^\tau x_{m(N+k\alpha)} &= x_{m(N+(k-1)\alpha)}^\tau \cdot \Phi^\tau(m(N+k\alpha), m(N+(k-1)\alpha)) \\ &\quad \cdot \Phi(m(N+k\alpha), m(N+(k-1)\alpha) x_{m(N+(k-1)\alpha)} \\ &= x_{m(N+(k-1)\alpha)}^\tau \cdot \rho_k \cdot x_{m(N+(k-1)\alpha)} = \rho_k. \end{aligned} \tag{20}$$

By (19) we have

$$
\begin{aligned}
x_{i+1}^{\tau}x_{i+1} &= x_i^{\tau}\left(I-\frac{\varphi_i\varphi_i^{\tau}}{r_i}\right)\left(I-\frac{\varphi_i\varphi_i^{\tau}}{r_i}\right)x_i \\
&= x_i^{\tau}x_i - x_i^{\tau}\frac{\varphi_i\varphi_i^{\tau}}{r_i}x_i - x_i^{\tau}\left(\frac{\varphi_i\varphi_i^{\tau}}{r_i}-\frac{\varphi_i\|\varphi_i\|^2\varphi_i^{\tau}}{r_i^2}\right)x_i \\
&\leqslant x_i^{\tau}x_i - x_i^{\tau}\frac{\varphi_i\varphi_i^{\tau}}{r_i}x_i.
\end{aligned} \tag{21}
$$

From here and (20) it follows that

$$
\sum_{i=m(N+(k-1)\alpha)}^{m(N+k\alpha)-1}\frac{\|\varphi_i^{\tau}x_i\|^2}{r_i}\leqslant x_{m(N+(k-1)\alpha)}^{\tau}x_{m(N+(k-1)\alpha)}-x_{m(N+k\alpha)}^{\tau}x_{m(N+k\alpha)}=1-\rho_k. \tag{22}
$$

For any $i\in[m(N+(k-1)\alpha),\ m(N+k\alpha)-1]$ we have from (19) that

$$
\begin{aligned}
\|x_i-x_{m(N+(k-1)\alpha)}\| &= \left\|\sum_{j=m(N+(k-1)\alpha)}^{i-1}\frac{\varphi_j\varphi_j^{\tau}}{r_j}x_j\right\| \\
&\leqslant\{\log r_{m(N+k\alpha)-1}\}^{1/8}\cdot\sum_{j=m(N+(k-1)\alpha)}^{m(N+k\alpha)-1}\frac{\|\varphi_j\|}{r_j^{1/2}\cdot\{\log r_{j-1}\}^{1/8}}\cdot\frac{\|\varphi_j^{\tau}x_j\|}{r_j^{1/2}} \\
&\leqslant\{\log r_{m(N+k\alpha)-1}\}^{1/8}\cdot\left\{\sum_{j=m(N+(k-1)\alpha)}^{m(N+k\alpha)-1}\frac{\|\varphi_j\|^2}{r_j(\log r_{j-1})^{1/4}}\right\}^{1/2}\cdot\left\{\sum_{j=m(N+(k-1)\alpha)}^{m(N+k\alpha)-1}\frac{\|\varphi_j^{\tau}x_j\|^2}{r_j}\right\}^{1/2} \\
&\leqslant\{\log r_{m(N+k\alpha)-1}\}^{1/8}\cdot\sqrt{\alpha+1}\cdot\sqrt{1-\rho_k}.
\end{aligned} \tag{23}
$$

The last inequality follows from (11), (12) and (22).

By using Lemma 1, (22) and (23) we have

$$
\begin{aligned}
\beta &\leqslant x_{m(N+(k-1)\alpha)}^{\tau}\cdot\sum_{i=m(N+(k-1)\alpha)}^{m(N+k\alpha)-1}\frac{\varphi_i\varphi_i^{\tau}}{r_i}x_{m(N+(k-1)\alpha)} \\
&\leqslant\left\|x_{m(N+(k-1)\alpha)}^{\tau}\cdot\sum_{i=m(N+(k-1)\alpha)}^{m(N+k\alpha)-1}\frac{\varphi_i\varphi_i^{\tau}}{r_i}(x_{m(N+(k-1)\alpha)}-x_i)\right\| \\
&\quad+\left\|x_{m(N+(k-1)\alpha)}^{\tau}\cdot\sum_{i=m(N+(k-1)\alpha)}^{m(N+k\alpha)-1}\frac{\varphi_i\varphi_i^{\tau}}{r_i}x_i\right\| \\
&\leqslant\{\log r_{m(N+k\alpha)-1}\}^{1/4}\cdot\sum_{i=m(N+(k-1)\alpha)}^{m(N+k\alpha)-1}\frac{\|\varphi_i\|^2}{r_i\cdot\{\log r_{i-1}\}^{1/4}}\|x_{m(N+(k-1)\alpha)}-x_i\| \\
&\quad+\{\log r_{m(N+k\alpha)-1}\}^{1/8}\cdot\sum_{i=m(N+(k-1)\alpha)}^{m(N+k\alpha)-1}\frac{\|\varphi_i\|}{r_i^{1/2}\cdot(\log r_{i-1})^{1/8}}\cdot\frac{\|\varphi_i^{\tau}x_i\|}{r_i^{1/2}} \\
&\leqslant\{\log r_{m(N+k\alpha)-1}\}^{1/4}\cdot(1+\alpha)\{\log r_{m(N+k\alpha)-1}\}^{1/8}\cdot\sqrt{\alpha+1}\cdot\sqrt{1-\rho_k} \\
&\quad+\{\log r_{m(N+k\alpha)-1}\}^{1/8}\cdot\sqrt{1+\alpha}\cdot\sqrt{1-\rho_k} \\
&=\{\log^{3/8}r_{m(N+k\alpha)-1}\cdot(\alpha+1)^{3/2}+\log^{1/8}r_{m(N+k\alpha)-1}(1+\alpha)^{1/2}\}\cdot\sqrt{1-\rho_k}.
\end{aligned} \tag{24}
$$

From (12) and (16) we see that

$$
t\geqslant t_{m(t)}\geqslant\frac{4}{3l}[\log^{3/4}r_{m(t)-1}-\log^{3/4}r_1] \tag{25}
$$

and consequently

$$
\log^{3/8}r_{m(N+k\alpha)-1}\leqslant\left\{\frac{3l}{4}(N+k\alpha)+\log^{3/4}r_1\right\}^{1/2}, \tag{26}
$$

$$
\log^{1/8}r_{m(N+k\alpha)-1}\leqslant\left\{\frac{3l}{4}(N+k\alpha)+\log^{3/4}r_1\right\}^{1/6}, \tag{27}
$$

From (24), (26) and (27) we obtain

$$\beta\leqslant\left\{(\alpha+1)^{3/2}+(\alpha+1)^{1/2}\left[\frac{3l}{4}(N+k\alpha)+\log^{3/4}r_1\right]^{-1/3}\right\}$$
$$\cdot\left\{\frac{3l\alpha}{4}k+\frac{3l}{4}N+\log^{3/4}r_1\right\}^{1/2}\cdot\sqrt{1-\rho_k}.$$

Clearly, we can find a positive constant c_0 which is independent of k such that

$$\beta\leqslant(c_0k)^{1/2}(1-\rho_k)^{1/2}.$$

Then

$$\rho_k\leqslant 1-\frac{\beta^2}{c_0k} \tag{28}$$

and the conclusion of Lemma 2 follows immediately from (18) and (28).

Proof of Theorem 1.

From (10) it is easy to see

$$0\leqslant\|\Phi(n+1,\ 0)\|\leqslant\|\Phi(n,\ 0)\| \tag{29}$$

and therefore

$$\|\Phi(n,0)\|\xrightarrow[n\to\infty]{} l_0\geqslant 0. \tag{30}$$

In view of $\sum_{k=1}^{\infty}\frac{\beta^2}{c_0k}=\infty$ and Lemma 2 we have

$$\|\Phi(m(N+k\alpha),0)\|\leqslant\prod_{i=1}^{k}\|\Phi(m(N+i\alpha),\ m(N+(i-1)\alpha))\|\cdot\|\Phi(m(N),0)\|$$
$$\leqslant\left\{\prod_{i=1}^{k}\left(1-\frac{\beta^2}{c_0i}\right)\right\}^{1/2}\xrightarrow[k\to\infty]{}0. \tag{31}$$

From here, (30) and $m(N+k\alpha)\xrightarrow[k\to\infty]{}\infty$ we conclude that

$$\Phi(n,0)\xrightarrow[n\to\infty]{}0 \tag{32}$$

and hence $\theta_n\to\theta$ by Theorem 3 in [6].

§ 3. Convergence Rate of Estimate

In this section we discuss the convergence rate of the estimate for a specific class of system noises.

Again, we consider system (1) but with (4) and (7) replaced respectively by

$$\theta^\tau=[-A_1\cdots-A_pB_1\cdots B_q] \tag{33}$$

and

$$\varphi_n^\tau=[y_n^\tau\cdots y_{n-p+1}^\tau u_n^\tau\cdots n_{n-q+1}^\tau]. \tag{34}$$

The system noise $\{\varepsilon_n\}$ discussed in this section is characterized by the following Condition A.

A. The series $\sum_{i=0}^{\infty}\frac{\varphi_i}{r_i}\varepsilon_{i+1}^\tau$ converges a.s. and there are constants $c>0$ and $\delta>0$, possibly depending on ω, such that

$$\left\|\sum_{i=n}^{\infty}\frac{\varphi_i}{r_i}\varepsilon_{i+1}^\tau\right\|\leqslant cr_n^{-\delta},\quad\forall n\geqslant 1. \tag{35}$$

The authors have proved in [6] that the noise satisfying Condition A covers the martingale difference sequence and other kinds of dependent random sequences. It is also shown in [6] that for strong consistency of θ_n the condition $\Phi(n,\ 0)\xrightarrow[n\to\infty]{}0$ is necessary and sufficient.

Lemma 3. The following estimates take place

1) $\|\Phi(n, j)\| \leqslant 1, \quad 0 \leqslant j \leqslant n, n \geqslant 0.$

2) $\frac{1}{r_n} = O(\|\Phi(n+1, 0)\|^d), \quad \forall n \geqslant 1$ (d is the dimension of φ_n).

3) $\|\Phi(n, m+1)\| = O(\|\Phi(n, 0)\| \cdot r_m), \quad \forall n \geqslant 0, \forall m \geqslant 0.$

4) $\sum_{j=n+1}^{\infty} \frac{\|\varphi_j\|^2}{r_j^{1+\delta}} \leqslant \frac{1}{\delta} r_n^{-\delta}, \quad \forall n \geqslant 0.$

Proof. Assertion 1) follows immediately from (10).

To prove 2) we denote by λ_i ($i=1, \cdots, d$) the eigenvalues of $\Phi(n+1, 0) \cdot \Phi^\tau(n+1, 0)$ and set $\lambda_{\max} = \max_i \{\lambda_i\}$. From (10) we have

$$\det \Phi(n+1, 0) = \det \prod_{i=0}^{n} \Phi(i+1, i) = \prod_{i=0}^{n} \det\left(I - \frac{\varphi_i \varphi_i^\tau}{r_i} \right)$$
$$= \prod_{i=1}^{n} \frac{r_{i-1}}{r_i} (1 - \|\varphi_0\|^2) = \frac{1}{r_n} (1 - \|\varphi_0\|^2)$$

and then

$$\frac{1}{r_n^2} (1 - \|\varphi_0\|^2)^2 = \det [\Phi(n+1, 0) \cdot \Phi^\tau(n+1, 0)]$$
$$= \prod_{i=1}^{d} \lambda_i \leqslant \lambda_{\max}^d = \|\Phi(n+1, 0)\|^{2d}. \tag{36}$$

Since the initial value φ_0 can be arbitrarily chosen, we may assume that $\|\varphi_0\| \neq 1$, and therefore 2) is valid.

Again by (10) we have

$$\|\Phi(n, m+1)\| \leqslant \|\Phi(n, 0)\| \cdot \|\Phi^{-1}(m+1, 0)\| \leqslant \|\Phi(n, 0)\| \cdot \prod_{i=1}^{m+1} \|\Phi^{-1}(i, i-1)\|$$
$$\leqslant \|\Phi(n, 0)\| \prod_{i=1}^{m+1} \left\| \left(I - \frac{\varphi_{i-1} \varphi_{i-1}^\tau}{r_{i-1}} \right)^{-1} \right\|$$
$$= \|\Phi(n, 0)\| \cdot \prod_{i=2}^{m+1} \frac{r_{i-1}}{r_{i-2}} \|(I - \varphi_0 \varphi_0^\tau)^{-1}\|$$
$$= \|\Phi(n, 0)\| \cdot r_m \cdot \|(I - \varphi_0 \varphi_0^\tau)^{-1}\|.$$

Hence 3) holds true.

Recalling (8) we have

$$\sum_{j=n+1}^{\infty} \frac{\|\varphi_j\|^2}{r_j^{1+\delta}} = \sum_{j=n+1}^{\infty} \int_{r_{j-1}}^{r_j} \frac{dt}{r_j^{1+\delta}} \leqslant \sum_{j=n+1}^{\infty} \int_{r_{j-1}}^{r_j} \frac{dt}{t^{1+\delta}}$$
$$\leqslant \int_{r_n}^{\infty} \frac{dt}{t^{1+\delta}} = \frac{1}{\delta} r_n^{-\delta}.$$

This completes the proof of the lemma.

Theorem 2. If $\{\varepsilon_n\}$ satisfies Condition A, then for any initial value θ_0

$$\theta_n \xrightarrow[n \to \infty]{} \theta \quad \text{a.s.}$$

if and only if

$$\Phi(n, 0) \xrightarrow[n \to \infty]{} 0 \quad \text{a.s.}$$

and in this case

$$\|\theta_n-\theta\|=O(\|\Phi(n,0)\|^{\delta/1+\delta})\quad \text{a.s.}\quad \text{as } n\to\infty.$$

Proof. The first assertion of the theorem has been given in [6], so we only need to prove the second part.

Set

$$\tilde{\theta}_n=\theta-\theta_n. \tag{37}$$

From (1), (33), (34) we know that

$$y_{n+1}=\theta^\tau\varphi_n+\varepsilon_{n+1}. \tag{38}$$

From this and (6), (37) it follows that

$$\tilde{\theta}_{n+1}=\tilde{\theta}_n-\frac{\varphi_n}{r_n}(\varphi_n^\tau\tilde{\theta}_n+\varepsilon_{n+1}^\tau).$$

Then

$$\tilde{\theta}_n=\Phi(n,0)\tilde{\theta}_0-\sum_{j=0}^{n-1}\Phi(n,j+1)\frac{\varphi_j}{r_j}\varepsilon_{j+1}^\tau. \tag{39}$$

Set

$$\alpha(t)=\max[m: r_m\leqslant t],\quad t\geqslant 0, \tag{40}$$

$$\lambda(n)=\alpha(\|\Phi(n,0)\|^{-\frac{1}{1+\delta}}),\quad n\geqslant 0. \tag{41}$$

By Lemma 3, 3) it is easy to see that

$$\|\Phi(n,\lambda(n)+1)\|=O(\|\Phi(n,0)\|\cdot r_{\lambda(n)})=O(\|\Phi(n,0)\|^{\delta/1+\delta}). \tag{42}$$

From this we see that $\lambda(n)<n-1$ for all large n.

Set

$$S=\sum_{i=0}^{\infty}\frac{\varphi_i}{r_i}\varepsilon_{i+1}^\tau,\quad S_n=\sum_{i=0}^{n}\frac{\varphi_i}{r_i}\varepsilon_{i+1}^\tau,\quad \tilde{S}_n=S-S_n,\quad S_{-1}=0.$$

By Lemma 3 and (40)—(42) we have the following estimates:

$$\begin{aligned}
&\left\|\sum_{j=0}^{n-1}\Phi(n,j+1)\frac{\varphi_j}{r_j}\varepsilon_{j+1}^\tau\right\|=\left\|\sum_{j=0}^{n-1}\Phi(n,j+1)(S_j-S_{j-1})\right\|\\
&=\left\|S_{n-1}-\sum_{j=0}^{n-1}[\Phi(n,j+1)-\Phi(n,j)]S_{j-1}\right\|\\
&=\Big\|S_{n-1}-\sum_{j=0}^{n-1}[\Phi(n,j+1)]-\Phi(n,j)]S\\
&\quad+\sum_{j=0}^{n-1}[\Phi(n,j+1)-\Phi(n,j)]\tilde{S}_{j-1}\Big\|\\
&\leqslant\|\tilde{S}_{n-1}\|+\|\Phi(n,0)S\|+\sum_{j=0}^{n-1}\|\Phi(n,j+1)(I-\Phi(j+1,j))\tilde{S}_{j-1}\|\\
&\leqslant O(r_n^{-\delta})+O(\|\Phi(n,0)\|^{\delta/1+\delta})+c\cdot\sum_{j=0}^{n-1}\|\Phi(n,j+1)\|\cdot\frac{\|\varphi_j\|^2}{r_j^{1+\delta}}\\
&=O(\|\Phi(n,0)\|^{\delta/1+\delta})+c\cdot\sum_{j=0}^{n-1}\|\Phi(n,j+1)\|\cdot\frac{\|\varphi_j\|^2}{r_j^{1+\delta}}
\end{aligned}\tag{43}$$

and

$$\begin{aligned}
&\sum_{j=0}^{n-1}\|\Phi(n,j+1)\|\cdot\frac{\|\varphi_j\|^2}{r_j^{1+\delta}}\\
&\leqslant\sum_{j=0}^{\lambda(n)}\|\Phi(n,\lambda(n)+1)\|\cdot\|\Phi(\lambda(n)+1,j+1)\|\cdot\frac{\|\varphi_j\|^2}{r_j^{1+\delta}}
\end{aligned}$$

$$+\sum_{j=\lambda(n)+1}^{n-1}\|\Phi(n,j+1)\|\cdot\frac{\|\varphi_j\|^2}{r_j^{1+\delta}}$$
$$=O(\|\Phi(n,0)\|^{\delta/1+\delta})\cdot\sum_{j=0}^{\infty}\frac{\|\varphi_j\|^2}{r_j^{1+\delta}}+\sum_{j=\lambda(n)+1}^{n-1}\frac{\|\varphi_j\|^2}{r_j^{1+\delta}}$$
$$=O(\|\Phi(n,0)\|^{\delta/1+\delta})+\frac{\|\varphi_{\lambda(n)+1}\|^2}{r_{\lambda(n)+1}^{1+\delta}}+\sum_{j=\lambda(n)+2}^{\infty}\frac{\|\varphi_j\|^2}{r_j^{1+\delta}}$$
$$=O(\|\Phi(n,0)\|^{\delta/1+\delta})+O(r_{\lambda(n)+1}^{-\delta})$$
$$=O(\|\Phi(n,0)\|^{\delta/1+\delta})+O(\|\Phi(n,0)\|^{\delta/1+\delta})$$
$$=O(\|\Phi(n,0)\|^{\delta/1+\delta}). \tag{44}$$

Hence it follows from (39), (43) and (44) that

$$\|\tilde{\theta}_n\|=O(\|\Phi(n,0)\|^{\delta/1+\delta}),\quad n\to\infty.$$

We now give results on the rate of convergence for both cases with and without persistent excitation condition, and we shall see that the rate can be expressed via simply characterizable quantities.

Theorem 3. Let Condition A be satisfied by $\{\varepsilon_n\}$ and let $r_n\to\infty$ and $\overline{\lim_{n\to\infty}}\, r_n/r_{n-1}<\infty$. Then

1) $$\|\tilde{\theta}_n\|=O(r_n^{-\delta_1})\ \text{a.s. with } \delta_1>0 \text{ as } n\to\infty \tag{45}$$

if

$$\lambda_{\max}^n/\lambda_{\min}^n\leqslant\gamma<\infty,\quad \forall n\geqslant 0.$$

2) $$\|\tilde{\theta}_n\|=O(\{\log r_n\}^{-\delta_2})\ \text{a.s. with } \delta_2>0 \text{ as } n\to\infty \tag{46}$$

if

$$\lambda_{\max}^n/\lambda_{\min}^n\leqslant M(\log r_n)^{1/4},\quad \forall n\geqslant N_0,$$

where γ, M and N_0 are all positive constants possibly depending on ω.

Proof. 1) By Theorem 2 in [5] we know that there exist $\alpha,\beta,N\in(0,\infty)$ such that

$$\sum_{i=m(t)}^{m(t+\alpha)-1}\frac{\varphi_i\varphi_i^\tau}{r_i}\geqslant\beta I,\quad\forall t\geqslant N \tag{47}$$

where

$$m(t)=\max[n: t_n\leqslant t],\quad t\geqslant 0, \tag{48}$$
$$t_n=\sum_{i=0}^{n-1}\frac{\|\varphi_i\|^2}{r_i},\quad t_0=0. \tag{49}$$

Following the notations introduced in the proof of Lemma 2, it is easy to see that (18)—(22) still hold with $m(t)$ defined by (12) replaced by $m(t)$ defined by (48).

By (19) we have for $i\in[m(N+(k-1)\alpha),\ m(N+k\alpha)-1]$

$$\|x_i-x_{m(N+(k-1)\alpha)}\|=\left\|\sum_{j=m(N+(k-1)\alpha)}^{i-1}\frac{\varphi_j\varphi_j^\tau}{r_j}x_j\right\|$$
$$\leqslant\left(\sum_{j=m(N+(k-1)\alpha)}^{m(N+k\alpha)-1}\frac{\|\varphi_j\|^2}{r_j}\right)^{1/2}\left(\sum_{j=m(N+(k-1)\alpha)}^{m(N+k\alpha)-1}\frac{\|\varphi_j^\tau x_j\|^2}{r_j}\right)^{1/2}\leqslant\sqrt{\alpha+1}\cdot\sqrt{1-\rho_k}. \tag{50}$$

From (47), (22) and (50) we obtain

$$\begin{aligned}\beta &\leqslant x^{\tau}_{m(N+(k-1)\alpha)}\cdot \sum_{i=m(N+(k-1)\alpha)}^{m(N+k\alpha)-1}\frac{\varphi_i\varphi_i^{\tau}}{r_i}\cdot x_{m(N+(k-1)\alpha)}\\ &\leqslant \sum_{i=m(N+(k-1)\alpha)}^{m(N+k\alpha)-1}\frac{\|\varphi_i\|^2}{r_i}\|x_{m(N+(k-1)\alpha)}-x_i\|+\sum_{i=m(N+(k-1)\alpha)}^{m(N+k\alpha)-1}\frac{\|\varphi_i\|\cdot\|\varphi_i^{\tau}x_i\|}{r_i}\\ &\leqslant(\alpha+1)\cdot\sqrt{\alpha+1}\cdot\sqrt{1-\rho_k}+\sqrt{1+\alpha}\cdot\sqrt{1-\rho_k}\\ &=\sqrt{\alpha+1}(\alpha+2)\sqrt{1-\rho_k}.\end{aligned}$$

From here we see that

$$\rho_k\leqslant 1-\frac{\beta^2}{(\alpha+1)(\alpha+2)^2}$$

and

$$\|\Phi(m(N+k\alpha),\ m(N+(k-1)\alpha))\|\leqslant\sqrt{1-\frac{\beta^2}{(\alpha+1)(\alpha+2)^2}}. \tag{51}$$

Hence, for any $k\geqslant 1$

$$\begin{aligned}\|\Phi(m(N+k\alpha),0)\|&\leqslant\prod_{i=1}^{k}\|\Phi(m(N+i\alpha),\ m(N+(i-1)\alpha))\|\cdot\|\Phi(m(N),0)\|\\ &\leqslant\left\{\prod_{i=1}^{k}\left(I-\frac{\beta^2}{(\alpha+1)(\alpha+2)^2}\right)\right\}^{1/2}.\end{aligned} \tag{52}$$

It is easy to see from (49) and $r_n\to\infty$ that $t_n\to\infty$ and then $m(t)\to\infty$. Therefore for any $n\geqslant m(N+\alpha)$ there exists a positive constant k such that

$$m(N+k\alpha)\leqslant n\leqslant m(N+(k+1)\alpha). \tag{53}$$

By the monotonicity of t_n we have

$$t_n\leqslant t_{m(N+(k+1)\alpha)}\leqslant N+(k+1)$$

and then

$$k\geqslant\frac{t_n-N-\alpha}{\alpha}. \tag{54}$$

From (52)—(54) we obtain

$$\begin{aligned}\|\Phi(n,0)\|&\leqslant\|\Phi(n,\ m(N+k\alpha))\|\cdot\|\Phi(m(N+k\alpha),0)\|\\ &\leqslant\|\Phi(m(N+k\alpha),0)\|\leqslant\rho^k\quad\left(\rho=\left(1-\frac{\beta^2}{(\alpha+1)(\alpha+2)^2}\right)^{1/2}<1\right)\\ &\leqslant\rho^{\frac{t_n-N-\alpha}{\alpha}}\leqslant\left(\frac{1}{\rho}\right)^{\frac{N+\alpha}{\alpha}}\cdot\rho^{\frac{t_n}{\alpha}}.\end{aligned} \tag{55}$$

It follows from $\overline{\lim_{n\to\infty}}\ r_n/r_{n-1}<\infty$ that there exists a constant $l\in(0,\infty)$ such that

$$r_n/r_{n-1}\leqslant l,\quad\forall n\geqslant 1.$$

Thus we have

$$\begin{aligned}t_n&=\sum_{i=0}^{n-1}\frac{\|\varphi_i\|^2}{r_i}\geqslant\frac{1}{l}\sum_{i=1}^{n-1}\int_{r_{i-1}}^{r_i}\frac{dt}{r_{i-1}}\\ &\geqslant\frac{1}{l}\sum_{i=1}^{n-1}\int_{r_{i-1}}^{r_i}\frac{dt}{t}=\frac{1}{l}\int_{1}^{r_{n-1}}\frac{dt}{t}=\frac{1}{l}\log r_{n-1}.\end{aligned}$$

From here and (55) we have

$$\|\Phi(n,0)\| \leqslant \left(\frac{1}{\rho}\right)^{\frac{N+\alpha}{\alpha}} \cdot \rho^{\frac{\log r_{n-1}}{\alpha l}} = c_0 \cdot (\sqrt[\alpha l]{\rho})^{\log r_{n-1}}, \quad \left(c_0 = \left(\frac{1}{\rho}\right)^{\frac{N+\alpha}{\alpha}}\right)$$

$$= c_0 \cdot r_{n-1}^{\log \sqrt[\alpha l]{\rho}} = c_0 \cdot r_{n-1}^{-\eta_1} \quad (\eta_1 = -\log \sqrt[\alpha l]{\rho} > 0)$$

$$\leqslant c_0 \cdot l^{\eta_1} \cdot r_n^{-\eta_1} = O(r_n^{-\eta_1}).$$

Hence by Theorem 2, (45) holds with $\delta_1 = \frac{\delta \eta_1}{1+\delta}$.

2) Making use of (31) and the following inequalities

$$1 - x \leqslant e^{-x}, \quad \forall x \geqslant 0,$$

$$\sum_{i=1}^{k} \frac{1}{i} \geqslant \log(k+1), \quad \forall\, k \geqslant 1,$$

we obtain

$$\|\Phi(m(N+k\alpha), 0)\| \leqslant e^{-\frac{\beta^2}{2c_0}\sum_{i=1}^{k}\frac{1}{i}} \leqslant e^{-\frac{\beta^2}{2c_0}\log(k+1)} = (k+1)^{-\frac{\beta^2}{2c_0}}. \tag{56}$$

For any $n \geqslant m(N+\alpha)$, by (14) we know that there exists some $k \geqslant 1$ such that

$$m(N+k\alpha) \leqslant n \leqslant m(N+(k+1)\alpha). \tag{57}$$

Then

$$t_n \leqslant t_{m(N+(k+1)\alpha)} \leqslant N + (k+1)\alpha$$

and therefore

$$k \geqslant \frac{t_n - N - \alpha}{\alpha}. \tag{58}$$

From (16), (56) and (58) we have

$$\|\Phi(m(N+k\alpha), 0)\| \leqslant \left(\frac{t_n - N}{\alpha}\right)^{-\frac{\beta^2}{2c_0}} \leqslant \alpha^{\frac{\beta}{2c_0}} \cdot \left\{\frac{4}{3l}(\log^{3/4} r_{n-1} - \log^{3/4} r_1) - N\right\}^{-\frac{\beta^2}{2c_0}}$$

$$= O(\{\log r_{n-1}\}^{-\eta_2}) = O(\{\log r_n\}^{-\eta_2}), \quad n \to \infty \tag{59}$$

where η_2 is some positive constant.

From (57) and (59) we have

$$\|\Phi(n,0)\| \leqslant \|\Phi(m(N+k\alpha), 0)\| = O(\{\log r_n\}^{-\eta_2}).$$

Hence, by Theorem 2, we see that (46) holds with $\delta_2 = \frac{\delta \eta_2}{1+\delta}$.

Remark. δ_1 in Theorem 3 should lie in the interval $\left(0, \frac{1}{d}\right)$ since $r_n^{-1/d} = O(\|\Phi(n+1, 0)\|)$ by 2) of Lemma 4 and $\|\Phi(n+1,0)\| = O(r_n^{-\delta_1})$ from the proof of the first part of Theorem 3.

§ 4. Convergence Rate in Adaptive Tracking Case

In order to track a deterministic reference signal y_n^* by the output of the system, the $\mathscr{F}_n$-measurable feedback control u_n is selected to satisfy the following equation (the existence of such a u_n will be discussed elsewhere)

$$\theta_n^\tau \varphi_n = y_{n+1}^* + v_n \tag{60}$$

where $\{v_n\}$ is a disturbance sequence specially introduced for the consistence of θ_n and $\mathscr{F}_n = \sigma\{w_i, v_i, i \leqslant n\}$.

We shall use the following conditions:

1° y_n^* is a bounded deterministic sequence.

2° $\{w_n\}$ and $\{v_n\}$ are two i.i.d. sequences with independent components and with continuous-type distributions, and their moments are as follows

$$Ew_n = Ev_n = 0,$$
$$Ew_n w_n^\tau = R_1 > 0, \qquad Ev_n v_n^\tau = R_2 > 0,$$
$$E\|w_n\|^4 < \infty, \qquad E\|v_n\|^4 < \infty.$$

3° B_1 and $B_1^+ B_q$ are of full rank, $B_1^+ B(z)$ is asymptotically stable and is left-coprime with $B_1^+ A(z)$ with $p \geqslant 1$, $q \geqslant 1$, $r = 0$ and $m \leqslant l$.

Theorem 4. For the system and the algorithm described by (1)—(9) let the control u_n be selected to satisfy (60) and let Conditions 1°—3° be fulfilled. Then there exists $\delta_1 \in \left(0, \frac{1}{d}\right) (d = mp + lq)$ such that

$$\|\tilde{\theta}_n\| = O(n^{-\delta_1}) \quad \text{a.s.} \quad \text{as } n \to \infty \tag{61}$$

and the long run average of the tracking errors has the expansion as follows

$$\frac{1}{n}\sum_{i=1}^{n}\|y_i - y_i^*\|^2 = \operatorname{tr}(R_1 + R_2) + O(n^{-s}) \quad \text{a.s. } n \to \infty, \quad \forall s \in (0, \delta_1). \tag{62}$$

Proof. From the demonstration of Theorem 3 in [2] we know that there are $\alpha_2 \geqslant \alpha_1 > 0$ and $N > 0$ such that

$$\alpha_1 I \leqslant \frac{1}{n}\left(\sum_{i=1}^{n} \varphi_i \varphi_i^\tau + \frac{1}{d}\right) \leqslant \alpha_2 I, \quad \forall n \geqslant N. \tag{63}$$

Consequently

$$\alpha_1 dn \leqslant r_n \leqslant \alpha_2 dn, \quad \forall n \geqslant N. \tag{64}$$

Hence (61) follows from (63), (64), 1) of Theorem 3 and the remark at the end of Section 3.

In view of $r = 0$, (1), (4) and (7) we see that

$$y_n = \theta^\tau \varphi_{n-1} + w_n.$$

This together with (60) gives

$$y_n - y_n^* = \tilde{\theta}_{n-1}^\tau \varphi_{n-1} + w_n + v_{n-1}.$$

Then

$$\begin{aligned}\|y_n - y_n^*\|^2 &= (\tilde{\theta}_{n-1}^\tau \varphi_{n-1} + w_n + v_{n-1})^\tau (\tilde{\theta}_{n-1}\varphi_{n-1} + w_n + v_{n-1}) \\ &= \|\tilde{\theta}_{n-1}^\tau \varphi_{n-1}\|^2 + \|w_n\|^2 + \|v_{n-1}\|^2 + 2\varphi_{n-1}^\tau \tilde{\theta}_{n-1} w_n + 2\varphi_{n-1}^\tau \tilde{\theta}_{n-1} v_{n-1} + 2w_n^\tau v_{n-1}.\end{aligned} \tag{65}$$

From (61) we have

$$\sum_{i=1}^{\infty} \frac{\|\tilde{\theta}_{i-1}^\tau \varphi_{i-1}\|}{r_i^{1-2s}} \leqslant c_0 \cdot \sum_{i=1}^{\infty} \frac{\|\varphi_{i-1}\|^2}{r_i^{1-2s+2\delta_1}} < \infty$$

and therefore

$$\frac{1}{r_n^{1-2s}} \sum_{i=1}^{n} \|\tilde{\theta}_{i-1}^\tau \varphi_{i-1}\|^2 \xrightarrow[n\to\infty]{} 0 \quad \text{a.s.}$$

i.e.

$$\frac{1}{n}\sum_{i=1}^{n}\|\tilde{\theta}_{i-1}^\tau \varphi_{i-1}\|^2 = o(n^{-2s}). \tag{66}$$

By the Hartman-Wintner Theorem[10] we have

$$\overline{\lim_{n\to\infty}}\sqrt{\frac{n}{\log\log n}}\left|\frac{1}{n}\sum_{i=1}^{n}\|w_i\|^2-\operatorname{tr}R_1\right|=\overline{\lim_{n\to\infty}}\frac{1}{\sqrt{n\cdot\log\log n}}\left|\sum_{i=1}^{n}(\|w_i\|^2-\operatorname{tr}R_1)\right|<\infty \quad \text{a.s.}$$

Hence

$$\frac{1}{n}\sum_{i=1}^{n}\|w_i\|^2=\operatorname{tr}R_1+O\left(\sqrt{\frac{\log\log n}{n}}\right) \quad \text{a.s.} \tag{67}$$

Similarly,

$$\frac{1}{n}\sum_{i=1}^{n}\|v_i\|^2=\operatorname{tr}R_2+O\left(\sqrt{\frac{\log\log n}{n}}\right) \quad \text{a.s.} \tag{68}$$

$$\frac{1}{n}\sum_{i=1}^{n}w_i^\tau v_{i-1}=O\left(\sqrt{\frac{\log\log n}{n}}\right) \quad \text{a.s.} \tag{69}$$

By (66)—(68) and the Schwarz inequality it is easy to see that

$$\frac{1}{n}\sum_{i=1}^{n}\varphi_{i-1}^\tau\tilde{\theta}_{i-1}w_i=o(n^{-\varepsilon}), \tag{70}$$

$$\frac{1}{n}\sum_{i=1}^{n}\varphi_{i-1}^\tau\tilde{\theta}_{i-1}v_{i-1}=o(n^{-\varepsilon}). \tag{71}$$

Then (62) follows immediately from (65)—(71) and $0<\varepsilon<\delta_1<\frac{1}{d}\leqslant\frac{1}{2}$. Thus the proof is completed.

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