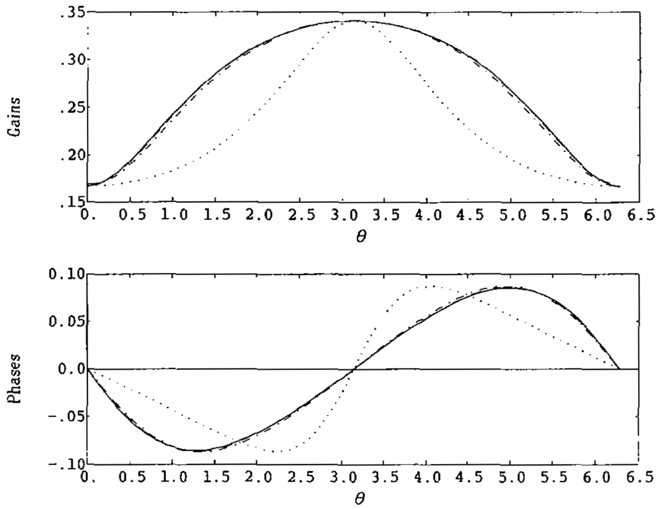

Fig. 1. Gain and phases of full and reduced models. Solid line: w. Dashed line: $\hat{w}_{Zk}$. Dotted/dashed line: $\hat{w}_1$. Dotted line: $\hat{w}_{JH}$.

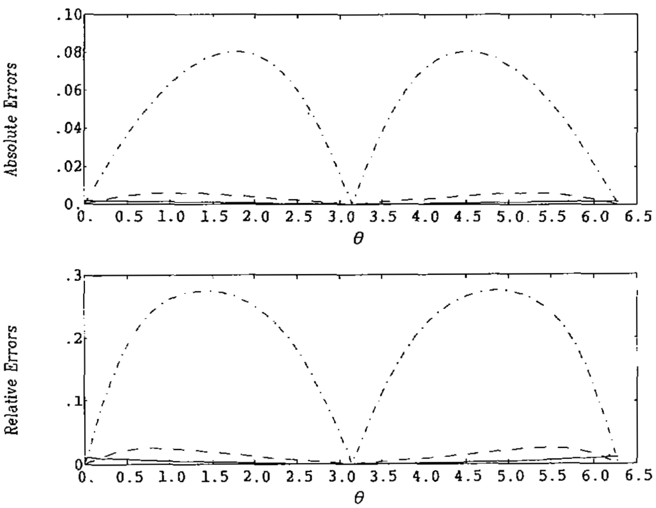

Fig. 2. Absolute and relative errors of reduced models. Dotted/dashed line: $\hat{w}_{JH}$. Dashed line: $\hat{w}_1$. Solid line: $\hat{w}_{Zk}$.

and $\hat{w}_1(e^{j\theta})$ are plotted versus θ in Fig. 1. The absolute errors $|w(e^{j\theta}) - \hat{w}(e^{j\theta})|$ and the relative errors $|(w(e^{j\theta}) - \hat{w}(e^{j\theta}))/w(e^{-j\theta})|$ are plotted versus θ in Fig. 2. Clearly, the removal of the white noise component results in a drastic improvement of the phase matched reduced model. Yet it appears, for this particular example, that $\hat{w}_{Zk}$ is a better reduced model than $\hat{w}_1$, although $\hat{w}_1$ is better than $\hat{w}_{Zk}$ over a narrow low frequency band.

III. Conclusion

It appears that in order to make the phase approximation procedure of Jonckheere and Helton [1] competitive with the procedure of Zhou and Khargonekar [2], it is imperative to remove the white noise component before approximating the phase of the outer spectral factor. In the continuous-time case, a similar recommendation applies; see [3].

With this technical fix, the reduced model of Jonckheere and Helton [1] yields an L^∞ bound on the *relative* error on the spectra (see [5]) as well as an L^∞ bound on the error on the phases of the spectral factors (see [6], [7], and [9]). Further, Green and Anderson [10] derived an L^∞-error bound on the *gain* of the spectral factor from the L^∞-error bound on its *phase*. On the other hand, the Zhou–Khargonekar procedure appears to be the only one that naturally provides a bound on the *absolute* error on the spectra.

Another way to avoid conflict between the structures at infinity of full and reduced models is to extend $\hat{a}_r$ in a nonoptimal way; see [4]. Interestingly, among all reduced models derived from suboptimal extensions lies the Desai–Pal reduced model [4].

A fairly exhaustive treatment of the structure at infinity of the full order spectral factor and its phase matched reduced models is to due Green and Anderson [8].

References

[1] E. A. Jonckheere and J. W. Helton, "Power spectrum reduction by optimal Hankel norm approximation of the phase of the outer spectral factor," *IEEE Trans. Automat. Contr.*, vol. AC-30, pp. 1192–1201, Dec. 1985.

[2] K. Zhou and P. P. Khargonekar, "Remarks on stochastic model reduction," in *Proc. Amer. Contr. Conf.*, Seattle, WA, June 18–20, 1986, TP6; also *IEEE Trans. Automat. Contr.*, vol. AC-32, pp. 356–358, Apr. 1987.

[3] P. Harshavardhana and E. A. Jonckheere, "Spectral factor reduction by phase matching—The continuous-time single-input single-output case," *Int. J. Contr.*, vol. 42, pp. 43–63, 1985.

[4] Ph. C. Opdenacker and E. A. Jonckheere, "A state space approach to approximation by phase matching," in *Proc. MTNS'85*, Stockholm, Sweden, June 1985; also in *Modelling, Identification and Robust Control*, Ch. Byrnes and A. Lindquist, Eds. Amsterdam, The Netherlands: North-Holland, 1986, pp. 349–362; also in *Int. J. Contr.*, vol. 45, no. 2, pp. 671–679, 1987.

[5] K. Glover and E. A. Jonckheere, "A comparison of two Hankel norm methods for approximating spectra," in *Proc. MTNS'85*, Stockholm, Sweden, June 1985; also *Modelling, Identification and Robust Control*, Ch. Byrnes and A. Lindquist, Eds. Amsterdam, The Netherlands: North-Holland, 1986, pp. 297–306.

[6] R. Li and E. A. Jonckheere, "An L^∞ error bound for the phase approximation problem," *IEEE Trans. Automat. Contr.*, 1986.

[7] E. A. Jonckheere and R. Li, "L^∞ error bound for phase matching approximation—The one-step-at-a-time Hankel norm model reduction version," *Int. J. Contr.*, to be published.

[8] M. Green and B. D. O. Anderson, "The approximation of power spectra by phase matching," in *Proc. IEEE Conf. Decision Contr.*, Athens, Greece, Dec. 10–12, 1986, pp. 1085–1090.

[9] M. Green, "Balanced stochastic realization," *Linear Algebra Appl.*, 1986.

[10] B. D. O. Anderson and M. Green, "Hilbert transform and gain/phase error bounds for rational functions," *IEEE Trans. Circuits Syst.*, 1986.

Consistent Estimation of the Order of Stochastic Control Systems

HAN-FU CHEN AND LEI GUO

Abstract—**A consistent estimate of the order of feedback control systems with unknown matrix coefficients estimated by the least-squares method is derived by minimizing a modified version of the Bayesian information criterion.**

I. Introduction

Over the last few years considerable progress has been made in the order estimation problem in time series analysis (e.g., [1]–[5]). But to the authors' knowledge there is no consistent estimate for the order of a linear stochastic system with feedback control which, obviously, depends on the driven noise.

In this note a multidimensional stochastic feedback control system with unknown coefficients and order is considered and the system noise is assumed uncorrelated.

The unknown coefficients, the number of which is obviously defined by the order $(p_o,\ q_o)$ of the system, are estimated by the least-squares

Manuscript received May 30, 1986; revised January 15, 1987. Paper recommended by Past Associate Editor, J. K. Tugnait. This work was supported by the Science Fund of the Chinese Academy of Sciences.

The authors are with the Institute of Systems Science, Academia Sinica, Beijing, People's Republic of China.

IEEE Log Number 8714285.

0018-9286/87/0600-0531$01.00 © 1987 IEEE

method. Then we introduce an information criterion denoted by $L_n(p, q)$, minimizing which gives estimates p_n, q_n for p_o and q_o, respectively, where n denotes the data size. It is shown that for consistency of p_n and q_n the key condition is $\log \lambda_{\max}^{p,q}(n)/\lambda_{\min}^{p,q}(n) \xrightarrow[n\to\infty]{} 0$, where $\lambda_{\max}^{p,q}(n)$ and $\lambda_{\min}^{p,q}(n)$ denote, respectively, the maximum and minimum eigenvalue of the matrix consisting of stochastic regressors. As is known from [6] this condition is satisfied when we apply the attenuating excitation control which leads to consistent parameter estimation and simultaneously to the optimization of the quadratic loss function. In other words, combining this note with the results given in [6] we thus have designed the optimal adaptive control minimizing the quadratic index and have developed an estimation method giving consistent estimates for both the order and the coefficients of the system.

II. Statement of the Problem

Let the l-input, m-output stochastic control system be described by

$$y_{n+1}=A_1 y_n+\cdots+A_{p_o}y_{n-p_o+1}+B_1u_n+\cdots+B_{q_o}u_{n-q_o+1}+w_{n+1}, \quad (1)$$

$$y_n=0,\ u_n=0, \qquad \text{for } n<0$$

with unknown order (p_o, q_o) and unknown matrix coefficients

$$\theta=[A_1 \cdots A_{p_o} B_1 \cdots B_{q_o}]^\tau.$$

We list the conditions used for the order estimation.

H_1: The system noise $\{w_n\}$ is a martingale difference sequence with respect to a nondecreasing family of σ-algebras $\{\mathfrak{F}_n\}$ such that

$$\sup_n E[\|w_n\|^\beta|\mathfrak{F}_{n-1}]<\infty, \qquad \beta>2, \text{ a.s.} \quad (2)$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} w_i w_i^\tau=R>0, \text{ a.s.} \quad (3)$$

H_2: The true order (p_o, q_o) belongs to a known finite set M

$$M=\{(p, q), p\in P, q\in Q\}.$$

H_3: A_{p_o} and B_{q_o} are of row-full rank.

H_4: A sequence of real numbers $\{a_n\}$ can be found such that $a_n>0$ and

$$a_n\to\infty, \quad a_n=o(n) \quad (4)$$

and

$$\frac{\log\lambda_{\max}^{p,q}(n)}{a_n}\xrightarrow[n\to\infty]{} 0,\ \frac{a_n}{\lambda_{\min}^{p,q}(n)}\xrightarrow[n\to\infty]{} 0, \qquad \forall(p,q)\in M \quad (5)$$

where $\lambda_{\max}^{p,q}(n)$ and $\lambda_{\min}^{p,q}(n)$ denote the maximum and minimum eigenvalues of $\sum_{i=1}^{n-1}\varphi_i(p,q)\varphi_i^\tau(p,q)$, respectively, and where

$$\varphi_n(p,q)=[y_n^\tau\cdots y_{n-p+1}^\tau u_n^\tau\cdots u_{n-q+1}^\tau]^\tau, \qquad \forall(p,q)\in M. \quad (6)$$

It is obvious that condition H_3 is automatically satisfied for the single-input and single-output systems.

Remark 1: If there are $c_1>0$, $b>0$, and $a>0$ (they possibly depend on ω) such that

$$\sum_{i=1}^{n}(\|y_i\|^2+\|u_i\|^2)=O(n^b), \qquad \text{a.s.}$$

$$\lambda_{\min}^{p,q}(n)\geqslant c_1\log^{l+a}n, \qquad \text{a.s. } \forall(p,q)\in M$$

then condition H_4 is satisfied and we can take $a_n=(\log n)\log\log n$.

For any fixed (p, q) the least-squares estimate

$$\theta_n(p,q)=[A_{1n}\cdots A_{pn}\ B_{1n}\cdots B_{qn}]^\tau \quad (7)$$

for θ at time n is given by

$$\theta_n(p,q)=\left(\sum_{i=0}^{n-1}\varphi_i(p,q)\varphi_i^\tau(p,q)\right)^{-1}\left(\sum_{i=0}^{n-1}\varphi_i(p,q)y_{i+1}^\tau\right). \quad (8)$$

For estimating the unknown order (p_o, q_o) we introduce an information criterion $L_n(p, q)$ which is a modified version of BIC

$$L_n(p,q)=n\log\sigma_n(p,q)+(p+q)a_n \quad (9)$$

where

$$\sigma_n(p,q)=\sum_{i=0}^{n-1}\|y_{i+1}-A_{1n}y_i-\cdots-A_{pn}y_{n-p+1} - B_{1n}u_n-\cdots-B_{qn}u_{i-q+1}\|^2 \quad (10)$$

and a_n is defined in H_4.

The estimate (p_n, q_n) for (p_o, q_o) is given by minimizing $L_n(p,q)$, i.e.,

$$(p_n, q_n)=\operatorname*{argmin}_{(p,q)\in M} L_n(p,q). \quad (11)$$

The main purpose of this note is to establish $(p_n, q_n)\xrightarrow[n\to\infty]{}(p_o, q_o)$.

III. Main Results

In this section we give the main results of the note.

Theorem 1: Under Conditions H_1–H_4 the order estimate (p_n, q_n) given by (11) is consistent

$$(p_n, q_n)\xrightarrow[n\to\infty]{}(p_o, q_o) \qquad \text{a.s.}$$

As is mentioned in the Introduction, in order to get both optimality of the control and consistency of the estimate, we often use the attenuating excitation control, by which we mean that the desired control action u_n^s is disturbed by a random dither v_n which tends to zero, namely, let $\{v_n\}$ be an l-dimensional mutually independent random vector sequence and let $\{v_n\}$ be independent of $\{w_n\}$ with properties

$$Ev_n=0,\ Ev_nv_n^\tau=\frac{1}{n^\epsilon}I, \qquad \|v_n\|^2\leqslant\frac{\sigma^2}{n^\epsilon}$$

where $\epsilon\in[0, 1/2(t+1))$, $t=mp^*+q^*-1$, $p^*=\max\{p: p\in P\}$, $q^*=\max\{q: q\in Q\}$, and σ^2 is a constant. Without loss of generality, assume

$$\mathfrak{F}_n=\sigma\{w_i, v_i, 0\leqslant i\leqslant n\}$$

and that the desired control u_n^s is $\sigma\{w_i, v_{i-1}, 0\leqslant i\leqslant n\}$ measurable ($v_{-1}=v_o=0$). Obviously, any feedback control is of this kind. Then the attenuating excitation control u_n is defined as

$$u_n=u_n^s+v_n \quad (12)$$

in which the additive disturbance v_n, as is shown in [6] does not influence the long run average loss function but gives sufficient excitation to the system for the estimation purpose.

Theorem 2: Suppose that the attenuating excitation control (12) is applied to system (1) and that conditions H_1–H_3 are satisfied and $0\bar{\in}Q$. If there is a positive number $\delta\in[0, (1-2\epsilon(t+1))/(2t+3))$ such that

$$\sum_{i=1}^{n}(\|y_i\|^2+\|u_i^s\|^2)=0(n^{1-\delta}), \qquad \text{a.s., } n\to\infty \quad (13)$$

then

$$(p_n, q_n)\xrightarrow[n\to\infty]{}(p_o, q_o), \qquad \text{a.s.} \quad (14)$$

$$\theta_n(p_{n-1}, q_{n-1})\xrightarrow[n\to\infty]{}\theta, \qquad \text{a.s.} \quad (15)$$

where $\theta_n(p_{n-1}, q_{n-1})$ and (p_n, q_n) are, respectively, given by (8) and (11) with $a_n=(\log n)\log\log n$.

IV. Proof of Theorems

We will need the following auxiliary estimate for the weighted sum of martingale difference sequence; for the proof we refer to [7, Lemma 2].

Lemma 1: Let H_1 be satisfied except condition (3), and let random vector φ_n be measurable with respect to $\mathfrak{F}_n$, $\forall n$. Then as $n \to \infty$

$$\left\| \left(\sum_{i=0}^{n-1} \varphi_i \varphi_i^\tau \right)^{-1/2} \sum_{i=1}^{n-1} \varphi_i w_{i+1}^\tau \right\| = 0(\sqrt{\log \lambda_{\max}(n)}), \text{ a.s.}$$

where $\lambda_{\max}(n)$ denotes the maximum eigenvalue of $\Sigma_{i=0}^{n-1} \varphi_i \varphi_i^\tau$ which is assumed nondegenerate for sufficiently large n (say for $n \geqslant n_o$).

Proof of Theorem 1: We need to show that any limit point of (p_n, q_n) coincides with (p_o, q_o). Let $(p', q') \in M$ be a limit point of (p_n, q_n), i.e., let it be the limit of a subsequence (p_{n_k}, q_{n_k})

$$(p_{n_k}, q_{n_k}) \underset{k\to\infty}{\longrightarrow} (p', q'). \tag{16}$$

For our purpose it suffices to prove the impossibility of the following situation: 1) $p' < p_o$; 2) $q' < q_o$; 3) $p' + q' > p_o + q_o$.

We note at once that p_{n_k} and q_{n_k} are integers, hence (16) means that

$$(p_{n_k}, q_{n_k}) \equiv (p', q')$$

for sufficiently large k.

Set

$$\tilde\theta_n(p, q) = [A_1 - A_{1n}, \cdots A_s - A_{sn}, B_1 - B_{1n}, \cdots B_t - B_{tn}]^\tau \tag{17}$$

where $A_i = A_{jn} = 0$ for $i > p_o$ and $j > p$, and $B_i = B_{jn} = 0$ for $i > q_o$ and $j > q$ with $t = q_o \vee q$, $s = p_o \vee p$.

We first show the impossibility of $p' < p_o$. If $p' < p_o$ were true, then from (10) and (17) it would follow that

$$\begin{aligned}(p_{nk}, q_{n_k}) = &\operatorname{tr} \sum_{i=0}^{n_k-1} \tilde\theta_{n_k}^\tau(p', q') \varphi_i(p_o, q_o \vee q') \\ &\cdot \varphi_i^\tau(p_o, q_o \vee q') \tilde\theta_{n_k}(p', q') \\ &+ 2 \operatorname{tr} \sum_{i=0}^{n_k-1} \tilde\theta_{n_k}^\tau(p', q') \varphi_i(p_o, q_o \vee q') w_{i+1}^\tau \\ &+ \sum_{i=1}^{n_k-1} \| w_{i+1} \|^2. \end{aligned} \tag{18}$$

Set

$$\begin{aligned} M_{n_k} = &\sum_{i=0}^{n_k-1} \tilde\theta_{n_k}^\tau(p', q') \varphi_i(p_o, q_o \vee q') \varphi_i^\tau(p_o, q_o \vee q') \tilde\theta_{n_k}(p', q') \\ &+ 2 \sum_{i=0}^{n_k-1} \tilde\theta_{n_k}^\tau(p', q') \varphi_i(p_o, q_o q') w_{i+1}^\tau \end{aligned}$$

and

$$\alpha_k = \left(\sum_{i=0}^{n_k-1} \varphi_i(p_o, q_o \vee q') \varphi_i^\tau(p_o, q_o \vee q') \right)^{1/2} \tilde\theta_{n_k}(p', q') x$$

for $x \in R^m$.

By $p' < p_o$ and condition H_3 we know that $\tilde\theta_{n_k}^\tau(p', q')$ is of row-full rank, hence,

$$\begin{aligned} x^\tau M_{n_k} x = &\alpha_k^\tau \Bigg[I + 2 \left(\sum_{i=0}^{n_k-1} \varphi_i(p_o, q_o \vee q') \varphi_i^\tau(p_o, q_o \vee q') \right)^{-1/2} \\ &\cdot \sum_{i=0}^{n_k-1} \varphi_i(p_o, q_o \vee q') w_{i+1}^\tau \tilde\theta_{n_k}^+(p', q') \\ &\cdot \left(\sum_{i=0}^{n_k-1} \varphi_i(p_o, q_o \vee q') \varphi_i^\tau(p_o, q_o \vee q') \right)^{-1/2} \Bigg] \alpha_k \\ = &\|\alpha_k\|^2 \left(1 + 0 \left(\|\tilde\theta_{n_k}^+(p', q')\| \sqrt{\frac{\log \lambda_{\max}^{p_o, q_o \vee q'}(n_k)}{\lambda_{\min}^{p_o, q_o \vee q'}(n_k)}} \right) \right) \\ = &\|\alpha_k\|^2 (1 + o(\|\tilde\theta_{n_k}^+(p', q')\|)) \end{aligned} \tag{19}$$

by Lemma 1 and H_4.

From (19) it follows that for large k

$$\begin{aligned} x^\tau M_{n_k} x = &\|\alpha_k\|^2 (1 + o(1)) \geqslant \frac{1}{2} \|\alpha_k\|^2 \\ = &\frac{1}{2} x^\tau \sum_{i=0}^{n_k-1} \tilde\theta_{n_k}^\tau(p', q') \varphi_i(p_o, q_o \vee q') \\ &\cdot \varphi_i^\tau(p_o, q_o \vee q') \tilde\theta_{n_k}(p', q') x \end{aligned} \tag{20}$$

since

$$\begin{aligned} \|\tilde\theta_{n_k}^+(p', q')\|^2 &\leqslant \operatorname{tr} (\tilde\theta_{n_k}^\tau(p', q') \tilde\theta_{n_k}(p', q'))^+ \\ &= \operatorname{tr} (\tilde\theta_{n_k}^\tau(p', q') \tilde\theta_{n_k}(p', q'))^{-1} \\ &\leqslant \operatorname{tr} (A_{p_o} A_{p_o}^\tau)^{-1} < \infty \end{aligned} \tag{21}$$

by the row-full rank of A_{p_o}.

Using (20) and (21) from (18) we see

$$\begin{aligned} \sigma_{n_k}(p_{n_k}, q_{n_k}) = &\operatorname{tr} M_{n_k} + \sum_{i=0}^{n_k-1} \| w_{i+1} \|^2 \\ \geqslant &\frac{1}{2} \operatorname{tr} \tilde\theta_{n_k}^\tau(p', q') \sum_{i=0}^{n_k-1} \varphi_i(p_o, q_o \vee q') \\ &\cdot \varphi_i^\tau(p_o, q_o \vee q') \tilde\theta_{n_k}(p', q') + \sum_{i=0}^{n_k-1} \| w_{i+1} \|^2 \\ \geqslant &\frac{1}{2} \operatorname{tr} A_{p_o} A_{p_o}^\tau \lambda_{\min}^{p_o, q_o \vee q'}(n_k) + \sum_{i=0}^{n_k-1} \| w_{i+1} \|^2. \end{aligned}$$

On the other hand, from (3), (10), and (11) it is easy to see

$$\begin{aligned} 0 \leqslant \sigma_{n_k}(p_o, q_o) = &-\operatorname{tr} \left(\sum_{i=0}^{n_k-1} \varphi_i(p_o, q_o) w_{i+1}^\tau \right)^\tau \\ &\cdot \left(\sum_{i=0}^{n_k-1} \varphi_i(p_o, q_o) \varphi_i^\tau(p_o, q_o) \right)^{-1} \\ &\cdot \left(\sum_{i=0}^{n_k-1} \varphi_i(p_o, q_o) w_{i+1}^\tau \right) + \sum_{i=0}^{n_k-1} \| w_{i+1} \|^2 \\ \leqslant &\, 2 n_k \operatorname{tr} R \end{aligned} \tag{23}$$

for sufficiently large k.

As a consequence of (5) we find

$$\frac{\log \lambda_{\max}^{p_o, q_o}(n)}{\lambda_{\min}^{p_o, q_o \vee q'}(n)} \underset{n\to\infty}{\longrightarrow} 0, \quad \text{a.s.} \tag{24}$$

Then by (9), (11), (22)–(24) and Lemma 1 we have

$$\begin{aligned} 0 \geqslant &L_{n_k}(p_{n_k}, q_{n_k}) - L_{n_k}(p_o, q_o) \\ = &n_k \log \left(1 + \frac{\sigma_{n_k}(p_{n_k}, q_{n_k}) - \sigma_{n_k}(p_o, q_o)}{\sigma_{n_k}(p_o, q_o)} \right) + (p' + q' - p_o - q_o) a_{n_k} \\ \geqslant &n_k \log \left(1 + \frac{\frac{1}{2} \operatorname{tr} A_{p_o} A_{p_o}^\tau \lambda_{\min}^{p_o, q_o \vee q'}(n_k) + 0(\log \lambda_{\max}^{p_o, q_o}(n_k))}{2 n_k \operatorname{tr} R} \right) \\ &+ (p' + q' - p_o - q_o) a_{n_k} \\ = &n_k \log \left(1 + \frac{\operatorname{tr} A_{p_o} A_{p_o}^\tau \lambda_{\min}^{p_o, q_o \vee q'}(n_k)(1 + 0(1))}{4 n_k \operatorname{tr} R} \right) \\ &+ (p' + q' - p_o - q_o) a_{n_k} \triangleq Q_{n_k}. \end{aligned} \tag{25}$$

We now prove (25) is impossible by showing that the limsup of its right-hand side Q_{n_k} diverges to infinity.

If $\liminf_{k\to\infty} \lambda_{\min}^{p_o,q_o\vee q'}(n_k)/n_k = \alpha > 0$, then by (4) and (25) we see that

$$Q_{n_k} \geqslant n_k \left[\log\left(1+\frac{\alpha \operatorname{tr} A_{p_o}A_{p_o}^{\tau}}{8 \operatorname{tr} R}\right)+(p'+q'-p_o-q_o)\frac{a_{n_k}}{n_k}\right]$$

$$= n_k \log\left(1+\frac{\alpha \operatorname{tr} A_{p_o}A_{p_o}^{\tau}}{8\operatorname{tr} R}\right)(1+o(1)) \xrightarrow[k\to\infty]{} \infty.$$

If $\liminf_{k\to\infty} \lambda_{\min}^{p_o,q_o\vee q'}(n_k)/n_k = 0$, then letting $\{m_k\}$ be a sequence of $\{n_k\}$ such that

$$\lim_{k\to\infty} \lambda_{\min}^{p_o,q_o\vee q'}(m_k)/m_k = 0$$

and noticing $\log(1+x) = x + o(x)$, as $x \to 0$, we see by H_4

$$Q_{m_k} \geqslant m_k \left[\frac{\operatorname{tr} A_{p_o}A_{p_o}^{\tau}\lambda_{\min}^{p_o,q_o\vee q'}(m_k)}{8m_k \operatorname{tr} R}+o\left(\frac{\lambda_{\min}^{p_o,q_o\vee q'}(m_k)}{m_k}\right)\right]$$

$$+(p'+q'-p_o-q_o)a_{m_k} = \lambda_{\min}^{p_o,q_o\vee q'}(m_k)$$

$$\cdot\left[\frac{\operatorname{tr} A_{p_o}A_{p_o}^{\tau}}{8 \operatorname{tr} R}+o(1)+(p'+q'-p_o-q_o)\frac{a_{mk}}{\lambda_{\min}^{p_o,q_o\vee q}}\right]$$

$$\xrightarrow[k\to\infty]{} \infty.$$

Impossibility of $q' < q_o$ is proved in the same manner but with $\operatorname{tr} A_{p_o}A_{p_o}^{\tau}$ replaced by $\operatorname{tr} B_{q_o}B_{q_o}^{\tau}$.

Thus what remains to show is to prove the impossibility of $p' + q' > p_o + q_o$.

Since we have proved $p' \geqslant p_o$, $q' \geqslant q_o$, it is reasonable to set

$$\bar{\theta} = [A_1 \cdots A_{p_o}\underbrace{0\cdots 0}_{p'-p_o} B_1 \cdots B_{q_o}\underbrace{0\cdots 0}_{q'-q_o}]^{\tau}.$$

From (8) it is easy to see

$$\theta_{n_k}(p',q') = \left(\sum_{i=0}^{n_k-1}\varphi_i(p',q')\varphi_i^{\tau}(p',q')\right)^{-1}$$

$$\cdot \sum_{i=0}^{n_k-1}\varphi_i(p',q')[\varphi_i^{\tau}(p',q')\bar{\theta}+w_{i+1}^{\tau}]$$

and

$$\tilde{\theta}_{n_k}(p',q') = -\left(\sum_{i=0}^{n_k-1}\varphi_i(p',q')\varphi_i^{\tau}(p',q')\right)^{-1}\sum_{i=0}^{n_k-1}\varphi_i(p',q')w_{i+1}^{\tau}.$$

Putting the last expression into (18) leads to

$$\sigma_{n_k}(p_{n_k},q_{n_k}) = -\operatorname{tr}\left(\sum_{i=0}^{n_k-1}\varphi_i(p',q')w_{i+1}^{\tau}\right)^{\tau}$$

$$\cdot\left(\sum_{i=0}^{n_k-1}\varphi_i(p',q')\varphi_i^{\tau}(p',q')\right)^{-1}$$

$$\cdot\left(\sum_{i=0}^{n_k-1}\varphi_i(p',q')w_{i+1}^{\tau}\right)+\sum_{i=0}^{n_k-1}\|w_{i+1}\|^2 \tag{26}$$

which together with (23), H_4, and Lemma 1 gives us the following estimate:

$$0 \geqslant L_{n_k}(p_{n_k},q_{n_k}) - L_{n_k}(p_o,q_o)$$

$$= n_k \log\frac{\sigma_{n_k}(p_{n_k},q_{n_k})}{\sigma_{n_k}(p_o,q_o)}+(p'+q'-p_o-q_o)a_{n_k}$$

$$\geqslant n_k \log\left(1+\frac{\sigma_{n_k}(p_{n_k},q_{n_k})-\sum_{i=0}^{n_k-1}\|w_{i+1}\|^2}{\sum_{i=0}^{n_k-1}\|w_{i+1}\|^2}\right)$$

$$+(p'+q'-p_o-q_o)a_{n_k}$$

$$= n_k\left(0\left(\frac{\log \lambda_{\max}^{p',q'}(n_k)}{n_k}\right)+o\left(\frac{\log\lambda_{\max}^{p',q'}(n_k)}{n_k}\right)\right)$$

$$+(p'+q'-p_o-q_o)a_{n_k}$$

$$= a_{n_k}[(p'+q'-p_o-q_o)+o(1)] \xrightarrow[k\to\infty]{} \infty$$

if $p' + q' > p_o + q_o$.

Thus, we have completed the proof.

Proof of Theorem 2: By (13) it is easy to see that

$$\lambda_{\max}^{p,q}(n) = 0(n^{1+\delta}), \qquad \text{a.s. } \forall (p,q)\in M,$$

therefore

$$\frac{\log \lambda_{\max}^{p,q}(n)}{a_n} = 0\left(\frac{1}{\operatorname{loglog} n}\right) = o(1).$$

So for proving (14) by Theorem 1 we need only to show

$$\frac{(\log n)\operatorname{loglog} n}{\lambda_{\min}^{p,q}(n)} \xrightarrow[n\to\infty]{} 0, \qquad \forall (p,q)\in M, \text{ a.s} \tag{27}$$

Let

$$\det A(z) = a_o + a_1 z + \cdots + a_{mp_o}z^{mp_o}$$

and set

$$\psi_n(p,q) = (\det A(z))\varphi_n(p,q), \qquad \forall (p,q)\in M.$$

By the Schwarz inequality and the fact $\varphi_i(p,q) = 0$, for $i < 0$, it is easy to see

$$\lambda_{\min}\left(\sum_{i=0}^{n-1}\psi_i(p,q)\psi_i^{\tau}(p,q)\right) = \inf_{\|x\|=1}\sum_{i=0}^{n-1}(x^{\tau}\psi_i(p,q))^2$$

$$\leqslant (mp_o+1)\sum_{j=0}^{mp_o}a_j^2\lambda_{\min}^{p,q}(n).$$

So for (27) it suffices to show that

$$\frac{(\log n)\operatorname{loglog} n}{\lambda_{\min}\left(\sum_{i=1}^{n}\psi_i(p,q)\psi_i^{\tau}(p,q)\right)} \xrightarrow[n\to\infty]{} 0, \qquad \text{a.s. } \forall (p,q)\in M \tag{28}$$

which is clearly implied by

$$\liminf_{n\to\infty}\frac{1}{n^{\alpha}}\lambda_{\min}\left(\sum_{i=1}^{n}\psi_i(p,q)\psi_i^{\tau}(p,q)\right)\neq 0 \tag{29}$$

where $\alpha \in ((1+\delta)/2, 1-(t+1)(\epsilon+\delta))$.

If (29) were not true, then there would exist a vector sequence $\{\eta_{n_k}\}$:

$$\eta_{n_k}=(\alpha_{n_k}^{0\tau}\cdots\alpha_{n_k}^{(p-1)\tau},\beta_{n_k}^{0\tau}\cdots\beta_{n_k}^{(q-1)\tau})^\tau$$

such that $\|\eta_{n_k}\| = 1$ and

$$\lim_{k\to\infty} n_k^{-\alpha}\left(\sum_{i=1}^{n_k}(\eta_{n_k}^\tau\psi_i(p,q))^2\right)=0. \tag{30}$$

Let

$$H_{n_k}(z)=\sum_{i=0}^{p-1}\alpha_{n_k}^{i\tau}z^i(\text{adj }A(z))[B(z),I]+\sum_{i=0}^{q-1}\beta_{n_k}^{i\tau}z^i[\det A(z)I_l,0]$$

$$\triangleq\sum_{j=0}^{t}[h_{n_k}^{j\tau},g_{n_k}^{j\tau}]z^j \tag{31}$$

where $t = mp^* + q^* - 1$ and $h_{n_k}^i$ and $g_{n_k}^j$ are l- and m-dimensional vectors, respectively.

Thus, (30) can be rewritten as

$$\lim_{k\to\infty} n_k^{-\alpha}\sum_{i=1}^{n_k}(h_{n_k}^{0\tau}u_i+\cdots+h_{n_k}^{t\tau}u_{i-t}+g_{n_k}^{0\tau}w_i+\cdots+g_{n_k}^{t\tau}w_{i-t})^2=0. \tag{32}$$

Noticing that (32) is the same as [6, eq. (49)], a similar argument as used in proving (61) and (63) of that paper leads to

$$h_{n_k}^i\xrightarrow[k\to\infty]{}0,\quad g_{n_k}^i\xrightarrow[k\to\infty]{}0,\qquad 0\leqslant i\leqslant t. \tag{33}$$

Hence, by (31) and (33) we see

$$\lim_{k\to\infty}\left[\sum_{i=0}^{p-1}\alpha_{n_k}^{i\tau}z^i(\text{adj }A(z))B(z)+\sum_{i=0}^{q-1}\beta_{n_k}^{i\tau}z^i\det A(z)I_l\right]=0$$

and

$$\lim_{k\to\infty}\sum_{i=0}^{p-1}\alpha_{n_k}^{i\tau}z^i\text{ adj }A(z)=0.$$

Consequently

$$\alpha_{n_k}^i\xrightarrow[k\to\infty]{}0,\quad \beta_{n_k}^j\xrightarrow[k\to\infty]{}0,\ 0\leqslant i\leqslant p-1,\ 0\leqslant j\leqslant q-1.$$

This contradicts $\|\eta_{n_k}\| = 1$, hence, (14) is valid.

To complete the proof of the theorem, we have to show (15), but this is a direct consequence of (14) and [6, Theorem 3].

V. Conclusion

For systems with uncorrelated noise we have given a consistent estimate of the system order. We emphasize that the system input is a general feedback control; hence, generally speaking, it depends on the driven noise. Also, the process y_n generated by the system is not necessarily stationary. It is desirable to generalize the results to systems with correlated noise and to develop a recursive algorithm for computing the order estimate.

References

[1] H. Akaike, "A new look at the statistical model identification," *IEEE Trans. Automat. Contr.*, vol. AC-19, pp. 716–723, 1974.

[2] E. J. Hannan and L. Kavalieris, "Multivariate linear time models," *Adv. Appl. Prob.*, vol. 16, pp. 492–561, 1984.

[3] E. J. Hannan, "The estimation of the order of an ARMA process," *Ann. Statist.*, vol. 8, pp. 1071–1081, 1980.

[4] E. J. Hannan and J. Rissanen, "Recursive estimation of ARMA order," *Biometrika*, vol. 69, pp. 81–94, 1982.

[5] H.-Z. An, Z.-G. Chen, and E. J. Hannan, "Autocorrelation, autoregression and autoregressive approximation," *Ann. Statist.*, vol. 10, pp. 926–936, 1982.

[6] H. F. Chen and L. Guo, "Convergence rate of least squares identification and adaptive control for stochastic systems," *Int. J. Contr.*, vol. 44, pp. 1459–1476, 1986.

[7] T. L. Lai and C. Z. Wei, "On the concept of excitation in least squares identification and adaptive control," *Stochastics*, vol. 16, pp. 227–254, 1986.

On Bang-Bang Solutions of Stochastic Differential Games

YASUHIRO FUJITA AND HIROAKI MORIMOTO

Abstract—**We consider two classes of scalar stochastic differential games with hard constraints on controls. The solutions are found to be bang-bang, by extending a technique developed earlier for stochastic optimal control problems.**

I. Introduction

In this note, we are concerned with two player zero-sum and nonzero-sum stochastic differential games with constraints. Let U be the set of all Borel measurable functions $u = u(x)$ on $\mathbb{R}$ taking values in $[-1, 1]$. For each $u, v \in U$, we consider the evolution of the system described by the stochastic differential equation

$$dx_t=au(x_t)dt+bv(x_t)dt+dW_t,\qquad x_0=0 \tag{1}$$

where a and b are nonzero constants, and $(W_t)_{t\geqq 0}$ is a standard Brownian motion on a complete probability space $(\Omega, \mathfrak{F}, P, \{\mathfrak{F}_t\}_{t\geqq 0})$ with $W_0 = 0$. Veretennikov [9] shows that (1) has a unique strong solution $(x_t)_{t\geqq 0}$. Let us denote by $C_b(\mathbb{R})$ the set of all bounded continuous functions $f:\mathbb{R}\to\mathbb{R}$ with its norm $\|f\| = \sup_{x\in\mathbb{R}}|f(x)|$. Given $\alpha > 0$ and $f, f_1, f_2 \in C_b(\mathbb{R})$, we define the payoff functions by

$$J(u,v)=E\left[\int_0^\infty e^{-\alpha s}f(x_s)\,ds\right] \tag{2}$$

and

$$J_1(u,v)=E\left[\int_0^\infty e^{-\alpha s}\{f_1+(1/2)u^2\}(x_s)ds\right]$$

$$J_2(u,v)=E\left[\int_0^\infty e^{-\alpha s}\{f_2+(1/2)v^2\}(x_s)\,ds\right]\qquad u,v\in U. \tag{3}$$

The purpose of this note is to present the synthesis of both a saddle point $(\hat{u}, \hat{v}) \in U \times U$ and a Nash equilibrium solution $(u^*, v^*) \in U \times U$, satisfying, respectively,

$$J(\hat{u},v)\leqq J(\hat{u},\hat{v})\leqq J(u,\hat{v}) \tag{4}$$

and

$$J_1(u^*,v^*)\leqq J_1(u,v^*)$$

$$J_2(u^*,v^*)\leqq J_2(u^*,v),\qquad u,v\in U. \tag{5}$$

These games are the same type of problems as linear-quadratic stochastic

Manuscript received May 19, 1986; revised November 3, 1986 and January 20, 1987.

Y. Fujita is with the Department of Mathematics Fundamentals, Division of System Science, Kobe University, Kobe, Japan.

H. Morimoto is with the Department of Mathematics, Faculty of General Education, Ehime University, Matsuyama, Japan.

IEEE Log Number 8714286.