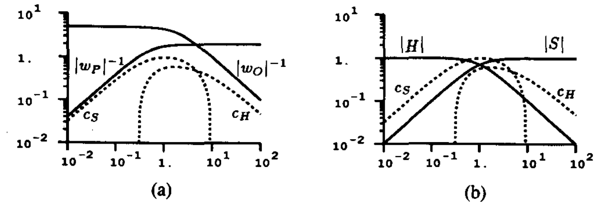

Fig. 3. Graphical representation of conditions (33) and (34). R.P. is guaranteed since $|S| < c_s$ for $\omega < 2$ and $|H| < c_H$ for $\omega > 1.4$.

The step from (A1) to (A2) follows from $M = N_{11} + N_{12}(I - TN_{22})^{-1}TN_{21}$ and Schurs formula

$$\det(A - BD^{-1}C) = \det\begin{bmatrix} A & B \\ C & D \end{bmatrix} \Big/ \det D \tag{A3}$$

and the assumption $\det D = \det(I - TN_{22}) \neq 0$.

Lemma 1: An equivalent statement of the lemma is as follows: Let $\tilde{\Delta} = \operatorname{diag}\{\Delta_1, \Delta_2\}$ where Δ_1 and Δ_2 have the same *size* as N_{11} and N_{22}, respectively. (Δ_1 and Δ_2 may have additional structure.) Then

$$\mu_{\tilde{\Delta}}\begin{bmatrix} 0 & N_{12} \\ cN_{21} & 0 \end{bmatrix} = \sqrt{c}\,\mu_{\tilde{\Delta}}\begin{bmatrix} 0 & N_{12} \\ N_{21} & 0 \end{bmatrix}. \tag{A4}$$

Proof of (A4):

$$\mu_{\tilde{\Delta}}\begin{bmatrix} 0 & N_{12} \\ cN_{21} & 0 \end{bmatrix} \leq 1/k_1 \tag{A5}$$

$$\Leftrightarrow \det\left(I + k_1\begin{bmatrix} \Delta_1 & \\ & \Delta_2 \end{bmatrix}\begin{bmatrix} 0 & N_{12} \\ cN_{21} & 0 \end{bmatrix}\right) \neq 0$$

$$\Leftrightarrow \det\begin{bmatrix} I & k_1\Delta_1 N_{12} \\ k_1 c\Delta_2 N_{21} & I \end{bmatrix} \neq 0 \tag{A6}$$

$$\Leftrightarrow \det(I - k_1^2 c\Delta_1 N_{12}\Delta_2 N_{21}) \neq 0 \tag{A7}$$

$$\Leftrightarrow \det\begin{bmatrix} I & \sqrt{k_1^2 c}\,\Delta_1 N_{12} \\ \sqrt{k_1^2 c}\,\Delta_2 N_{21} & I \end{bmatrix} \neq 0 \tag{A8}$$

$$\Leftrightarrow \mu_{\tilde{\Delta}}\begin{bmatrix} 0 & N_{12} \\ N_{21} & 0 \end{bmatrix} \leq 1/\sqrt{k_1^2 c}. \tag{A9}$$

The conditions involving det ( ) $\neq 0$ must hold for $\forall \Delta_1$ s.t. $\bar{\sigma}(\Delta_1) < 1$ and $\forall \Delta_2$ s.t. $\bar{\sigma}(\Delta_2 < 1)$. The step from (A6) to (A7) and back to (A8) follows from (A3). Since (A5) and (A9) must hold for *any* value of k_1, (A4) follows.

Theorem 2: From Theorem 1 and Lemma 1 for the case $N_{11} = N_{22} = 0$

$$\mu_\Delta(N_{12}TN_{21}) \leq k \text{ if } \bar{\sigma}(T)\mu_{\tilde{\Delta}}^{\;2}\begin{bmatrix} 0 & N_{12} \\ N_{21} & 0 \end{bmatrix} \leq k. \tag{A10}$$

Since (A10) must hold for *any* choice of k it is equivalent to

$$\mu_\Delta(N_{12}TN_{21}) \leq \bar{\sigma}(T)\mu_{\tilde{\Delta}}^{\;2}\begin{bmatrix} 0 & N_{12} \\ N_{21} & 0 \end{bmatrix}.$$

Theorem 2 follows by choosing $N_{12} = A$, $N_{21} = B$.

Special Cases of Theorem 2: Let Δ_1 and Δ_2 have the same *structure* as Δ and T in Theorem 2. Define $\tilde{\Delta} = \operatorname{diag}\{\Delta_1, \Delta_2\}$. Then

$$\mu_{\tilde{\Delta}}^{\;2}\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \leq 1/k \tag{A11}$$

$$\Leftrightarrow \mu_{\tilde{\Delta}}\begin{bmatrix} 0 & kA \\ B & 0 \end{bmatrix} \leq 1 \tag{A12}$$

$$\Leftrightarrow \det\left(I + \begin{bmatrix} \Delta_1 & \\ & \Delta_2 \end{bmatrix}\begin{bmatrix} 0 & kA \\ B & 0 \end{bmatrix}\right) \neq 0 \ \forall \Delta_1, \Delta_2$$

$$\Leftrightarrow \det(I - k\Delta_2 B\Delta_1 A) = \det(I - k\Delta_1 A\Delta_2 B) \neq 0 \ \forall \Delta_1, \Delta_2$$

$$\Leftrightarrow \mu_{\Delta_2}(B\Delta_1 A) \leq 1/k \ \forall \Delta_1 \tag{A13}$$

$$\Leftrightarrow \mu_{\Delta_1}(A\Delta_2 B) \leq 1/k \ \forall \Delta_2 \tag{A14}$$

$$\Leftrightarrow \rho(\Delta_1 A\Delta_2 B) = \rho(\Delta_2 B\Delta_1 A) \leq 1/k \ \forall \Delta_1, \Delta_2. \tag{A15}$$

By $\forall \Delta_i$ is understood all Δ_i s.t. $\bar{\sigma}(\Delta_i) < 1$. The step from (A11) to (A12) follows from (A4).

Case 1): Follows from (A15): Use the SVD of $A = U_A\Sigma_A V_A^H$ and $B = U_B\Sigma_B V_B^H$. Since Δ_1 and Δ_2 are full, Δ_1 may be chosen such that $\Delta_1 U_A = V_B$ and Δ_2 such that $V_A^H\Delta_2 = U_B^H$. Then $\rho(\Delta_1 A\Delta_2 B) = \rho(V_B\Sigma_1\Sigma_2 V_B^H) = \rho(\Sigma_1\Sigma_2) = \bar{\sigma}(A)\bar{\sigma}(B)$. [The generalization to the case when A and B are nonsquare is straightforward and involves lining up the directions corresponding to $\bar{\sigma}(A)$ and $\bar{\sigma}(B)$.]

Case 2): Follows from (A14).

Case 3): Follows from (A13).

Cases 4), 5): Follow from (A15).

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Recursive Algorithm for the Computation of the H^∞-Norm of Polynomials

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***Abstract*—A recursive algorithm for computing the H^∞-norm of polynomials is developed. The algorithm is shown to converge monotonically and the convergence rate is also established. Some examples are presented to illustrate the algorithm.**

I. INTRODUCTION

In recent years, H^∞-norm and its optimization have been used more and more frequently in many areas of control theory and applications. For example, H^∞-norm optimal controller synthesis approach [1], [2], model/

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controller reduction, and even some problems in system identification are closely related to H^∞-norms. The model/controller reduction is often best posed as a frequency weighted H^∞ optimal approximation problem [3]. For a given transfer function $G(z)$, many approaches give a reduced-order transfer function $G_r(z)$, normally, which is not optimal in the H^∞ sense. Certainly, it is desirable to know the value of the H^∞-approximation error $\|G(z) - G_r(z)\|_\infty$. In system identification, if a monic polynomial $C(z)$ is the moving average noise process transfer function in an ARMAX model, it is well known that for the convergence of the extended least-squares algorithm, a key condition is that $C^{-1}(z) - 1/2$ is strictly positive real (e.g., [4], [5]). It is easy to see that this condition is equivalent to the requirement $\|C(z) - 1\|_\infty < 1$. However, in practice, to calculate the value of the H^∞-norm is not a pleasant task. It is usually done by a rather trivial method, i.e., plotting the absolute value of the function concerned on the unit circle.

In this note, we propose a theoretical recursive algorithm for the computation of the H^∞-norm of polynomials or FIR transfer functions (Section II). In Section III we give the derivation of the algorithm and show that the guaranteed convergence rate of the algorithm is $O(\log n/n)$. Simulation results of some examples are provided in Section IV. Section V concludes the note with some remarks.

Before pursuing further, we need some concepts and definitions as follows.

Let $f(z)$ be a complex-valued function on the unit circle bounded almost everywhere; the set of all such functions is denoted by L^∞, with norm

$$\|f(z)\|_\infty = \operatorname*{ess\,sup}_{\theta\in[0,2\pi]} |f(e^{i\theta})|. \tag{1}$$

The Hardy space H^∞ consists of all complex-valued functions which are analytic and of bounded modulus on $|z| < 1$, with norm

$$\|f(z)\|_\infty = \sup_{|z|<1} |f(z)|. \tag{2}$$

It is known that each f in H^∞ yields a unique L^∞ boundary function with the two norms equal. The set of such boundary functions is the subspace of L^∞-functions with Fourier coefficients zero for negative indexes, and we can regard H^∞ as a closed-subspace of space L^∞.

We also need the concept of space L^p ($p > 0$). It consists of all measurable complex functions $f(z)$ defined on the unit circle $|z| = 1$ such that $|f(e^{i\theta})|p$ is integrable with respect to Lebesgue measure, with norm

$$\|f(z)\|_p = \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta\right\}^{1/p}. \tag{3}$$

II. Algorithm Description and Main Results

Let $C(z)$ be a polynomial with real coefficients and with degree r

$$C(z) = C_0 + C_1 z + \cdots + C_r z^r, \qquad C_0 C_r \neq 0. \tag{4}$$

Define a function $f(z)$ as

$$f(z) = C(z)C(z^{-1}) \triangleq \gamma_0 + \sum_{j=1}^{r} \gamma_j (z^j + z^{-j}) \tag{5}$$

where

$$\gamma_j = \sum_{k=0}^{r} C_k C_{k+j}, \qquad (C_k = 0, \text{ for } k > r). \tag{6}$$

To describe our algorithm, we need the following auxiliary variables:

$$\{\chi_i(n),\ 1 \leqslant i \leqslant 2r,\ n \geqslant 1\} \text{ and } \{T(n),\ n \geqslant 1\}$$

which are recursively defined by

$$\chi_{k+r}(n-1) = \left[\sum_{j=1}^{r} (nj-k)\gamma_j \chi_{k-j}(n-1) - \sum_{j=0}^{r-1} (nj+k)\gamma_j \chi_{k+j}(n-1)\right] \Big/ (nr+k)\gamma_r \qquad 1 \leqslant k \leqslant r \tag{7}$$

$$\chi_k(n) = (n/k) \sum_{j=1}^{r} j\gamma_j [\chi_{k-j}(n-1) - \chi_{k+j}(n-1)] \cdot \left[\gamma_0 + 2\sum_{j=1}^{r} \gamma_j \chi_j(n-1)\right]^{-1} \qquad 1 \leqslant k \leqslant r \tag{8}$$

$$T(n) = \frac{n-1}{n} T(n-1) + \frac{1}{2n} \log \left[\gamma_0 + 2\sum_{j=1}^{r} \gamma_j \chi_j(n-1)\right] \tag{9}$$

where by definition

$$\chi_0(n) = 1 \text{ and } \chi_{-i}(n) = \chi_i(n), \qquad 1 \leqslant i \leqslant 2r,\ n \geqslant 1$$

and where the initial conditions are

$$\chi_j(1) = \gamma_j/\gamma_0,\ 1 \leqslant j \leqslant r; \qquad T(1) = \frac{1}{2}\log \gamma_0.$$

The nth approximation for the norm $\|C(z)\|_\infty$ is defined by

$$J(n) = \exp\{T(n)\}, \qquad n \geqslant 1. \tag{11}$$

The asymptotic properties of the above algorithm are summarized in the following theorem.

Theorem 1: For any polynomial $C(z)$ defined as in (4), the quantity $J(n)$ given by (7)–(11) increases monotonically and converges to $\|C(z)\|_\infty$ as $n \to \infty$, with convergence rate

$$\|C(z)\|_\infty - J(n) \leqslant (\|C(z)\|_\infty) \frac{\log n}{2n} + 0(1/n). \tag{12}$$

III. Convergence Analysis

For the proof of Theorem 1, we first establish the following lemmas.

Lemma 1: For $T(n)$ given by (9)

$$T(n) = \log(\|C(z)\|_{2n})$$

holds for any $n \geqslant 1$.

Proof: Define

$$M_k(n) = \frac{1}{2\pi}\int_0^{2\pi} f^n(e^{i\theta}) e^{ki\theta} \, d\theta \tag{13}$$

for $n \geqslant 1$, $-2r \leqslant k \leqslant 2r$, where $f(e^{i\theta})$ is given by (5).

It is easy to see that for any $n \geqslant 1$

$$M_{-k}(n) = M_k(n), \qquad k = 0, 1, \cdots, 2r \tag{14}$$

and

$$M_0(n) = \frac{1}{2\pi}\int_0^{2\pi} f^n(e^{i\theta}) \, d\theta = \|f^n(z)\|_n^n = \|C(z)\|_{2n}^{2n}. \tag{15}$$

So for the proof of the lemma, we need only to show that

$$T(n) = \frac{1}{2n} \log M_0(n). \tag{16}$$

We proceed as follows.

By (5), (13), and (14)

$$M_0(n)=\frac{1}{2\pi}\int_0^{2\pi} f^{n-1}(e^{i\theta})f(e^{i\theta})\,d\theta$$
$$=\frac{1}{2\pi}\int_0^{2\pi} f^{n-1}(e^{i\theta})\left[\gamma_0+\sum_{k=1}^{r}\gamma_k(e^{ki\theta}+e^{-ki\theta})\right]d\theta$$
$$=\gamma_0 M_0(n-1)+2\sum_{j=1}^{r}\gamma_j M_j(n-1)$$
$$=M_0(n-1)\left\{\gamma_0+2\sum_{j=1}^{r}\gamma_j[M_j(n-1)/M_0(n-1)]\right\} \tag{17}$$

consequently, we have

$$\left[\frac{1}{2n}\log M_0(n)\right]=\frac{n-1}{n}\left[\frac{1}{2(n-1)}\log M_0(n-1)\right]+\frac{1}{2n}\log\left\{\gamma_0+2\sum_{j=1}^{r}\gamma_j[M_j(n-1)/M_0(n-1)]\right\}.$$

Comparing this to (9), we see that for (16) it suffices to show that

$$\chi_j(n)=M_j(n)/M_0(n), \qquad 1\leqslant j\leqslant r. \tag{18}$$

Now, by integral parts from (13) and the fact that $f(z)=f(z^{-1})$, we have

$$M_k(n)=\frac{1}{2\pi}\int_0^{2\pi} f^n(e^{-i\theta})e^{ki\theta}\,d\theta$$
$$=\frac{1}{2\pi ki}\int_0^{2\pi} f^n(e^{-i\theta})\,de^{ki\theta}$$
$$=\frac{1}{2\pi ki}\left\{f^n(e^{-i\theta})e^{ki\theta}\Big|_0^{2\pi}-\int_0^{2\pi}(e^{ki\theta})\,d[f^n(e^{-i\theta})]\right\}$$
$$=\frac{n}{2\pi k}\int_0^{2\pi} f^{n-1}(e^{-i\theta})\cdot f'(e^{-i\theta})e^{(k-1)i\theta}\,d\theta \tag{19}$$

where

$$f'(e^{-i\theta})\,\frac{df(z)}{dz}\Big|_{z=e^{-i\theta}}=\sum_{j=1}^{r} j\gamma_j[e^{(1-j)i\theta}-e^{(j+1)i\theta}]. \tag{20}$$

For (19) and (20) we immediately have ($1\leqslant k\leqslant r$)

$$M_k(n)=\frac{n}{k}\sum_{j=1}^{r} j\gamma_j[M_{k-j}(n-1)-M_{k+j}(n-1)]. \tag{21}$$

Multiplying $1/M_0(n)$ on both sides of this equality and using (17), we know that the recursion (8) is true with $\chi_k(n)$ replaced by $M_k(n)/M_0(n)$.

To conclude (18), we still need to show that the recursion (7) also holds with $\chi_k(n)$ replaced by $M_k(n)/M_0(n)$. To this end, consider the following decomposition for $f'(e^{-i\theta})$:

$$f'(e^{-i\theta})=g_1(e^{-i\theta})-rf(e^{-i\theta})e^{i\theta} \tag{22}$$

where

$$g_1(e^{-i\theta})=r\gamma_0 e^{i\theta}+\sum_{j=1}^{r}\gamma_j[(r+j)e^{(1-j)i\theta}+(r-j)e^{(j+1)i\theta}]. \tag{23}$$

Substituting (22) into (19), we get

$$M_k(n)=\frac{n}{2\pi k}\int_0^{2\pi} f^{n-1}(e^{-i\theta})[g_1(e^{-i\theta})-rf(e^{-i\theta})e^{i\theta}]e^{(k-1)i\theta}\,d\theta$$
$$=\frac{-nr}{2\pi k}\int_0^{2\pi} f^n(e^{-i\theta})e^{ki\theta}\,d\theta+\frac{n}{2\pi k}\int_0^{2\pi} f^{n-1}(e^{-i\theta})g_1(e^{-i\theta})e^{(k-1)i\theta}\,d\theta$$
$$=\frac{-nr}{k}M_k(n)+\frac{n}{2\pi k}\int_0^{2\pi} f^{n-1}(e^{-i\theta})\cdot g_1(e^{-i\theta})e^{(k-1)i\theta}\,d\theta.$$

By this identity, we obtain for $1\leqslant k\leqslant r$

$$M_k(n)=\frac{n}{nr+k}\cdot\frac{1}{2\pi}\int_0^{2\pi} f^{n-1}(e^{-i\theta})g_1(e^{-i\theta})e^{(k-1)i\theta}\,d\theta$$
$$=\frac{n}{nr+k}\left\{r\gamma_0 M_k(n-1)+\sum_{j=1}^{r}\gamma_j[(r+j)M_{k-j}(n-1)+(r-j)M_{k+j}(n-1)]\right\} \tag{24}$$

which in conjunction with (21) gives the recursive formula for $M_{k+r}(n-1)$

$$M_{k+r}(n-1)=\frac{1}{(nr+k)\gamma_r}\left[\sum_{j=1}^{r}(nj-k)\gamma_j M_{k-j}(n-1)-\sum_{j=0}^{r-1}(nj+k)\gamma_j M_{k+j}(n-1)\right].$$

From here it is easy to see that (7) is true with $\chi_k(n-1)$ replaced by $M_k(n-1)/M_0(n-1)$. This proves the assertion (18), and hence the conclusion of the lemma.

Lemma 2: Let a complex function $f(z)\epsilon L^\infty$, if $d/d\theta\,[|f(e^{i\theta})|^2]\epsilon L^\infty$; then

$$0\leqslant\|f(e^{i\theta})\|_\infty-\|f(e^{i\theta})\|_n\leqslant(\|f(e^{i\theta})\|_\infty)\frac{\log n}{n}+O\left(\frac{1}{n}\right).$$

Proof: By (1) and (3) it is evident that for any $n\geqslant 1$

$$\|f(e^{i\theta})\|_n\leqslant\|f(e^{i\theta})\|_\infty.$$

Now, denote

$$g(\theta)=|f(e^{i\theta})|^2, \qquad \theta\in[0, 2\pi].$$

Since $g(\theta)$ is a continuous function of θ, there exists a $\theta_0\epsilon[0, 2\pi]$ such that

$$g(\theta_0)=\max_{\theta\in[0,2\pi]} g(\theta)=\|f(e^{i\theta})\|_\infty^2.$$

Without loss of generality, assume $\theta_0\epsilon(0, 2\pi)$.

By the Taylor expansion we know that

$$g(\theta)=g(\theta_0)+g'(\xi)(\theta-\theta_0)$$

where ξ is some point between θ and θ_0.

From here we have, for sufficiently large n,

$$\begin{aligned}
&\|f(e^{i\theta})\|_n \\
&= \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^n \, d\theta\right\}^{1/n} \\
&= \left\{\frac{1}{2\pi}\int_0^{2\pi} [g(\theta)]^{n/2} \, d\theta\right\}^{1/n} \\
&= \left\{\frac{1}{2\pi}\int_0^{2\pi} g^{n/2}(\theta_0)\left[1+\frac{g'(\xi)}{g(\theta_0)}(\theta-\theta_0)\right]^{n/2} d\theta\right\}^{1/n} \\
&= [g(\theta_0)]^{1/2}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left[1+\frac{g'(\xi)}{g(\theta_0)}(\theta-\theta_0)\right]^{n/2} d\theta\right\}^{1/n} \\
&\geqslant \|f(z)\|_\infty \left\{\frac{1}{2\pi}\int_{\theta_0-1/n}^{\theta_0+1/n}\left[1+\frac{g'(\xi)}{g(\theta_0)}(\theta-\theta_0)\right]^{n/2} d\theta\right\}^{1/n} \\
&\geqslant \|f(z)\|_\infty \left\{\frac{1}{2\pi}\int_{\theta_0-1/n}^{\theta_0+1/n}\left[1-\frac{\|g'(\theta)\|_\infty}{g(\theta_0)}\cdot\frac{1}{n}\right]^{n/2} d\theta\right\}^{1/n} \\
&= \|f(z)\|_\infty \left(\frac{1}{\pi n}\right)^{1/n}\left[1-\frac{\|g'(\theta)\|_\infty}{g(\theta_0)}\cdot\frac{1}{n}\right]^{1/2} \\
&= \|f(z)\|_\infty \cdot \exp\left\{\frac{1}{n}\log\frac{1}{n\pi}\right\}\cdot\left[1+O\left(\frac{1}{n}\right)\right] \\
&= \|f(z)\|_\infty \cdot \left[1-\frac{\log \pi n}{n}+O\left(\frac{\log^2 n}{n^2}\right)\right]\left[1+O\left(\frac{1}{n}\right)\right] \\
&= \|f(z)\|_\infty - (\|f(z)\|_\infty)\frac{\log n}{n}+O\left(\frac{1}{n}\right).
\end{aligned}$$

This completes the proof of the lemma.

Proof of Theorem 1: By (11) and Lemma 1, we know that

$$J(n)=\|C(z)\|_{2n}. \tag{25}$$

By the Hölder inequality, it is easy to see that the L^p-norm $\|\cdot\|_p$ is monotonically increasing in p, and hence $J(n)$ is monotonically increasing in n. The other results follow from (25) and Lemma 2.

IV. Example Studies

To illustrate the algorithm works, two examples are studied. They are as follows:

i) $C(z)=1-z-z^2$

ii) $C(z)=1+2z+3z^2$.

It is easy to show in example ii) that $\|C(z)\|_\infty = 6$. However, it is not straightforward to see in example i) that $\|C(z)\|_\infty = \sqrt{5}$. After 1500 iterations, the H_∞-norm is approximated with relative error under 0.00154 in both cases, which are depicted in Figs. 1 and 2, respectively.

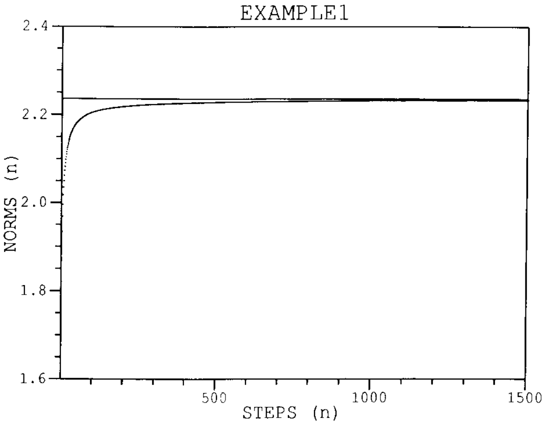

Fig. 1.

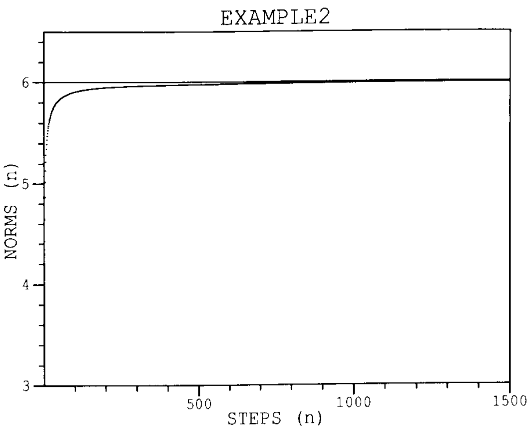

Fig. 2.

V. Conclusions and Remarks

a) The proposed algorithm has itself theoretical interests as well as its application importance. Various algorithms for minimization (maximization) of functions exist [6]–[8], but to the authors' knowledge, theoretical algorithms for computing the H^∞-norm have not yet been studied elsewhere.

b) It is interesting to note that the principal part of the relative error of the algorithm is independent of the polynomial $C(z)$ (i.e., $(\log n)/2n$). Furthermore, the error is monotonically decreasing to zero. So, for a given relative error, we can roughly decide the iteration step n to achieve the desired accuracy.

c) In this note, we have only considered the scalar polynomial case. Of course, for a given stable scalar rational function, one can first approximate it by an rth-order polynomial (with exponential decaying error $0(\lambda^r)$, $0 < \lambda < 1$) and then use the above method to approximate the H^∞-norm of the rational function. It is desirable to extend the results of this note to the general matrix transfer function case.

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