Example: Assume that θ belongs to $F = \{0, 1\}$ and $A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Take $B_0 = B_1 = I$, then (A_i, B_i) are controllable, $i = 0, 1$.

Then $K_0 = \begin{bmatrix} 0 & k_0 \\ k_0 & 0 \end{bmatrix}$, $k_0 \in R$ is the general solution of $A_0^T K_0 + K_0 A_0 = 0$ and K_0 has eigenvalues $\lambda_{01} = k_0$, $\lambda_{02}, = -k_0$, so that K_0 is an indefinite matrix.

Also $K_1 = \begin{bmatrix} k_1 & 0 \\ 0 & k_1 \end{bmatrix}$, $k_1 \in R$ is the general solution of $A_1^T K_1 + K_1 A_1 = 0$ and K_1 has eigenvalues $\lambda_{11} = k_1$ with multiplicity two, so that K_1 is positive or negative semidefinite for $k_1 \geq 0$ or $k_1 \leq 0$, respectively.

The triples (A_i, B_i, C_i), $i = 0, 1$ are minimal with $C_i = F_i = B_i^T K_i$ if and only if the pairs (A_0, C_0) and (A_1, C_1) are observable. Since A_0 and A_1 are nonsingular matrices we require that K_0 and K_1 each be of rank 2. which is true iff $k_0 \neq 0$ and $k_1 \neq 0$.

The triples (A_0, B_0, C_0) and (A_1, B_1, C_1) are uniformly stabilizable with $F = I$ iff $\bar{A}_0 = A_0 - B_0 F C_0$ and $\bar{A}_1 = A_1 - B_1 F C_1$ are stable. Now $\bar{A}_0 = A_0 - K_0 = \begin{bmatrix} 1 & -k_0 \\ -k_0 & -1 \end{bmatrix}$ which has real eigenvalues located at $\pm\sqrt{1 + k_0^2}$.

Thus, the system is not uniformly nor adaptively stabilizable.

We also note that $\bar{A}_1 = A_1 - K_1 = \begin{bmatrix} -k_1 & 1 \\ -1 & k_1 \end{bmatrix}$ which has eigenvalues $\lambda_{11} = -k_1 + i$ and $\lambda_{12} = -k_1 - i$ so that K_1 stabilizes A_1, if $k_1 > 0$.

Take $\pi_1(0) = 1$, so that the problem becomes a standard LQG problem.

Now, since K_1 stabilizes A_1, we obtain

$$J^u(x_0) = \inf_{k_1} (x_0^T K_1 x_0 + trBK_1),\ K_1 = \operatorname{diag}\{k_1\}$$

s.t. $k_1 > 0$ which does not have a unique positive definite "minimizing" solution.

Conclusions

It has been shown that the continuous-time version of the DUL controller is optimal for a cost functional that includes a quadratic term and a nonquadratic term referred to as the "dual or learning cost." Since the DUL controller is a "passively learning" controller it becomes clear that there must be a probing effect induced by the original quadratic cost functional that is removed by subtracting the "dual cost." We conclude that no extra terms need to be added to the cost functional for active probing of the system if a "standard" quadratic cost functional is used.

The dual cost has been defined in [4]. A general theory for discrete-time problems will be given in a companion paper which will appear at a later time.

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Least-Squares Identification for ARMAX Models without the Positive Real Condition

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***Abstract*—The main purpose of this note is to study recursive identification problems of linear stochastic feedback control systems described by ARMAX models, *without imposing the strictly positive real (SPR) condition* on the noise model. The key ingredient in the present method is the introduction of increasing lag regressors to formulate the least-squares estimates for the noise process, while the main techniques in the convergence analysis are limit theorems for *double array* martingales.**

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I. Introduction

Let us consider the following linear stochastic control systems described by the ARMAX model:

$$A(z)y_n = B(z)u_n + C(z)w_n, \qquad n \geq 0 \tag{1}$$

where y_n, u_n, and w_n are the m-, l-, and m-dimensional system output, input, and noise sequences, respectively, $A(z)$, $B(z)$, and $C(z)$ are matrix polynomials in backwards-shift operator z:

$$A(z) = I + A_1 z + \cdots \cdots + A_p z^p, \qquad p \geq 0, \tag{2}$$

$$B(z) = B_1 z + B_2 z^2 + \cdots \cdots + B_q z^q, \qquad q \geq 0, \tag{3}$$

$$C(z) = I + C_1 z + \cdots \cdots + C_r z^r, \qquad r \geq 0 \tag{4}$$

with *unknown* coefficients A_i, B_j, C_k ($i = 1, \cdots p$, $j = 1, \cdots q$, $k = 1 \cdots r$), and known upper bounds p, q, and r for the true orders. Let us organize the unknown coefficients into a parameter θ

$$\theta = [-A_1 \cdots -A_p,\ B_1 \cdots B_q,\ C_1 \cdots C_r]^\tau. \tag{5}$$

We assume that the innovation process $\{w_n\}$ is a martingale difference sequence with respect to a family $\{F_n\}$ of nondecreasing σ-algebras, and that the input u_n is any F_n-measurable vector for $n \geq 0$, i.e.,

$$E[w_{n+1} | F_n] = 0, \qquad u_n \in F_n,\ \forall n \geq 0. \tag{6}$$

Thus, the input sequence may include any feedback control signal. Furthermore, we assume that

$$\sup_n E[\|w_{n+1}\|^4 | F_n] < \infty,\ \liminf_{n\to\infty} \lambda_{\min}\left\{\frac{1}{n}\sum_{i=0}^{n-1} w_i w_i^\tau\right\} \neq 0, \qquad \text{a.s.,} \tag{7}$$

$$\|w_n\| = 0(d(n)), \qquad \text{a.s.,} \tag{8}$$

where $\{d(n)\}$ is a positive nondecreasing deterministic sequence. Here and thereafter the norm for a real matrix X is defined as $\|X\| = \{\lambda_{\max}(XX^\tau)\}^{1/2}$, and the maximum[minimum] eigenvalue of a square matrix X is denoted by $\lambda_{\max}(X)[\lambda_{\min}(X)]$.

Note that (7) implies that $\|w_n\| = 0(n^c)$ a.s. $\forall c > 1/4$, by the conditional Borel–Cantelli lemma [1]. Better bounds are also obtainable if there are further assumptions, for example, $\|w_n\| = 0(\{\log n\}^{1/2})$ when $\{w_n\}$ is Gaussian and white.

Since $\{w_n\}$ is the innovation sequence, it is natural to assume that the noise model $C(z)$ is stable (e.g., [2]), i.e.,

$$\det C(z) \neq 0, \qquad \forall z: |z| \leq 1, \tag{9}$$

however, further *a priori* information on $C(z)$ is generally unavailable [3].

Estimation and its related adaptive control problems for system (1) have been extensively studied over the last decade in the engineering literature. Many identification algorithms have been proposed and analyzed (e.g., [2], [4], [5]). However, most of the existing recursive identification and adaptive control algorithms need the noise model to be strictly positive real (SPR) in the convergence analysis. Specifically, for the standard extended least-squares (ELS) algorithm, it is required that

$$C^{-1}(e^{i\lambda}) + C^{-\tau}(e^{i\lambda}) - I > 0, \qquad \forall \lambda \in [0, 2\pi],\ (i^2 = -1).$$

This condition is obviously not verifiable *a priori*, and necessarily implies that $\|[C_1 \cdots C_r]\| < 1$. Hence, it is a much stronger condition than (9). Qualitatively, it means that the noise process $C(z)w_n$ is not "too colored."

It is also known that if the SPR condition fails, counterexamples can be constructed such that the ELS algorithm does not converge [6].

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Several attempts have been taken to relax the SPR condition in adaptive estimation and control. It has been suggested in [7] that in the scalar case, if there is a known polynomial $D(z)$ such that $D(z)C^{-1}(z) - 1/2$ is SPR, then by modifying the ELS algorithm (incorporating prefiltering), strong consistency of parameter estimates can be guaranteed (e.g., [2], [7]). Unfortunately, the prefilter $D(z)$ is generally unavailable. Another interesting idea used in [3] is to guarantee the noise model SPR condition by overparametrization, or to suitably increase the lag of the regression vectors. However, this method needs *a priori* knowledge for the stability margin of the $C(z)$ polynomial and may not give consistent parameter estimates, as is mentioned by the authors [3]. A further attempt was made in [8] where the "pre-whitening" idea was proposed. This idea is to add a white noise sequence to the output data in the implementation of the usual estimation algorithms in order to estimate the unknown parameters A_i and B_j, while the noise parameters C_k are estimated by a parallel algorithm involving on-line spectral factorization. This approach applies to a special class of linear regression models [9], but fails for general ARMAX models as recognized in [9].

Similar estimation problems have also received extensive attention in a related area of time series analysis. The main interest, however, is in open-loop identification of stationary processes, since it is usually assumed that *either* the system is subjected to no control actions, e.g., the standard ARMA models [10]–[12], *or* the input sequence $\{u_n\}$ is stationary and independent of the noise process $\{w_n\}$ (e.g., [13]–[15]). This latter restriction excludes the application of the results in [13]–[15] to general feedback control systems. This is because any real feedback controller depends essentially on the system output and hence the system noise, and in general is nonstationary. Anyway, in the following we will see that some of the techniques in this area do turn out to be helpful in solving our present problems.

In this note, we present an identification scheme for system (1) without imposing the traditionally used noise model SPR condition for its convergence. This scheme consists of two steps. In the first step, estimates for the noise process are formed by using increasing lag least squares. The parameter estimates for θ are then formed in the second step by using an ELS algorithm with regressors formed by using the noise estimates of the first step. This algorithm is similar to those used in time series analysis (e.g., [12]–[14]). The convergence analysis here, however, turns out to be completely different from those in [12]–[14] due to the nonstationarity and dependency of the system signals $\{y_n, u_n, w_n\}$ *in the present case*. In fact, the recently established limit theorems for *double array* martingales in [16] are the crucial analytical tools in this note.

The note is organized as follows. In Section II we describe the estimation algorithm and state the main results. The convergence analysis is given in Section III, and finally, Section IV concludes the note with some remarks.

II. Main Results

Algorithm Description: Note that our objective here is, at any time n, to give an estimate $\hat{\theta}(n)$ for the unknown parameter θ based on the observation data $\{y_i, u_{i-1}, 0 \le i \le n\}$ only.

Let $\{p_n\}$ be a sequence of positive integers such that $1 \le p_n \le p_{n+1} \le \cdots$ and $p_n = o(n)$. The *estimation algorithm* is divided into the following two steps.

Step 1: For any $n > 0$, let us define the following regressors:

$$\psi_i(n) = [y_i^\tau \cdots y_{i-p_n+1}^\tau, u_i^\tau, \cdots u_{i-p_n+1}^\tau]^\tau, \qquad 0 \le i \le n-1 \tag{10}$$

so that we can use the least-squares method to obtain the estimates $\{\hat{w}_i(n), 0 \le i \le n-1\}$ for the noise process $\{w_i, 0 \le i \le n-1\}$ as follows:

$$\hat{w}_i(n) = y_i - \hat{\alpha}_i^\tau(n)\psi_{i-1}(n), \qquad 0 \le i \le n-1, \tag{11}$$

$$\hat{\alpha}_{i+1}^\tau(n) = \hat{\alpha}_i^\tau(n) + b_i(n)P_i(n)\psi_i(n)[y_{i+1} - \psi_i^\tau(n)\hat{\alpha}_i(n)], \qquad 0 \le i \le n-1, \tag{12}$$

$$P_{i+1}(n) = P_i(n) - b_i(n)P_i(n)\psi_i(n)\psi_i^\tau(n)P_i(n),$$

$$b_i(n) = \{1 + \psi_i^\tau(n)P_i(n)\psi_i(n)\}^{-1} \tag{13}$$

where the initial values $\hat{\alpha}_0^\tau(n) = 0$, $P_0(n) = \beta I$, $\beta > 0$.

Step 2: The estimate $\hat{\theta}(n)$ for the parameter θ at time n is then obtained by an extended least-squares method with regressors formed by using the noise estimates obtained above

$$\hat{\theta}(n) = \left[\sum_{i=0}^{n-1} \varphi_i(n)\varphi_i^\tau(n) + \beta I\right]^{-1} \sum_{i=0}^{n-1} \varphi_i(n) y_{i+1}^\tau \tag{14}$$

$$\varphi_i(n) = [y_i^\tau \cdots y_{i-p+1}^\tau, u_i^\tau \cdots u_{i-q+1}^\tau, \hat{w}_i^\tau(n) \cdots \hat{w}_{i-r+1}^\tau(n)]^\tau, \qquad 0 \le i \le n-1. \tag{15}$$

Remark 1: Neither of the two steps in the above algorithm are surprising. The first step corresponds to the estimation problems of the equivalent model $C^{-1}(z)A(z)y_n = C^{-1}(z)B(z)u_n + w_n$, while the second step corresponds to the standard ELS method. The use of increasing lag least squares as in Step 1 was first suggested by Durbin [17]. However, rigorous theoretical results for even the stationary case have become available only in recent years (e.g., [12]–[15]). Of course, more complicated algorithms similar to the Hannan–Rissanen method [12] or that in [13] involving on-line order determination can also be considered. In this case, the computations will become more complicated.

Remark 2: In [18] we also used the idea of a "two step" algorithm. The second step is similar. However, in the first step an enlarged lag ELS is used instead of an increasing lag least squares as here. Of course, if we know *a priori* the stability margin of $C(z)$, then there may be no need to introduce increasing lag regressors as in Step 1. In fact, in this case, by the overparametrization method in [3], we may first choose a suitably overparametrized system with an SPR noise model, and then use the standard ELS method to form the noise estimates. So, in this case, Step 1 can be replaced by a suitably large but fixed lag ELS algorithm. The key idea behind this is that although overparametrization may not give consistent parameter estimates, it does give good noise estimates (see [18] or [20] eq. (29) and (31), for example).

The following main results of the note will be proven in Section III.

Theorem 1: Consider the system described by (1)–(9) and the "two step" least-squares algorithm defined by (10)–(15). Then as $n \to \infty$, the estimation error satisfies

$$\|\hat{\theta}(n) - \theta\|^2 = O\left(\frac{1}{\lambda_{\min}(n)}\{p_n \log r_n + [d(n)\log n]^{2+\epsilon} + [p_n \log r_n]^{(1/2)+\epsilon} d^2(n) \log^{1+\epsilon} n + n r_n \exp[-\lambda p_n]\}\right), \text{ a.s.}$$

for any $\epsilon > 0$, where $\lambda > 0$ is some constant and r_n, $\lambda_{\min}(n)$ are defined as

$$r_n = 1 + \sum_{i=0}^{n-1} [\|y_i\|^2 + \|u_i\|^2] \tag{16}$$

$$\lambda_{\min}(n) \triangleq \lambda_{\min}\left\{\sum_{i=0}^{n-1} \varphi_i(n)\varphi_i^\tau(n) + \beta I\right\}. \tag{17}$$

Let us now introduce the following regressors:

$$\varphi_i^0 = [y_i^\tau \cdots y_{i-p+1}^\tau, u_i^\tau \cdots u_{i-q+1}^\tau, w_i^\tau \cdots w_{i-r+1}^\tau]^\tau, \qquad i \ge 0 \tag{18}$$

which, in contrast to $\varphi_i(n)$ defined by (15), is free of estimates and depends explicitly on the three system signals $\{y_i, u_i, w_i\}$. Similar to (17) we set

$$\lambda_{\min}^0(n) \triangleq \lambda_{\min}\left\{\sum_{i=0}^{n-1} \varphi_i^0 \varphi_i^{0\tau} + \beta I\right\}. \tag{19}$$

Corollary 1: If, in addition to the assumptions of Theorem 1, the regression lag p_n in (10) is taken as $p_n = [\log^a(n+e)]$, $a > 1$, and the growth rate of r_n and $\lambda_{\min}^0(n)$ defined, respectively, by (16) and (19) satisfy

$$r_n = O(n^b), \text{ a.s.}, \qquad \text{for some } b \ge 1 \tag{20}$$

and

$$\log^{1+a} n + [d(n)\log n]^{2+\epsilon} + d^2(n)[\log n]^{(3+a)/2+\epsilon} = o(\lambda_{\min}^0(n)) \tag{21}$$

for some $\epsilon > 0$, then as $n \to \infty$, the estimation error satisfies

$$\|\hat{\theta}(n)-\theta\|^2 = O\left(\frac{1}{\lambda^0_{\min}(n)}\{[\log n]^{1+a}+[d(n)\log n]^{2+\epsilon} + d^2(n)[\log n]^{(3+a)/2+\epsilon}\}\right).$$

Proof: See Section III.

Thus, for example, if the noise is Gaussian, then $d(n)$ given by (8) can be taken as $\{\log n\}^{1/2}$ since in this case $\|w_n\| = O(\{\log n\}^{1/2})$. It follows from Corollary 1 that

$$\|\hat{\theta}(n)-\theta\|^2 = O\left(\frac{\log^{c(\epsilon)} n}{\lambda^0_{\min}(n)}\right),$$

$$c(\epsilon)=\max\left\{1+a, \frac{5+a}{2}+\epsilon\right\}, \quad \forall \epsilon>0.$$

Now, a key problem is: How fast is the growth rate of $\lambda^0_{\min}(n)$. This, however, cannot be solved if there is no further assumptions on the input sequence $\{u_n\}$. We now consider the "attenuating excitation controller" described in [19], [20], which includes a large class of feedback controllers, and has been used in solving adaptive LQ control as well as robust adaptive tracking problems (e.g., [19]–[21]). This kind of controller is described as follows.

Let $\{v_n\}$ be a sequence of l-dimensional mutually independent random vectors independent of $\{w_n\}$ with properties

$$Ev_n=0,\ Ev_nv_n^\tau=\gamma/n^\mu,\ \sup_n E\|v_n\|^4<\infty,\ \mu \in \left[0, \frac{1}{2(d+1)}\right) \tag{22}$$

where $d \triangleq \max(p, q, r) + mp - 1$ and $\gamma > 0$ is an arbitrary constant.

Without loss of generality we may assume that $F_n = \sigma\{w_i, v_i, i \le n\}$. Let u_n^0 be an l-dimensional and $F_n' \triangleq \sigma\{w_i, v_{i-1}, i \le n\}$ -measurable desired controller. Obviously, any feedback control law is of this kind, and in the adaptive control case u_n^0 is usually given by the "certainty equivalence principle." The excitation techniques used in [19], [20] suggest that the actual input for the system is

$$u_n = u_n^0 + v_n \tag{23}$$

instead of $u_n = u_n^0$. We note that when μ in (22) is taken as 0, then (23) is similar to the "continuously disturbed controller" proposed in [22].

The following result is proved in [18], which is a generalization of assertion (45) in [19].

Proposition 1: Suppose that for system (1)–(7), $A(z)$, $B(z)$, and $C(z)$ have no common left factor, and that *at least one* of the three matrices $\{A_p, B_q, C_r\}$ is of full row rank. If the attenuating excitation controller (23) is applied to system (1) and that

$$\sum_{i=0}^{n-1}\{\|y_i\|^2+\|u_i\|^2\}=O(n^{1+\delta}),$$

$$\text{a.s., for some } \delta \in \left[0, \frac{1-2\mu(d+1)}{2d+3}\right]$$

where d and μ are as in (22). Then for $\lambda^0_{\min}(n)$ defined by (19),

$$\liminf_{n\to\infty} \lambda^0_{\min}(n)/n^\alpha > 0, \qquad \text{a.s., } \alpha \triangleq 1-(d+1)(\mu+\delta).$$

Remark 3: We mention that if $\mu = \delta = 0$, then $\alpha = 1$. Consequently, the familiar persistence of the excitation condition is achieved for a class of feedback control systems. Note that the control law (23) includes also a large class of open-loop inputs, e.g., ARMA processes with innovations $\{v_n\}$.

Remark 4: Several applications of the results of this note are straightforward. For example, combining the estimation algorithm of this note with the adaptive LQ controller in [20] or [19], we may get simultaneously the minimality of control performance and consistency of parameter estimates. It is also true that if we use the innovation estimate (11) to replace that in [18], then similar order estimation results are also obtainable. Certainly, in these applications the SPR condition will no longer be required.

III. Convergence Analysis

In the convergence analysis of the algorithm, we will need estimations for double array sums of the form

$$S_i(n)=\sum_{j=0}^{i-1} f_{jn}w_{j+1}, \qquad 0\le i\le n,\ n\ge 1.$$

Lemma 1: Let $\{w_i\}$ and $\{f_{in}, 1 \le i \le n\}$, $n \ge 1$ be any m- and p_n-dimensional random sequences, respectively. Denote $M_i(n) = \sum_{j=0}^{i-1} f_{jn}f_{jn}^\tau + \beta I$, $\beta > 0$. Then,

$$\sum_{j=0}^{n-1} f_{jn}^\tau[M_{j+1}(n)]^{-1}f_{jn} = O(p_n \log^+ \lambda_{\max}(M_n(n))) \tag{24}$$

and

$$\operatorname{tr}\{S_n^\tau(n)[M_n(n)]^{-1}S_n(n)\} \le \sum_{j=0}^{n-1}\|w_{j+1}\|^2, \qquad \forall n \ge 1. \tag{25}$$

Furthermore, if $\{w_i\}$ is a *martingale difference* sequence satisfying (6)–(8) and f_{in} is F_i-measurable for any $0 \le i \le n$, $n \ge 0$. Then as $n \to \infty$

$$\operatorname{tr}\{S_n^\tau(n)[M_n(n)]^{-1}S_n(n)\} = O(p_n \log^+\lambda_{\max}(M_n(n))) + 0(\{d(n)\log n\}^{2+\epsilon}) + 0(\{p_n \log[e+\lambda_{\max}(M_n)]\}^{(1/2)+\epsilon}[\log^{1+\epsilon} n]d^2(n), \qquad \forall \epsilon > 0 \tag{26}$$

and

$$\sum_{i=0}^{n-1}\|S_{i+1}^\tau(n)[M_{i+1}(n)]^{-1}f_{in}\|^2 = O(p_n\log^+\lambda_{\max}(M_n(n))) + 0(\{d(n)\log n\}^{2+\epsilon}) + 0(\{p_n\log[e+\lambda_{\max}(M_n)]\}^{(1/2)+\epsilon}[\log^{1+\epsilon} n]d^2(n)), \quad \forall\epsilon>0 \tag{27}$$

where $\log^+\{\cdot\}$ means the positive part of $\log\{\cdot\}$.

Proof: The key steps in the proof are first to use the matrix inverse formula to get the following recursion ($c_j(n) = [1 + f_{jn}^\tau[M_j(n)]^{-1}f_{jn}]^{-1}$):

$$\begin{aligned}\operatorname{tr}\{S_{i+1}^\tau(n)[M_{i+1}(n)]^{-1}S_{i+1}(n)\} &= \operatorname{tr}\{S_i^\tau(n)[M_i(n)]^{-1}S_i(n)\} \\ &\quad + 2c_i(n)w_{i+1}^\tau S_i^\tau(n)[M_i(n)]^{-1}f_{in} - c_i(n)\|S_1^\tau(n)[M_i(n)]^{-1}f_{in}\|^2 \\ &\quad + c_i(n)f_{jn}^\tau[M_j(n)]^{-1}f_{jn}\|w_{i+1}\|^2\end{aligned}$$

then to sum up both sides of the above equality and use estimations for double array martingales to get the desired results. For details see [16, Lemmas 3.4 and 3.6] together with their proofs.

We remark that when f_i stands for the usual regressors (nondouble array case), the quantity $\operatorname{tr}\{S_n^\tau(n)[M_n(n)]^{-1}S_n(n)\}$ is nothing but the standard stochastic Lyapunov function frequently used in the literature (e.g., [2], [4], [5], [20], [23], [24]).

We now prove the main results of the note.

Proof of Theorem 1: Let us consider the estimation error for the noise process first. Denote

$$C^{-1}(z)A(z)=I+\sum_{i=1}^{\infty}G_iz^i,\ C^{-1}(z)B(z)=\sum_{i=1}^{\infty}H_iz^i \tag{28}$$

and set

$$\alpha(n)=[-G_1, \cdots -G_{p_n}, H_1 \cdots H_{p_n}]^\tau,$$

$$\epsilon_i(n)=\sum_{t=p_n+1}^{\infty}[-G_ty_{i-t+1}+H_tu_{i-t+1}]. \tag{29}$$

Then by (1) and (28), (29) we see that

$$y_i = \alpha^\tau(n)\psi_{i-1}(n) + \epsilon_{i-1}(n) + w_i \tag{30}$$

which in conjunction with (11) yields

$$\hat{w}_i(n) - w_i = [\alpha^\tau(n) - \hat{\alpha}_i^\tau(n)]\psi_{i-1}(n) + \epsilon_{i-1}(n). \tag{31}$$

Note that for any fixed n, (12), (13) is the standard least-squares recursion, so we have

$$\hat{\alpha}_i(n) = \left\{\sum_{j=0}^{i-1} \psi_j(n)\psi_j^\tau(n) + \beta I\right\}^{-1} \left\{\sum_{j=0}^{i-1} \psi_j(n) y_{j+1}^\tau\right\}. \tag{32}$$

Substituting (30) into this we get

$$\hat{\alpha}_i(n) - \alpha(n) = \left\{\sum_{j=0}^{i-1} \psi_j(n)\psi_j^\tau(n) + \beta I\right\}^{-1} \cdot \left\{\sum_{j=0}^{i-1} \psi_j(n)[w_{j+1}^\tau + \epsilon_j^\tau(n)] - \beta\alpha(n)\right\}$$

consequently, by noting $\|\psi_{i-1}^\tau(n)\{\sum_{j=0}^{i-1} \psi_j(n)\psi_j^\tau(n) + \beta I\}^{-1/2}\|^2 \le 1$, we know that

$$\begin{aligned}
&\|\psi_{i-1}^\tau(n)[\hat{\alpha}_i(n) - \alpha(n)]\|^2 \\
&\le 3\|\psi_{i-1}^\tau(n)\left\{\sum_{j=0}^{i-1} \psi_j(n)\psi_j^\tau(n) + \beta I\right\}^{-1} \sum_{j=0}^{i-1} \psi_j(n) w_{j+1}^\tau\|^2 \\
&\quad + 3\|\psi_{i-1}^\tau(n)\left\{\sum_{j=0}^{i-1} \psi_j(n)\psi_j^\tau(n) + \beta I\right\}^{-1} \sum_{j=0}^{i-1} \psi_j(n)\epsilon_j^\tau(n)\|^2 \\
&\quad + 3\|\psi_{i-1}^\tau(n)\left\{\sum_{j=0}^{i-1} \psi_j(n)\psi_j^\tau(n) + \beta I\right\}^{-1} \beta\alpha(n)\|^2 \\
&\le 3\|\psi_{i-1}^\tau(n)\left\{\sum_{j=0}^{i-1} \psi_j(n)\psi_j^\tau(n) + \beta I\right\}^{-1} \sum_{j=0}^{i-1} \psi_j(n) w_{j+1}^\tau\|^2 \\
&\quad + 3\|\left\{\sum_{j=0}^{i-1} \psi_j(n)\psi_j^\tau(n) + \beta I\right\}^{-1/2} \sum_{j=0}^{i-1} \psi_j(n)\epsilon_j^\tau(n)\|^2 \\
&\quad + O(\|\psi_{i-1}^\tau(n)\left\{\sum_{j=0}^{i-1} \psi_j(n)\psi_j^\tau(n) + \beta I\right\}^{-1/2}\|^2).
\end{aligned}$$

Summing up from 1 to n, and applying (24), (25), and (27), we get

$$\begin{aligned}
&\sum_{i=1}^{n} \|\psi_{i-1}^\tau(n)[\hat{\alpha}_i(n) - \alpha(n)]\|^2 \\
&= O\left(p_n \log^+ \lambda_{\max}\left\{\sum_{j=0}^{n-1} \psi_j(n)\psi_j^\tau(n)\right\}\right) + o(\{d(n) \log n\}^{2+\epsilon}) \\
&\quad + o\left(\left\{p_n \log^+ \lambda_{\max}\left[\sum_{j=0}^{n-1} \psi_j(n)\psi_j^\tau(n)\right]\right\}^{(1/2)+\epsilon} [\log^{1+\epsilon} n] d^2(n)\right) \\
&\quad + O\left(\sum_{i=1}^{n}\sum_{j=0}^{i-1} \|\epsilon_j(n)\|^2\right).
\end{aligned} \tag{33}$$

Note that by (9), (28), (29), and the Schwarz inequality it follows that:

$$\begin{aligned}
&\sum_{i=0}^{n-1} \|\epsilon_i(n)\|^2 \\
&\le 2\sum_{i=0}^{n-1}\left\{\sum_{j=p_n+1}^{\infty} \|G_j\| \sum_{j=p_n+1}^{\infty} \|G_j\|\, \|y_{i-j+1}\|^2 + \sum_{j=p_n+1}^{\infty} \|H_j\| \sum_{j=p_n+1}^{\infty} \|H_j\|\, \|u_{i-j+1}\|^2\right\} \\
&\le 2\left(\sum_{j=p_n+1}^{\infty} \|G_j\|\right)^2 \sum_{i=0}^{n-1} \|y_i\|^2 + 2\left(\sum_{j=p_n+1}^{\infty} \|H_j\|\right)^2 \sum_{i=0}^{n-1} \|u_i\|^2 \\
&= O(r_n \exp\{-\lambda p_n\}), \qquad \text{for some } \lambda > 0.
\end{aligned} \tag{34}$$

Similarly, by (1), (7), (28), and the Schwarz inequality it is easy to verify that

$$\begin{aligned}
0 &\ne \liminf_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \|w_i\|^2 \\
&\le \liminf_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1}\left\{\sum_{j=0}^{\infty} (\|G_j y_{i-j}\| + \|H_j u_{i-j}\|)\right\}^2 \\
&\le 2\left\{\sum_{j=0}^{\infty} (\|G_j\| + \|H_j\|)\right\}^2 \liminf_{n\to\infty} r_n/n = O(\liminf_{n\to\infty} r_n/n)
\end{aligned} \tag{35}$$

and then we have

$$\log^+ \lambda_{\max}\left\{\sum_{i=0}^{n-1} \psi_i(n)\psi_i^\tau(n)\right\} = O(\log\{p_n r_n\}) = O(\log r_n). \tag{36}$$

Finally, substituting (34) and (36) into (33) we see from (31) that for any $\epsilon > 0$,

$$\begin{aligned}
&\sum_{i=0}^{n-1} \|\hat{w}_{i+1}(n) - w_{i+1}\|^2 \\
&= O\left(\sum_{i=1}^{n} \|\psi_{i-1}^\tau(n)[\hat{\alpha}_i(n) - \alpha(n)]\|^2\right) + O\left(\sum_{i=1}^{n} \|\epsilon_{i-1}(n)\|^2\right) \\
&= O(p_n \log r_n) + o(\{d(n) \log n\}^{2+\epsilon}) \\
&\quad + o(\{p_n \log r_n\}^{(1/2)+\epsilon} d^2(n) \log^{1+\epsilon} n) + O(n r_n \exp\{-\lambda p_n\}).
\end{aligned} \tag{37}$$

Next, we consider the estimation error for parameter estimates. By (5) and (18) we may rewrite system (1) as

$$y_{i+1} = \theta^\tau \varphi_i^0 + w_{i+1} = \theta^\tau \varphi_i(n) + \theta^\tau[\varphi_i^0 - \varphi_i(n)] + w_{i+1}. \tag{38}$$

Substituting this into (14) we obtain that

$$\begin{aligned}
\hat{\theta}(n) - \theta &= \left[\sum_{i=0}^{n-1} \varphi_i(n)\varphi_i^\tau(n) + \beta I\right]^{-1} \sum_{i=0}^{n-1} \varphi_i(n)[\varphi_i^0 - \varphi_i(n)]^\tau\theta \\
&\quad + \left[\sum_{i=0}^{n-1} \varphi_i(n)\varphi_i^\tau(n) + \beta I\right]^{-1} \sum_{i=0}^{n-1} \varphi_i(n) w_{i+1}^\tau \\
&\quad - \left[\sum_{i=0}^{n-1} \varphi_i(n)\varphi_i^\tau(n) + \beta I\right]^{-1} \beta\theta.
\end{aligned} \tag{39}$$

Note that by (15), (18), and (37),

$$\sum_{i=0}^{n-1} \|\varphi_i^0 - \varphi_i(n)\|^2 = O(p_n \log r_n) + o(\{d(n) \log n\}^{2+\epsilon}) + o(\{p_n \log r_n\}^{(1/2)+\epsilon} d^2(n) \log^{1+\epsilon} n) + O(n r_n \exp\{-\lambda p_n\}). \quad (40)$$

Consequently by noting (35)

$$\log^+ \lambda_{\max}\left\{\sum_{i=0}^{n-1} \varphi_i(n)\varphi_i^T(n)\right\} = O\left(\log^+\left\{\sum_{i=0}^{n-1}\|\varphi_i^0\|^2\right\}\right) + O\left(\log^+\left\{\sum_{i=0}^{n-1}\|\varphi_i^0 - \varphi_i(n)\|^2\right\}\right) = O(\log r_n) + O(\log n) = O(\log r_n), \text{ a.s.} \quad (41)$$

Applying (25) and (26) to (39) and noting (40), (41), we finally get

$$\|\hat{\theta}(n) - \theta\|^2 = O\left(\frac{1}{\lambda_{\min}(n)}\left\{\left\|\left[\sum_{i=0}^{n-1}\varphi_i(n)\varphi_i^T(n) + \beta I\right]^{-1/2}\sum_{i=0}^{n-1}\varphi_i(n)[\varphi_i^0 - \varphi_i(n)]^T\right\|^2\right\}\right)$$
$$+ O\left(\frac{1}{\lambda_{\min}(n)}\left\|\left[\sum_{i=0}^{n-1}\varphi_i(n)\varphi_i^T(n) + \beta I\right]^{-1/2}\sum_{i=0}^{n-1}\varphi_i(n)w_{i+1}^T\right\|^2\right)$$
$$+ O\left(\frac{1}{\lambda_{\min}(n)}\right) = O\left(\frac{1}{\lambda_{\min}(n)}\sum_{i=0}^{n-1}\|\varphi_i^0 - \varphi_i(n)\|^2\right)$$
$$+ O\left(\frac{1}{\lambda_{\min}(n)}\left\|\left[\sum_{i=0}^{n-1}\varphi_i(n)\varphi_i^T(n)\right]^{-1/2}\sum_{i=0}^{n-1}\varphi_i(n)w_{i+1}^T\right\|^2\right)$$
$$= O\left(\frac{1}{\lambda_{\min}(n)}\{p_n \log r_n + [d(n)\log n]^{2+\epsilon} + [p_n \log r_n]^{(1/2)+\epsilon} d^2(n)\log^{1+\epsilon} n + n r_n \exp[-\lambda p_n]\}\right).$$

This completes the proof of Theorem 1. #

Proof of Corollary 1: We need only to show that $\lambda_{\min}^0(n) = O(\lambda_{\min}(n))$, but this is straightforward by (21) and (40), since with some simple manipulations,

$$\lambda_{\min}^0(n) = \lambda_{\min}\left\{\sum_{i=0}^{n-1}\varphi_i^0\varphi_i^{0T}\right\} + \beta$$
$$\leq 2\lambda_{\min}\left\{\sum_{i=0}^{n-1}\varphi_i(n)\varphi_i^T(n) + \beta I\right\} + 2\sum_{i=0}^{n-1}\|\varphi_i^0 - \varphi_i(n)\|^2$$
$$\leq 2\lambda_{\min}(n) + O(\log^{1+\epsilon} n) + O([d(n)\log n]^{2+\epsilon}) + O(d^2(n)[\log n]^{(3+a)/2+\epsilon}) = 2\lambda_{\min}(n) + o(\lambda_{\min}^0(n)). \quad \#$$

IV. Conclusion

By use of a "two step" least-squares algorithm, we developed a strongly consistent parameter estimator for ARMAX processes without requiring the standard strictly positive real condition on the noise model. An increasing lag least squares is used in the first step to estimate the noise process, while the parameter estimate is formed in the second step by an extended least squares. In the present algorithm, there is an increase in computational cost in comparison to traditional algorithms. However, the results of this note do not need any *a priori* information or conditions on the noise model except that of stability, and are applicable to the identification of general feedback control systems.

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Optimal Priority Assignment: A Time Sharing Approach

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***Abstract*—Jobs of several types arrive to their respective infinite capacity discrete-time queues. During each time slot service of a single job is attempted.**

Nonstationary "time sharing" policies are introduced to obtain optimal controls for new constrained optimization problems. The criteria are expected time averages of sizes of the queues. These policies and their cost are computed through linear programs. The achievable region of the vector of queues' length is characterized. Other applications of time sharing policies are discussed.

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