SIAM J. CONTROL AND OPTIMIZATION
Vol. 28, No. 3, pp. 513-527, May 1990

002

CONTINUOUS-TIME STOCHASTIC ADAPTIVE TRACKING—ROBUSTNESS AND ASYMPTOTIC PROPERTIES*

HAN-FU CHEN† AND LEI GUO†

Abstract. Adaptive estimation and control problems are considered for continuous-time stochastic systems containing both modeled and unmodeled dynamics. The least squares method is used to estimate unknown parameters included in the modeled part, which are used to update an adaptive control law. It is shown that both the estimation error and the tracking error are bounded, and that the bounds are proportional to constants dominating the unmodeled dynamics. Moreover, convergence rates of the tracking errrors are established in the case where no unmodeled dynamics exist.

Key words. continuous-time stochastic system, adaptive tracking, least squares, robustness, unmodeled dynamics

AMS(MOS) subject classifications. 93C40, 93E12

1. Introduction. In recent years, much attention has been devoted to the analysis of adaptive algorithms when unmodeled dynamics are contained in the system. It is known that (see, e.g., [1]-[3]) unmodeled dynamics or even small disturbances may cause instability in many adaptive algorithms when precautions are not taken. This inspired the study of robust adaptive control where the primary purpose is to maintain stability of the closed-loop system under violations of ideal assumptions. There is already a vast literature on this topic, especially in the deterministic framework (e.g., [4]-[6]).

In the stochastic case, robustness results are much more difficult to obtain. This results from the following "stochastic features": (i) a priori upper bounds for the noise sequence are usually not available, (ii) optimal or at least close to optimal rejection of the noise effects is required, and (iii) traditionally used supermartingale methods fail due to unmodeled dynamics. An initial attempt toward robustness analysis for discrete-time stochastic adaptive systems was made in [7], where an a priori assumption on the input-output data was required. This assumption was later removed in [8] for a large class of stochastic systems represented by a full ARMAX model plus unmodeled dynamics.

While discrete-time adaptive theory is well developed, the corresponding continuous-time analogue becomes a natural concern. There is no doubt that results of this kind are interesting and important in many situations. Unfortunately, it seems that they have received less attention in the literature, and that only some initial works in the adaptive estimation aspect are available (see, e.g., [9]-[12]).

In this paper, we consider both estimation and control problems for stochastic systems described by stochastic differential/integral equations. The adaptive control law is defined based on a continuous-time analogue of the least-squares estimation algorithm. We show the following:

(i) That the least squares method has some degree of robustness when unmodeled dynamics are contained in the model, provided that the system is "persistently excited."

(ii) That the closed-loop adaptive system is stable, with a tracking error upper bound. This bound implies that the tracking error will decrease when upper bounds on the unmodeled dynamics decrease.

* Received by the editors January 21, 1987; accepted for publication (in revised form) August 11, 1989.

† Institute of Systems Science, Academia Sinica, Beijing, People's Republic of China. This project was supported by the National Natural Science Foundation of China and the TWAS research grant 87-43.

(iii) That if there are no unmodeled dynamics, then the least squares estimation results parallel those obtained in the discrete-time case (see, e.g., [13], [14]); furthermore, in the present paper we provide a precise convergence rate for the tracking error, which in the discrete-time case still remains a standing issue.

We state here that the above-mentioned results are established under the assumption that the strong solution of the stochastic differential equations describing the closed-loop system exists. And for the time being, we know of no way to verify or sidestep this assumption. However, we believe that many of the ideas, techniques, and results presented in this paper are necessary preliminaries for future study.

2. The system description. Let $\{F_t\}$ be a family of nondecreasing σ-algebras defined on a probability space (Ω, F, P), and let the system to be considered be described by the following stochastic differential/integral equation:

$$
\begin{aligned}
[I+\mu_1 SH_1(S)]A(S)y_t &= [I+\mu_2 H_2(S)]SB(S)u_t + [I+\mu_3 SH_3(S)]C(S)v_t \\
&\quad + \mu_4 S\xi_t(y,u), \qquad t \geqq 0, \quad y_0=0, \quad u_0=0, \quad \xi_0=0
\end{aligned} \tag{1}
$$

where s denotes the integral operator (e.g., $Sy_t = \int_0^t y_z\, dz$), and y_t and u_t adapted to $\{F_t\}$ are m-dimensional output and l-dimensional input, respectively. The quantities μ_i, $i=1,\cdots,4$, are small constants, $H_i(S)$, $i=1,2,3$, are unmodeled matrix transfer functions, and $\xi_t(y,u)$, dependent on the previous observation $\{y_s, u_2, 0 \leqq s \leqq t\}$, is an unknown nonanticipative measurable process characterizing the unmodeled dynamics. Finally, v_t is the system noise that is generated via a known filter $D^{-1}(S)$ from a standard Wiener process (w_t, F_t):

$$D(S)v_t = w_t, \qquad t \geqq 0. \tag{2}$$

Assume that $A(S)$, $B(S)$, and $C(S)$ are matrix polynomials in S, with unknown coefficients but known upper bounds for the true orders:

$$A(S) = I + A_1 S + \cdots + A_p S^p, \qquad p \geqq 0, \tag{3}$$

$$B(S) = B_1 + B_2 S + \cdots + B_q S^{q-1}, \qquad q \geqq 1, \tag{4}$$

$$C(S) = I + C_1 S + \cdots + C_r S^r, \qquad r \geqq 1, \tag{5}$$

$$D(S) = I + D_1 S + \cdots + D_r S^r. \tag{6}$$

Note that (1) may be rewritten in the form

$$A(S)y_t = SB(S)u_t + C(S)v_t + \eta_t, \tag{7}$$

$$
\begin{aligned}
\eta_t &= \mu_4 S\xi_t(y,u) - \mu_1 SH_1(S)A(S)y_t + \mu_2 SH_2(S)B(S)u_t \\
&\quad + \mu_3 SH_3(S)C(S)v_t.
\end{aligned} \tag{8}
$$

We remark that, if the unmodeled dynamics are removed, i.e., $\eta_t = 0$, for all $t \geqq 0$, then the model (7) is reduced to the one considered in [9]-[12]. Clearly, in this case model (7) may be rewritten in the standard linear state space form, and the output process $\{y_t\}$ can be uniquely determined by the process $\{u_t, w_t\}$. In the general case, it is natural to assume that $\{y_t\}$ can also be determined by $\{u_t, w_t\}$ via (7)-(8).

We denote the collection of unknown matrix coefficients of $A(S)$, $B(S)$, and $C(S)$ by θ:

$$\theta^\tau = [-A_1 \cdots -A_p \quad B_1 \cdots B_q \quad C_1 \cdots C_r]. \tag{9}$$

In the sequel, θ is estimated by the continuous-time extended least square algorithm [10]-[12]:

$$d\theta_t = P_t\phi_t D(S)(dy_t^\tau - \phi_t^\tau \theta_t\, dt), \qquad \theta_0 = 0, \tag{10}$$

$$dP_t = -P_t\phi_t\phi_t^\tau P_t\, dt, \quad P_0 = aI \quad (a = \dim \text{ of } \phi_t), \tag{11}$$

$$\phi_t = [y_t^\tau, Sy_t^\tau \cdots S^{p-1}y_t^\tau, u_t^\tau, Su_t^\tau \cdots S^{q-1}u_t^\tau, \hat{v}_t^\tau \cdots S^{r-1}\hat{v}_t^\tau]^\tau, \tag{12}$$

$$\hat{v}_t = y_t - S\theta_t^\tau \phi_t. \tag{13}$$

Obviously, if $r=0$, then (10) and (11) can be expressed as

$$\theta_t = P_t \int_0^t \phi_s\, dy_s^\tau + P_t(P_0)^{-1}\theta_0, \tag{14}$$

$$P_t = \left(\int_0^t \phi_s\phi_s^\tau\, ds + a^{-1}\right)^{-1}, \tag{15}$$

and the right-hand side of (14) is completely determined by the observations $\{u_s, y_s, s \leqq t\}$.

In the general $r>0$ case, however, the regressor ϕ_t depends on $\{\theta_s, s \leqq t\}$. Then (10) and (11) constitute a system of nonlinear stochastic differential equations for θ_t. The existence of the solution is far from obvious since the typical Lipschitz condition, which plays a vital role in the standard theory of stochastic differential equations (see, [15, Chap. 4], for example), is hard to verify in the present case. For that study, the introduction of new techniques seems to be necessary, although our differential equations are well motivated.

Henceforth, we assume that the stochastic differential/integral equation (10)-(11) has a unique strong solution $\{\theta_t, t \geqq 0\}$ in the sense of [15, pp. 127].

Set

$$\phi_t^0 = [y_t^\tau, Sy_t^\tau \cdots S^{p-1}y_t^\tau, u_t^\tau, Su_t^\tau \cdots S^{q-1}u_t^\tau, v_t^\tau \cdots S^{r-1}v_t^\tau]^\tau, \tag{16}$$

$$\tilde{\phi}_t = [0 \cdots 0, 0 \cdots 0, \tilde{v}_t^\tau \cdots S^{r-1}\tilde{v}_t^\tau]^\tau \qquad \tilde{v}_t = v_t - \hat{v}_t, \tag{17}$$

$$Y_t = [y_t^\tau \cdots S^{p-1}y_t^\tau]^\tau, \qquad U_t = [u_t^\tau \cdots S^{q-1}u_t^\tau]^\tau, \tag{18}$$

$$V_t = [v_t^\tau \cdots S^{r-1}v_t^\tau]^\tau, \quad \hat{V}_t = [\hat{v}_t^\tau \cdots S^{r-1}\hat{v}_t^\tau]^\tau, \quad \tilde{V}_t = V_t - \hat{V}_t. \tag{19}$$

Then it follows that

$$\phi_t = [Y_t^\tau, U_t^\tau, \hat{V}_t^\tau]^\tau, \quad \phi_t^0 = [Y_t^\tau, U_t^\tau, V_t^\tau]^\tau, \quad \tilde{\phi}_t = [0, 0, \tilde{V}_t^\tau]^\tau. \tag{20}$$

Furthermore, we set

$$F_d = \begin{bmatrix} -D_1 & \cdots & \cdots & -D_r \\ I & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & I & 0 \end{bmatrix}, \qquad F_c = \begin{bmatrix} -C_1 & \cdots & \cdots & -C_r \\ I & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & I & 0 \end{bmatrix}. \tag{21}$$

By use of these kinds of matrices it is easy to represent an input-output equation in state space form. For example, from (2) and (19) we may write V_t as

$$dV_t = F_d V_t\, dt + [I, 0 \cdots 0]^\tau\, dw_t. \tag{22}$$

In the sequel, similar representations will be used without additional explanation.

On the unmodeled dynamics η_t, we make an assumption similar to that used for discrete-time systems in [8].

Assumption 1. There is a real number $\varepsilon \geqq 0$ such that

$$\int_0^t \|\dot{\eta}_s\| \, ds \leqq \varepsilon r_t, \qquad t \geqq 0 \tag{23}$$

where

$$\dot{\eta}_t = \mu_4 \xi_t(y, u) - \mu_1 H_1(S) A(S) y_t + \mu_2 H_2(S) B(S) u_t + \mu_3 H_3(S) C(S) v_t \tag{24}$$

and

$$r_t = e + \int_0^t \|\phi_s\|^2 \, ds. \tag{25}$$

We also need the following condition on the noise model, which in the discrete-time case is a standard assumption.

Assumption 2. $D(S)$ is stable and the transfer matrix $D(S)C^{-1}(S) - I/2$ is strictly positive real.

At first sight, Assumption 1 is somewhat hard to understand and rather restrictive. However, the following examples show that there is at least one substantial and important class of dynamical systems that does satisfy this condition.

Example 1. Let the single-input and single-output system be described by the following system with additive noise:

$$y_t = G_0(S)[I + \mu G_1(S)] S u_t + v_t, \tag{26}$$

where $G_0(S) = B(S)/A(S)$ represents the nominal transfer function, whereas $G_1(S)$ is the unmodeled transfer function and is assumed to be stable and proper.

When the additive noise v_t is identically equal to zero, then the system is reduced to the deterministic one, and it coincides with the model considered (e.g., [6]) in the robustness analysis for deterministic systems.

Putting the expression for $G_0(S)$ into (26) leads to

$$A(S) y_t = B(S) u_t + \mu S G_1(S) B(S) u_t + A(S) v_t.$$

Comparing this to (7) shows that in the present case

$$\dot{\eta}_t = \mu G_1(S) B(S) u_t. \tag{27}$$

We now prove that Assumption 1 is satisfied for the system (26). For this the following auxiliary result is needed. We formulate it as a lemma, as it will also be used in the proof of the main results to follow.

LEMMA 1. *Let $E(S)$ and $F(S)$ be matrix polynomials in the integral operator S, such that the transfer matrix $F(S)E^{-1}(s)$ is stable and proper. Then*

$$\int_0^t \|F(S)E^{-1}(S) x_z\|^2 \, dz \leqq c \int_0^t \|x_z\|^2 \, dz$$

for any square integrable function $\{x_t\}$, where c is a constant depending on $E(s)$ and $F(S)$ only.

Proof. Let us write

$$E(S) = I + ES + \cdots + E_d S^d, \qquad F(S) = I + F_1 S + \cdots + F_d S^d$$

and set

$$z_t = E^{-1}(S)x_t, \qquad Z_t = [z_t^\tau \cdots S^{d-1}z_t^\tau]^\tau.$$

Similar to (22) we have

$$Z_t = F_e S Z_t + [x_t^\tau, 0 \cdots 0]^\tau \tag{28}$$

with

$$F_e = \begin{bmatrix} -E_1 & \cdots & \cdots & -E_d \\ I & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & I & 0 \end{bmatrix}.$$

The above linear differential equation has the solution

$$Z_t = F_e \int_0^t \exp\{F_e(t-s)\}[x_s^\tau, 0 \cdots 0]^\tau \, ds + [x_t^\tau, 0 \cdots 0]^\tau.$$

Since F_e is stable, there are constants $c_1 \geqq 1$ and $\rho > 0$ such that

$$\|\exp\{F_e t\}\| \leqq c_1 e^{-\rho t} \quad \forall t \geqq 0$$

where here and hereafter c_i, $i = 1, 2, \cdots$, denote constants.

It then follows that

$$\begin{aligned}
\int_0^t \|Z_z\|^2 \, dz &\leqq 2\|F_e\|^2 \int_0^t \left\{ \left\| \int_0^z \exp[F_e(z-s)][x_s^\tau, 0 \cdots 0]^\tau \, ds \right\|^2 + \|x_z\|^2 \right\} dz \\
&\leqq 2(c_1)^2 \|F_e\|^2 \\
&\quad \cdot \int_0^t \left\{ \int_0^z \exp[-\rho(z-s)] \, ds \int_0^z \exp[-\rho(z-s)] \|x_s\|^2 \, ds + \|x_z\|^2 \right\} dz \\
&\leqq 2(c_1)^2 \rho^{-1} \|F_e\|^2 \int_0^t \left\{ \int_0^z \exp[-\rho(z-s)] \|x_s\|^2 \, ds + \|x_z\|^2 \right\} dz \\
&\leqq 2(c_1)^2 \rho^{-1} \|F_e\|^2 \left\{ \int_0^t \|x_z\|^2 \, dz + \rho^{-1} \int_0^t \|x_s\|^2 \, ds \right\} \\
&\leqq 2(c_1)^2 \rho^{-1} \|F_e\|^2 (1 + \rho^{-1}) \int_0^t \|x_s\|^2 \, ds.
\end{aligned} \tag{29}$$

Furthermore, by (28) it follows that

$$SZ_t = (F_e)^{-1} \left\{ Z_t - \begin{bmatrix} x_t \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\}$$

and

$$\int_0^t \|SZ_z\|^2 \, dz \leqq c_2 \int_0^t \|x_z\|^2 \, dz \tag{30}$$

by (29).

Finally, the lemma follows from (29) and (30):

$$\int_0^t \|F(S)E^{-1}(S)x_s\|^2\,ds = \int_0^t \|z_s + F_1 S z_s + \cdots + F_d S^d z_s\|^2\,ds$$
$$= \int_0^t \|[I, 0 \cdots 0]z_s + [F_1 \cdots F_d]SZ_s\|^2\,ds$$
$$\leqq c\int_0^t \|x_s\|^2\,ds. \qquad \square$$

We now turn back to show that $\dot{\eta}_t$ given by (27) satisfies Assumption 1.
Set

$$x_t = \mu B(S)u_t.$$

We then have

$$\int_0^t \|x_s\|^2\,ds \leqq \mu^2 \int_0^t \|B_1 u_s + \cdots + B_q S^{q-1} u_s\|^2\,ds \leqq \mu^2 c_3 r_t,$$

where r_t is defined by (25) and c_3 is a constant.

Finally, applying Lemma 1 to (27), we find that

$$\int_0^t \|\dot{\eta}_s\|^2\,ds \leqq \int_0^t \|G_1(S)x_s\|^2\,ds \leqq c\int_0^t \|x_s\|^2\,ds \leqq \mu^2 c c_3 r_t,$$

which verifies (23) with $\varepsilon = \mu^2 c c_3$. $\square$

Example 2. Consider the following system:

$$A(S)y_t = SB(S)u_t + C(S)v_t + S\xi_t(y, u).$$

When the last term is identically zero, this model becomes the continuous-time analogue of an ARMAX model (see, e.g., [9]-[12]).

It is clear that Assumption 1 is verified if the nonlinear part $\xi_t(y, u)$ is one of the following forms:

$$\xi_t(y, u) = \varepsilon_1 y_t \sin(t) + \varepsilon_2 u_t \cos(t),$$
$$\xi_t(y, u) = \varepsilon_1 \sin(y_t) + \varepsilon_2 \sin(u_t), \qquad \varepsilon_i \in [0, \varepsilon], \quad i = 1, 2,$$

and so on.

3. Robustness of parameter estimation. We now show that the estimation error is proportional to the constant ε defined in (23) if the input-output data is persistently exciting.

THEOREM 1. *If Assumptions* 1 *and* 2 *are satisfied, then*

$$\limsup_{t\to\infty} \|\theta_t - \theta\| \leqq \alpha k \varepsilon, \quad a.s.$$

where $\alpha \in (0, \infty)$ *is a constant,* ε *is defined in* (23), *and*

$$k = \limsup_{t\to\infty} r_t / \lambda_{\min}(t) < \infty$$

where $\lambda_{\min}(t)$ *denotes the minimum eigenvalue of* P_t^{-1}.

For the proof of this theorem we need the following lemmas.

LEMMA 2. *Under the conditions of Theorem* 1, *there is a constant* $k_0>0$ *such that*

$$\operatorname{tr}\tilde{\theta}_t^\tau P_t^{-1}\tilde{\theta}_t \leqq O(1)+O\left(\left\{\int_0^t \|g_s\|^2\,ds\right\}^{(1/2)+\eta}\right)+O(\log r_t)$$
$$+2\left\{-\left(k_0-\frac{c}{2}\right)\int_0^t \|g_s\|^2\,ds+\frac{1}{2c}\int_0^t \|D(S)C^{-1}(S)\dot{\eta}_s\|^2\,ds\right\},$$
$$\forall \eta>0,\quad c>0,$$

where $\tilde{\theta}_t=\theta-\theta_t$, $g_t=\tilde{\theta}_t^\tau\phi_t$.

Proof. By (7) and (16) it is easy to see that

$$\begin{aligned} dy_t &= \theta^\tau\phi_t^0\,dt+dv_t+\dot{\eta}_t\,dt\\ &=\theta^\tau\tilde{\phi}_t\,dt+\theta^\tau\phi_t\,dt+dv_t+\dot{\eta}_t\,dt \end{aligned}$$

and hence

$$\begin{aligned} \theta^\tau\tilde{\phi}_t\,dt &= dy_t-\theta_t^\tau\phi_t\,dt+(\theta_t-\theta)^\tau\phi_t\,dt-dv_t-\dot{\eta}_t\,dt\\ &= d\hat{v}_t-\tilde{\theta}_t^\tau\phi_t\,dt-dv_t-\dot{\eta}_t\,dt \qquad (31)\\ &= -d\tilde{v}_t-\tilde{\theta}_t^\tau\phi_t\,dt-\dot{\eta}_t\,dt \end{aligned}$$

or

$$C(S)\left(\frac{d\tilde{v}_t}{dt}\right)=-g_t-\dot{\eta}_t \quad\text{or}\quad \left(\frac{d\tilde{v}_t}{dt}\right)=-C^{-1}(S)(g_t+\dot{\eta}_t). \qquad (32)$$

Let us now set

$$f_t=\left\{\frac{[C(S)-D(S)]}{S}\right\}\tilde{v}_t+\frac{g_t}{2}; \qquad (33)$$

it then follows that

$$f_t=\left[D(S)C^{-1}(S)-\frac{I}{2}\right]g_t+\left\{\frac{[D(S)C^{-1}(S)-I]}{S}\right\}\eta_t. \qquad (34)$$

From this and Assumption 2 there are constants $k_0>0$, $k_1>0$ such that

$$\int_0^t g_s^\tau\{f_s+[I-D(S)C^{-1}(S)]\dot{\eta}_s-k_0g_s]\,ds+k_1>0. \qquad (35)$$

From (10), (32), and (33) it follows that

$$\begin{aligned} d\tilde{\theta}_t &= -P_t\phi_tD(S)[dy_t^\tau-\phi_t^\tau\theta_t\,dt]\\ &= -P_t\phi_tD(S)[dv_t-d\tilde{v}_t]^\tau\\ &= -P_t\phi_t[dw_t-d\tilde{v}_t-D_1\tilde{v}_t\,dt-\cdots-D_rS^{r-1}\tilde{v}_t\,dt]^\tau \qquad (36)\\ &= -P_t\phi_t\left[g_t\,dt+\dot{\eta}_t\,dt+\frac{C(S)-D(S)}{S}\tilde{v}_t\,dt+dw_t\right]^\tau\\ &= -P_t\phi_t\left(f_t\,dt+\frac{1}{2}g_t\,dt+\dot{\eta}_t\,dt+dw_t\right)^\tau. \end{aligned}$$

Applying Ito's formula, we obtain

$$\begin{aligned} d[\operatorname{tr}\tilde\theta_t^\tau P_t^{-1}\tilde\theta_t] &= -2g_t^\tau[f_t\,dt+\dot\eta_t\,dt+dw_t]+\phi_t^\tau P_t\phi_t\,dt \\ &= -2g_t^\tau\{f_t+[I-D(S)C^{-1}(S)]\dot\eta_t-k_0g_t\}\,dt \\ &\quad +2g_t^\tau[1-D(S)C^{-1}(S)]\dot\eta_t\,dt-2k_0\|g_t\|^2\,dt \\ &\quad -2g_t^\tau\dot\eta_t\,dt-2g_t^\tau\,dw_t+\phi_t^\tau P_t\phi_t\,dt; \end{aligned}$$

then by (35)

$$\begin{aligned} 0 &\leqq \operatorname{tr}\tilde\theta_t^\tau P_t^{-1}\tilde\theta_t \\ &\leqq \operatorname{tr}\tilde\theta_0^\tau P_0^{-1}\tilde\theta_0+\int_0^t \phi_s^\tau P_s\phi_s\,ds+2k_1 \qquad (37)\\ &\quad +2\left\{-k_0\int_0^t\|g_s\|^2\,ds-\int_0^t g_s^\tau D(S)C^{-1}(S)\dot\eta_s\,ds-\int_0^t g_s^\tau\,dw_s\right\}. \end{aligned}$$

Noting the following elementary facts:

$$2\int_0^t a_s^\tau b_s\,ds\leqq c\int_0^t\|a_s\|^2\,ds+c^{-1}\int_0^t\|b_s\|^2\,ds\quad\forall c>0,$$

$$\begin{aligned}\int_0^t\phi_s^\tau P_s\phi_s\,ds &= \int_0^t\operatorname{tr}[P_s\phi_s\phi_s^\tau\,ds]=\int_0^t\operatorname{tr}[P_s\,dP_s^{-1}] \\ &= \int_0^t\frac{d(\det P_s^{-1})}{\det P_s^{-1}}=O(\log r_t),\end{aligned}$$

and applying the following estimate for the Ito integral (see, e.g., [16, Lemma 4]):

$$\int_0^t x_s^\tau\,dw_s=O(1)+o\left(\left\{\int_0^t\|x_s\|^2\,ds\right\}^{1/2+\eta}\right)\quad\text{a.s. }\forall\eta>0, \tag{38}$$

for any predictable process (x_t, F_t), we can easily conclude the lemma from (37). □

LEMMA 3. *If F_d defined by* (21) *is stable, then*

$$\frac{1}{t}\int_0^t V_sV_s^\tau\,ds\to R\quad\text{a.s. as } t\to\infty \tag{39}$$

where V_t is defined in (19) *and*

$$R\triangleq\int_0^\infty\exp\{F_d\lambda\}\begin{bmatrix} I & 0\\ 0 & 0\end{bmatrix}\exp\{F_d^\tau\lambda\}\,d\lambda.$$

Proof. Since F_d is stable, there exists a positive-definite matrix $P>0$ such that

$$PF_d+F_d^\tau P=-I.$$

By this and the Ito formula we see from (22) that

$$\begin{aligned} d[V_t^\tau PV_t] &= V_t^\tau(PF_d+F_d^\tau P)V_t\,dt+\operatorname{tr}\left\{\begin{bmatrix} I\\ 0\\ \vdots\\ 0\end{bmatrix}[I,0\cdots0]P\,dt+2V_t^\tau P\begin{bmatrix} I\\ 0\\ \vdots\\ 0\end{bmatrix}dw_t\right. \\ &= -\|V_t\|^2\,dt+\operatorname{tr}\begin{bmatrix} I & 0\\ 0 & 0\end{bmatrix}P\,dt+2V_t^\tau P\begin{bmatrix} I\\ 0\\ \vdots\\ 0\end{bmatrix}dw_t. \end{aligned}$$

So it follows by applying (38) that

$$V_t^\tau P V_t + \int_0^t \|V_s\|^2\, ds = \operatorname{tr}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} Pt + o\left(\left\{\int_0^t \|V_s\|^2\, ds\right\}^{1/2+\eta}\right). \tag{40}$$

Consequently, we conclude that

$$\int_0^t \|V_s\|^2\, ds = O(t), \quad \text{a.s.} \tag{41}$$

Again, by the Ito formula we get

$$V_t V_t^\tau = \left(\int_0^t V_s V_s^\tau\, ds\right) F_d^\tau + F_d\left(\int_0^t V_s V_s^\tau\, ds\right) + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} t + \int_0^t \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix} dw_s V_s^\tau + \int_0^t V_s\, dw_s^\tau [I, 0 \cdots 0]$$

and hence

$$\int_0^t V_s V_s^\tau\, ds = \int_0^t \exp[F_d(t-z)] \int_0^z \left\{ \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix} dw_s V_s^\tau + V_s\, dw_s^\tau [I, 0 \cdots 0] \right\} \exp[F_d^\tau(t-z)]\, dz + \int_0^t \exp[F_d(t-z)] \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} z \exp[F_d^\tau(t-z)]\, dz. \tag{42}$$

We now consider the first term on the right-hand side (42). By (38), (41), and the stability of F_d, it is easy to see that there is a constant $\rho > 0$ such that

$$\left\| \int_0^t \exp[F_d(t-z)] \int_0^t \left\{ \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix} dw_s V_s^\tau + V_s\, dw_s^\tau [I, 0 \cdots 0] \right\} \exp[F_d^\tau(t-z)]\, dz \right\|$$

$$= O\left(\int_0^t \exp[-2\rho(t-z)] \left\{\int_0^z \|V_s\|^2\, ds\right\}^{1/2+\eta} dz\right) + O(1)$$

$$= O\left(\int_0^t \exp[-2\rho(t-z)] z^{1/2+\eta}\, dz\right) = O(t^{1/2+\eta}) \quad \forall \eta > 0.$$

Hence the lemma follows immediately from this and (42).

Proof of Theorem 1. Since $D(S)C^{-1}(S)$ is strictly positive real, $C(S)$ is stable. Then by Lemma 1 and Assumption 1, it follows that

$$\int_0^t \|D(S)C^{-1}(S)\dot{\eta}_s\|^2\, ds \leqq \varepsilon c_0 r_t \quad \text{for some } c_0 > 0.$$

Taking $c < 2k_0$ in Lemma 2, we see that

$$0 \leqq \operatorname{tr} \tilde{\theta}_t^\tau P_t^{-1} \tilde{\theta}_t \leqq O(1) - \left(k_0 - \frac{c}{2}\right) \int_0^t \|g_s\|^2\, ds + O(\varepsilon r_t) + O(\log r_t). \tag{43}$$

Then for sufficiently large t

$$\operatorname{tr} \tilde{\theta}_t^\tau \tilde{\theta}_t \leqq \frac{\operatorname{tr}(\tilde{\theta}_t^\tau P_t^{-1} \tilde{\theta}_t)}{\lambda_{\min}(t)}$$

$$\leqq \frac{1}{\lambda_{\min}(t)}\left\{O(1)-\left(k_0-\frac{c}{2}\right)\int_0^t \|g_s\|^2\,ds+O(\varepsilon r_t)+O(\log r_t)\right\} \tag{44}$$

$$\leqq O\left(\frac{\log r_t}{r_t}\right)-\frac{k}{r_t}\left(k_0-\frac{c}{2}\right)\int_0^t \|g_s\|^2\,ds+O(\varepsilon k).$$

Since $c<2k_0$, the desired result will follow if we can show that $r_t\to\infty$, as $t\to\infty$. We prove this as follows.

From (32) it follows that

$$\tilde{V}_t=-\int_0^t \exp\{F_c(t-s)\}[g_s+\dot{\eta}_s]\,ds.$$

Then by (43) and Assumption 1, we have for some $\rho>0$ and $c_1>0$

$$\int_0^t \|\tilde{V}_z\|^2\,dz \leqq \int_0^t \left\|\int_0^z \exp\{F_c(z-s)\}[g_s+\dot{\eta}_s]\,ds\right\|^2 dz$$

$$\leqq (c_1)^2\int_0^t\left\{\int_0^z \exp[-\rho(z-s)][\|g_s\|+\|\dot{\eta}_s\|]\,ds\right\}^2 dz$$

$$\leqq 2(c_1)^2\int_0^t\int_0^z \exp[-\rho(z-s)]\,ds\int_0^z \exp[-\rho(z-s)][\|g_s\|^2+\|\dot{\eta}_s\|^2]\,ds\,dz$$

$$\leqq 2\rho^{-1}(c_1)^2\int_0^t\int_z^t \exp[-\rho(z-s)]\,dz\,[\|g_s\|^2+\|\dot{\eta}_s\|^2]\,ds \tag{45}$$

$$\leqq 2\rho^{-2}(c_1)^2\int_0^t [\|g_s\|^2+\|\dot{\eta}_s\|^2]\,ds$$

$$\leqq 2\rho^{-2}(c_1)^2\{O(\log r_t)+O(\varepsilon r_t)+\varepsilon r_t\}$$

$$=O(\log r_t)+O(\varepsilon r_t).$$

Assume the converse were true, i.e., r_t was bounded in t; then from (45) it would follow that $\int_0^t \|\tilde{V}_z\|^2\,dz$ would be bounded. But by (20) and (25) it is clear that

$$r_t \geqq \int_0^t \|\hat{V}_z\|^2\,dz=\int_0^t \|V_z\|^2\,dz+\int_0^t \|\tilde{V}_z\|^2\,dz-2\int_0^t V_z^\tau \tilde{V}_z\,dz.$$

From this and the boundedness of r_t and $\int_0^t \|\tilde{V}_z\|^2\,dz$, it follows that

$$\int_0^t \|V_z\|^2\,dz \text{ is bounded.}$$

This contradicts Lemma 3. Hence $r_t\to\infty$, a.s., and Theorem 1 holds. □

Remark 1. If in (7) the unmodeled dynamics $\{\eta_t\}$ are identically zero, then we may take ε as zero in Assumption 1. In this case, it follows from (44) that

$$\|\tilde{\theta}_t\|^2=O\left(\frac{\log r_t}{\lambda_{\min}(t)}\right)\quad \text{a.s.}$$

This result is the continuous-time version of that obtained in the discrete-time case (see, e.g., [13]-[14]). See also [12] for related results.

4. Robustness of adaptive tracking. Let $\{u_t^*\}$ be a bounded deterministic and differentiable reference signal with $u_0^*=0$. Our objective here is to design the adaptive control u_t, so that the output $\{y_t\}$ tracks the output of the following reference model:

$$E(S)y_t^*=u_t^*$$

where $E(S)=I+E_1S+\cdots+E_pS^p$ is a stable matrix polynomial.

Similar to (18), we set

$$Y_t^*=[y_t^* \cdots S^{p-1}y_t^*]^\tau.$$

By a representation similar to (22), it is easy to see that $\{Y_t^*\}$ is a bounded sequence.

From now on, we assume that the upper bound for the order of the polynomial $A(S)$ is equal to that of $C(S)$, i.e., $p=r$.

Similar to the discrete-time case, we need the following standard minimum phase condition.

Assumption 3. $B(S)$ is stable.

Let us define the adaptive control u_t via the following equation:

$$\theta_t^\tau\phi_t=\frac{dy_t^*}{dt}. \tag{46}$$

This together with (1), (10), and (11) form a system of nonlinear stochastic differential equations, for which the existence and uniqueness of the strong solution is assumed.

THEOREM 2. *Consider the system* (1)-(6) *with* $p=r$, *and the estimation algorithm* (10)-(11). *If Assumptions* 1-3 *hold, and the control law is defined from* (46), *then there exists* $\varepsilon_1>0$ *such that whenever* ε *in* (23) *lies in the interval* $[0,\varepsilon_1)$, *the following properties hold*:

$$\limsup_{T\to\infty}\frac{1}{T}\int_0^T(\|Y_t\|^2+\|U_t\|^2)\,dt<\infty \quad a.s. \tag{47}$$

and

$$\limsup_{T\to\infty}\frac{1}{T}\int_0^T\|Y_t-Y_t^*\|^2\,dt=\operatorname{tr}R+\delta \quad a.s. \tag{48}$$

where $|\delta|=O(\varepsilon^{1/2})$, *and* R *is defined in Lemma* 3.

Proof. From (13) and (46) it is easy to see that

$$y_t-y_t^*=\hat{v}_t, \qquad Y_t^*-Y_t=\tilde{V}_t-V_t. \tag{49}$$

Note that

$$r_T=\int_0^T(\|Y_t\|^2+\|U_t\|^2+\|\hat{V}_t\|^2)\,dt+e. \tag{50}$$

We have by (39), (45), and (49) that

$$\int_0^T\|Y_t\|^2\,dt=O(T)+\varepsilon c_3r_T+O(\log r_T), \qquad c_3>0. \tag{51}$$

From this, (7) and the stability of $B(S)$, we have

$$\int_0^T\|U_t\|^2\,dt=O(T)+\varepsilon c_4r_T+O(\log r_T), \qquad c_4>0. \tag{52}$$

Note also that by (45) and Lemma 3, we have

$$\int_0^T \|\hat{V}_t\|^2\,dt \leqq O(T)+2\int_0^T \|\tilde{V}_t\|^2\,dt \leqq O(T)+\varepsilon c_5 r_T + O(\log r_T).$$

Hence, combining (50)-(52), we have

$$r_t \leqq O(t)+\varepsilon c_6 r_t + O(\log r_t), \qquad c_6>0,$$

which yields

$$\limsup_{t\to\infty}\frac{r_t}{t}<\infty \quad \text{for any } \varepsilon\in[0,\varepsilon_1)$$

with $\varepsilon_1 = 1/c_6$. Thus (47) is true.

We now proceed to prove (48). From (45) we have for any $\varepsilon\in[0,\varepsilon_1)$,

$$\frac{1}{T}\int_0^T \|\tilde{V}_t\|^2\,dt = O\left(\frac{\log T}{T}\right)+O(\varepsilon); \tag{53}$$

then by (49)

$$\begin{aligned}\frac{1}{T}\int_0^T &(Y_t-Y_t^*)(Y_t-Y_t^*)^\tau\,dt\\ &=\frac{1}{T}\int_0^T (V_t-\tilde{V}_t)(V_t-\tilde{V}_t)^\tau\,dt\\ &=\frac{1}{T}\int_0^T V_tV_t^\tau\,dt+\frac{1}{T}\int_0^T \tilde{V}_t\tilde{V}_t^\tau\,dt-\frac{1}{T}\int_0^T (V_t\tilde{V}_t^\tau+\tilde{V}_tV_t^\tau)\,dt,\\ &=R+\left[\frac{1}{T}\int_0^T V_tV_t^\tau\,dt-R\right]+O\left(\frac{\log T}{T}\right)+O(\varepsilon)+O\left(\left\{\frac{\log T}{T}+\varepsilon\right\}^{1/2}\right).\end{aligned} \tag{54}$$

Hence (48) is also true. □

Remark 2. If the initial value of the reference signal is not zero, i.e., $u_0^*\neq 0$, then we may replace (46) by

$$\theta_t^\tau\phi_t=\frac{dz_t^*}{dt},$$

where $z_t^* = E^{-1}(S)\{u_t^*-\exp(-t^2)u_0^*\}$. In this case, Theorem 2 is true for $\{z_t^*\}$, which approximates $\{y_t^*\}$ exponentially.

5. Asymptotic behavior of adaptive tracking. In this section we assume $\eta_t=0$ in (7). For this ideal case we give the convergence rate for the adaptive tracking errors. It is worth noting that the corresponding discrete-time results have not yet been established (see, e.g., [17], for related discussions).

LEMMA 4. *Let* $\{x_t\}$ *be any measurable process adapted to* $\{F_t\}$, *satisfying*

$$\limsup_{T\to\infty}\frac{1}{T}\int_0^T \|x_t\|^2\,dt\leqq k_1<\infty \quad a.s.$$

for some constant k_1. *Then*

$$\limsup_{T\to\infty}\frac{1}{(T\log\log T)^{1/2}}\left\|\int_0^T x_t\,dw_t^\tau\right\|<\infty \quad a.s. \tag{55}$$

Proof. Without loss of generality, we assume that x_t and w_t are scalars. Taking a constant k_2 so large that

$$(k_2)^2 - k_2 + 2(1-k_2)k_1 > 0, \qquad k_2 > 1,$$

we have

$$\begin{aligned}\int_0^T (k_2+x_t)^2\,dt &\geqq \int_0^T (x_t)^2\,dt + (k_2)^2 T - 2k_2\int_0^T |x_t|\,dt\\ &\geqq \int_0^T (x_t)^2\,dt + (k_2)^2 T - k_2\left[T + \int_0^T (x_t)^2\,dt\right]\\ &\geqq k_2(k_2-1)T + 2(1-k_2)k_1 T.\end{aligned}$$

Consequently,

$$\int_0^\infty (k_2+x_t)^2\,dt = \infty \quad \text{a.s.}$$

Now, define the following stopping time:

$$\tau(t) = \inf\left\{s : \int_0^s (k_2+x_z)^2\,dz = t\right\}.$$

It is known that

$$\int_0^{\tau(t)} (k_2+x_s)\,dw_s$$

is a Brownian motion (see, e.g., [18, Thm. 4.5]). Then by the law of the iterated logarithm for Wiener processes, we have

$$\frac{1}{(t\log\log t)^{1/2}}\left|\int_0^{\tau(t)} (k_2+x_s)\,dw_s\right| = O(1) \quad \text{a.s.} \tag{56}$$

Denoting

$$a(t) = \int_0^t (k_2+x_z)^2\,dz, \tag{57}$$

it is evident that $a(\tau(t)) = t$. Then (56) and (57) imply

$$\frac{1}{[a(T)\log\log a(T)]^{1/2}}\left|\int_0^T (k_2+x_s)\,dw_s\right| = O(1) \quad \text{a.s.}$$

as $T\to\infty$. From this and the fact that $a(T)/T = O(1)$, it follows that

$$\frac{1}{[T\log\log T]^{1/2}}\left|\int_0^T (k_2+x_s)\,dw_s\right| = O(1) \quad \text{a.s.}$$

and hence

$$\begin{aligned}&\frac{1}{[T\log\log T]^{1/2}}\left|\int_0^T x_s\,dw_s\right|\\ &\quad\leqq \frac{1}{[T\log\log T]^{1/2}}\left\{k_2|w_T| + \left|\int_0^T (k_2+x_s)\,dw_s\right|\right\}\\ &\quad= O(1) \quad \text{a.s. as } T\to\infty,\end{aligned}$$

completing the proof. □

We are now in a position to prove the following main result of this section.

THEOREM 3. *Consider the system described by* (7) *with* $\eta_t=0$ *and* $p=r$, *and estimation algorithm* (10)-(11). *If Assumptions* 2 *and* 3 *are satisfied, and if the adaptive control is defined from* (46), *then*

$$\|R_T-R\|^2=O\left(\frac{\log T}{T}\right) \quad \text{a.s. as } T\to\infty, \tag{58}$$

where R is given in Lemma 3 *and*

$$R_T=\frac{1}{T}\int_0^T (Y_t-Y_t^*)(Y_t-Y_t^*)^\tau\,dt.$$

Proof. We first consider the convergence rate of $1/T\int_0^T V_tV_t^\tau\,dt$.

From (40) it is clear that

$$\limsup_{T\to\infty}\frac{1}{T}\int_0^T \|V_s\|^2\,ds\leqq 2\operatorname{tr}\begin{bmatrix} I & 0\\ 0 & 0\end{bmatrix}P.$$

Then Lemma 4 implies

$$\limsup_{T\to\infty}\frac{1}{(T\log\log T)^{1/2}}\left\|\int_0^T V_t\,dw_t^\tau\right\|<\infty \quad \text{a.s.},$$

and hence

$$\left\|\int_0^t \exp[F_d(t-z)]\int_0^z\left\{\begin{bmatrix} I\\ 0\\ \vdots\\ 0\end{bmatrix} dw_s\,V_s^\tau+V_s\,dw_s^\tau[I,0\cdots 0]\right\}\exp[F_d^\tau(t-z)]\,dz\right\|$$

$$=O(\{t\log\log t\}^{1/2}) \quad \text{a.s.}$$

Consequently, it follows from (42) that

$$\begin{aligned}&\left\|\frac{1}{T}\int_0^T V_sV_s^\tau\,ds-\int_0^\infty \exp\{F_ds\}\begin{bmatrix} I & 0\\ 0 & 0\end{bmatrix}\exp\{F_d^\tau s\}\,ds\right\|\\ &\quad=O\left(\left\{\frac{\log\log T}{T}\right\}^{1/2}\right).\end{aligned}\tag{59}$$

Setting $\varepsilon=0$ in (53) and (54), and using (59), we see that

$$\begin{aligned}R_T&=\int_0^\infty \exp\{F_ds\}\begin{bmatrix} I & 0\\ 0 & 0\end{bmatrix}\exp\{F_d^\tau s\}\,ds\\ &\quad+O\left(\left\{\frac{\log\log T}{T}\right\}^{1/2}\right)+O\left(\frac{\log T}{T}\right)+O\left(\left\{\frac{\log T}{T}\right\}^{1/2}\right)\\ &=\int_0^\infty \exp\{F_ds\}\begin{bmatrix} I & 0\\ 0 & 0\end{bmatrix}\exp\{F_d^\tau s\}\,ds+O\left(\left\{\frac{\log T}{T}\right\}^{1/2}\right),\end{aligned}$$

which verifies (58). Hence the proof is complete. □

REFERENCES

[1] B. EGARDT, *Stability analysis of adaptive control systems with disturbances*, Proc. JACC, San Francisco, CA, 1980.

[2] C. E. ROHRS, L. VALAVANI, M. ATHANS, AND C. STEIN, *Robustness of adaptive control algorithm in the presence of unmodeled dynamics*, Preprints of 21st IEEE Conference on Decision and Control, Orlando, FL, 1982.

[3] B. D. RIEDLE AND P. V. KOKOTOVIC, *Disturbance instabilities in an adaptive system*, IEEE Trans. Automat. Control, 29 (1984), pp. 822–824.

[4] R. ORTEGA, L. PRALY, AND I.D. LANDAU, *Robustness of discrete-time direct adaptive controllers*, IEEE Trans. Automat. Control, 30 (1985).

[5] G. KREISSELMEIER AND B. D. O. ANDERSON, *Robust model reference adaptive control*, IEEE Trans. Automat. Control, 31 (1986).

[6] P. A. IOANNOU AND K. TSAKLIS, *A robust direct adaptive controller*, IEEE Trans. Automat. Control, 31 (1986).

[7] H. F. CHEN AND L. GUO, *Robustness analysis of identification and adaptive control for stochastic systems*, Systems Control Lett., 9 (1987), pp. 131–140.

[8] ———, *A robust stochastic adaptive controller*, IEEE Trans. Automat. Control, 33 (1988), pp. 1035–1043.

[9] H. F. CHEN, *Quasi-least-squares identification and its strong consistency*, Internat. J. Control, 34 (1981), pp. 921–936.

[10] J. H. VAN SCHUPPEN, *Convergence results for continuous time adaptive stochastic filtering algorithms*, J. Math. Anal. Appl., 96 (1983), pp. 209–225.

[11] H. F. CHEN, *Recursive Estimation and Control for Stochastic Systems*, John Wiley, New York, 1985.

[12] H. F. CHEN AND J. B. MOORE, *Convergence rate of continuous time stochastic* ELS *parameter estimation*, IEEE Trans. Automat. Control, 32 (1987), pp. 267–269.

[13] T. L. LAI AND C. Z. WEI, *Extended least squares and their application to adaptive control and prediction in linear systems*, IEEE Trans. Automat. Control, 31 (1986), pp. 898–906.

[14] H. F. CHEN AND L. GUO, *Convergence rate of least squares identification and adaptive control for stochastic systems*, Internat. J. Control, 44 (1986), pp. 1459–1476.

[15] R. S. LIPSTER AND A. N. SHIRYAYEV, *Statistics of Random Processes*, I. *General Theory*, Springer-Verlag, New York, 1977.

[16] N. CHRISTOPEIT, *Quasi-least-squares estimation in semimartingale regression models*, Stochastics, 16 (1986), pp. 255–278.

[17] P. R. KUMAR, *Convergence of adaptive control schemes using least-squares parameter estimates*, 1989, submitted.

[18] A. FRIEDMAN, *Stochastic Differential Equations and Applications*, Vol. 1, Academic Press, New York, 1975.