significantly simplify the results of [2] for testing Hurwitz and Schur stability of symmetric interval matrices.

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Stability, Convergence, and Performance of an Adaptive Control Algorithm Applied to a Randomly Varying System

S. P. Meyn and L. Guo

Abstract—**The stability and performance of a stochastic adaptive control algorithm applied to a randomly varying linear system is investigated. We demonstrate that**

i) loss functions on the input-output process converge to their expectation with respect to an invariant probability at a geometric rate. Hence a form of stochastic exponential asymptotic stability is established;

ii) when the parameter variation and measurement noise is small, it is shown that the performance is nearly optimal. If an excitation signal is added in the control law, near consistency of the parameter estimates is obtained.

Further results include central limit theorems and the law of large numbers for the input-output and parameter processes.

I. Introduction

In this note, we consider an adaptive control algorithm applied to a stochastic time-varying system.

We extend the stability proof of [3] to show that a projected version of the gradient estimation algorithm induces a controller which is stabilizing for a broad class of linear time-varying stochastic systems. This result is used together with recent results from the ergodic theory of Markov chains to show that loss functions on the input-output process converge to their expectation with respect to an invariant measure at a geometric rate. Related ergodic results such as convergence of loss functions on the sample paths is also established.

We also demonstrate that the parameter estimates are nearly consistent under an appropriate modification to the control law.

These results constitute a broad extension of the ideas initiated in [11] and [4]. The stability proof presented in these papers is fairly involved, however, the system model is highly specialized and it is hard to see how the results can be extended to more complex models. In the present note we will examine a standard adaptive control scheme, and our results will depend on general and natural hypotheses on the system and parameter process.

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Methodology

The techniques to be introduced in the following are generalizations of the Lyapunov approach of for instance [7] to the stochastic framework. These techniques are extremely general and may be applied to a large class of problems in stochastic systems theory.

It has been widely accepted that associated supermartingales should play the role of stochastic Lyapunov functions. However, in the stability analysis to follow it may be seen that the supermartingale approach which has previously been successful in stochastic adaptive control is not appropriate here. Our approach is to take a version of the Lyapunov function which may be used to establish ultimate boundedness in the deterministic case, and use this test function to prove that the distributions governing the state process are generated by an exponentially asymptotically stable dynamical system whose state space consists of probability measures. In the terminology of [13], the state process is geometrically ergodic. This result may then be used to infer stability results for the input-output process, and associated variables in the control process.

A general technique introduced in this note is the collection of results described in Appendix A, and applied in the proof of the consistency and optimality results (16)-(18) in Theorem 1. As discussed immediately after the proof of Theorem 1, we do not at present know of any proof which does not rely on the methods introduced in Appendix A.

Overview

The note is organized as follows: in Section II we describe the system and control algorithm subject to analysis, and present our main results.

In Section III we construct a specific Lyapunov function on a Markovian state process for the controlled system. A further analysis of the state process is made in Section IV. In particular, it is shown here that the existence of the Lyapunov function implies that the distributions governing the system, and hence also loss functions on the input-output process converge to their stationary values at a geometric rate. This, together with some results from the Appendix, provides a proof of the main result.

II. System Description and Main Results

We consider the stochastic time-varying system

$$y_{k+1} = a_1(k) y_k + \cdots + a_p(k) y_{k-p+1} + u_k + v_{k+1}, \qquad k \geq 1 \quad (1)$$

where y_k, u_k, and v_k are the (scalar) output, input, and disturbance processes, respectively, and the parameters $a_i(k)$, $1 \leq i \leq p$, $k \geq 0$, are partially observed through the input-output process $(\boldsymbol{u}, \boldsymbol{y})$.

Our goal is to choose a control law which stabilizes this system in a mean square sense.

First, we collect together our assumptions on the disturbance and parameter processes. Observe that the system (1) may be written in the regression form $y_{k+1} = \theta_k^{\mathrm{T}} \varphi_k + u_k + v_{k+1}$ where

$$\varphi_k^{\mathrm{T}} = (y_k, \cdots, y_{k-p+1}) \text{ and } \theta_k^{\mathrm{T}} = (a_1(k), \cdots, a_p(k)).$$

Assumptions

A1: The parameter process θ_k, $k \geq 0$, is a constant plus a moving average process of the form

$$\theta_k = \theta^0 + \eta_{k+1} + B_1 \eta_k + \cdots + B_{n_0} \eta_{k-n_0+1}.$$

A2: The joint process $\boldsymbol{w} \triangleq (\boldsymbol{\eta}, \boldsymbol{v})$ is an independent and identi-

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cally distributed (i.i.d.) process on $\mathbb{R}^{p+1}$, and the distribution μ_w of w_k, $k \geq 0$, possesses a lower semicontinuous density which is positive at the origin.

A3: For some M_0, $\epsilon_0 > 0$, δ_0, $M_v < \infty$

$$\mathrm{E}\left[\exp\left(\epsilon_0 v_k^2\right)\right] \leq \exp\left(M_v\right)$$

$$\mathrm{E}\left[\exp\left(M_0 |\Delta_k|^2\right)\right] \leq \exp\left(\delta_0\right) \qquad k \geq n_0$$

where Δ_k, $k \geq 1$, is defined to be the parameter variation process $\Delta_k \triangleq \theta_k - \theta_{k-1}$.

The constants defined in condition A3 are not chosen arbitrarily. The constant M_0 is assumed to be larger than some bound M_0^*, and the constant δ_0 will be assumed to be smaller than a bound δ_0^*. Their precise values may be found from an examination of the proofs below.

We now describe the estimation algorithm. Let $L > 0$ and $d > 0$ be two constants. We define the domain $D \subset \mathbb{R}^p$ as

$$D = \{x \in \mathbb{R}^p : |x_i| \leq L, \quad 1 \leq i \leq p\} \tag{2}$$

and $\Pi_D\{x\}$ is defined to be the nearest point from x to D under the Euclidian norm.

The parameter estimates $\hat{\theta}$ of the parameter process θ are generated by the following projected gradient algorithm:

$$\hat{\theta}_{k+1} = \Pi_D\left\{\hat{\theta}_k + \frac{\varphi_k}{d + |\varphi_k|^2}\left(y_{k+1} - u_k - \varphi_k^{\mathrm{T}}\hat{\theta}_k\right)\right\}. \tag{3}$$

We stress that the parameter process θ is not assumed to evolve in the set D. However, we assume that $\theta^0 \in D$, and the stability proof below requires the assumption that L is sufficiently large, so that there is a high probability that θ_k lies in D for each $k \geq 1$. A precise lower bound on the magnitude of the constant L may be obtained from the stability proof.

We collect the initial conditions in a vector

$$\Phi_0^{\mathrm{T}} \triangleq \left(\varphi_0^{\mathrm{T}}, \hat{\theta}_0^{\mathrm{T}}, \eta_0, \cdots, \eta_{1-n_0+1}\right)$$

which we assume is a deterministic constant, or more generally, is independent of the process w defined in condition A2.

Given the parameter estimates (3), we apply the certainty equivalent minimum variance adaptive control law

$$u_k = -\varphi_k^{\mathrm{T}}\hat{\theta}_k. \tag{4}$$

Incorporation of Excitation and ϵ-Parameterizations

When the disturbance sequence w is small in a certain sense, then we will see in Theorem 1 that, after a transient period, the input-output process will be small in L^2 norm. A natural question arises: is it true that the norm of the parameter estimation error $|\tilde{\theta}|$ will also be small after a transient period? A complete answer to this question is not at present available. However, by modifying the control law (4) by the addition of a "dither sequence" as in, for example, [1], we may obtain near consistency of the parameter estimates.

In a portion of the results to follow we will assume that the control law takes the form

$$u_k = -\varphi_k^{\mathrm{T}}\hat{\theta}_k + e_{k+1} \tag{5}$$

where we will require the hypothesis.

A4: The stochastic process e is i.i.d. and independent of (Φ_0, w), and the distribution μ_e of e_k, $k \geq 1$, is uniform on the interval $[-\sqrt{3\sigma_e^2}, \sqrt{3\sigma_e^2}]$.

To discuss the behavior of the closed-loop system when the disturbance w is small, it will be helpful to define an ϵ-parameterized family of systems of the form described above. Suppose that for each $\epsilon \in (0, 1]$ a disturbance process $w^\epsilon = (v^\epsilon, \eta^\epsilon)$ is defined, and take $w^0 \equiv \mathbf{0}$. In this case, we will assume that the following substitute for conditions A2 and A3 holds.

A5: For each $\epsilon \in (0, 1]$ Condition A2 holds, and there exist constants ϵ_0 and M_0 such that

$$\lim_{\epsilon \to 0} \mathrm{E}\left[\exp\left(\epsilon_0 (v_1^\epsilon)^2\right)\right] = \lim_{\epsilon \to 0} \mathrm{E}\left[\exp\left(M_0 |\eta_1^\epsilon|^2\right)\right] = 1.$$

Under condition A5 it follows that $w_k^\epsilon \to w_k^0$ as $\epsilon \to 0$ in every L^p space, and hence also in probability.

Construction of a Markovian State Process

Under conditions A1–A4, the system description (1), either control law (4) or (5), and the form of the parameter estimator (3), it follows that the stochastic process Φ defined as

$$\Phi_k = \left(\varphi_k^{\mathrm{T}}, \hat{\theta}_k^{\mathrm{T}}, \eta_k, \cdots, \eta_{k-n_0+1}\right)^{\mathrm{T}} \qquad k \in \mathbb{Z}_+, \tag{6}$$

is a Feller–Markov chain with stationary transition probabilities P^k, $k \geq 1$, defined as

$$\mathrm{P}\{\Phi_k \in A \mid \Phi_0\} = P^k(\Phi_0, A) \qquad k \geq 1, A \in \mathscr{B}(X).$$

This fact will be proven here. For definitions of these terms, the reader is refered to [16].

Lemma 1.1: Suppose that conditions A1–A4 hold and that either control law (4) or (5) is applied. Then the stochastic process Φ evolving on $X \triangleq \mathbb{R}^p \times D \times \mathbb{R}^{n_0}$ defined in (6) is a Feller–Markov chain.

Proof: For simplicity we consider only the case where the control law (4) is applied. To prove the lemma recall that $w = (v, \eta)$. We will construct a continuous function F: $X \times \mathbb{R}^{p+1} \to X$ such that Φ appears as $\Phi_{k+1} = F(\Phi_k, w_{k+1})$. By assumption A2 the result will follow.

First, observe that setting $\theta_k = \theta_k - \hat{\theta}_k$, $k \in \mathbb{Z}_+$, we have by (1)

$$\varphi_{k+1} = \begin{bmatrix} -\tilde{\theta}_k^{\mathrm{T}} & - \\ I & 0 \end{bmatrix}\varphi_k + \left(v_{k+1}, 0, \cdots, 0\right)^{\mathrm{T}}. \tag{7}$$

By (1), (3), and (4) we have

$$\hat{\theta}_{k+1} = \Pi_D\Bigg\{\hat{\theta}_k + \frac{\varphi_k\varphi_k^{\mathrm{T}}\left(\theta_0 + \eta_{k+1} + B_1\eta_k + \cdots + B_{n_0}\eta_{k-n_0+1}\right)}{d + |\varphi_k|^2} - \frac{\varphi_k\varphi_k^{\mathrm{T}}\hat{\theta}_k}{d + |\varphi_k|^2} + \frac{\varphi_k v_{k+1}}{d + |\varphi_k|^2}\Bigg\} \tag{8}$$

which shows that the process Φ is of the form $\Phi_{k+1} = F(\Phi_k, w_{k+1})$ where the function F: $X \times \mathbb{R}^{p+1} \to X$ is continuous. □

We may now state our main results.

Main Results

A fundamental object in the theory of Markov chains is the invariant probability. A probability π on $\mathscr{B}(X)$ is called *invariant* if Φ_k has distribution π for all $k \geq 1$ when Φ_0 has distribution π [13].

It is a well known fact (see, for example, [6] or [17]) that the distributions governing a Feller–Markov chain are generated by a (semi) dynamical system whose state space consists of probabilities. The following result uses the fact, which we shall prove below, that in the present example this dynamical system is exponentially asymptotically stable.

When the initial condition of a Markov chain Φ is a deterministic constant $x \in X$, it is customary to denote the expectation of a random variable Y as $\mathrm{E}_x[Y]$. In the sequel, we will follow this convention and when Φ_0 has distribution π, we will let $\mathrm{E}_\pi[Y]$ denote the expectation of the random variable Y.

Theorem 1: Suppose that either i) conditions A1–A3 hold and the control law (4) is applied, or ii) conditions A1–A4 hold and the control (5) is applied. Then the state process Φ defined in (6) possesses a unique invariant probability π, and the following limits hold.

Exponential Stability: There exists a function C on X and a constant $\rho_0 < 1$, such that for all $\epsilon > 0$ and every initial condition $x \in X$

$$|\mathrm{P}_x\{y_k^2 + u_k^2 > \epsilon\} - \mathrm{P}_\pi\{y_0^2 + u_0^2 > \epsilon\}| \le C(x)\rho_0^k \quad (9)$$

$$|\mathrm{E}_x[y_k^2 + u_k^2] - \mathrm{E}_\pi[y_0^2 + u_0^2]| \le C(x)\rho_0^k \quad (10)$$

$$|\mathrm{E}_x[|\tilde\theta_k|^2] - \mathrm{E}_\pi[|\tilde\theta_0|^2]| \le C(x)\rho_0^k \quad (11)$$

Sample Path Convergence: We have for every initial condition $x \in X$.

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N}(y_k^2 + u_k^2) = \mathrm{E}_\pi[y_0^2 + u_0^2] \qquad \text{a.s.} \quad (12)$$

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N}|\tilde\theta_k|^2 = \mathrm{E}_\pi[|\tilde\theta_0|^2] \qquad \text{a.s.} \quad (13)$$

$$\lim_{N\to\infty} \mathrm{P}_x\left\{\frac{1}{\sqrt{N}}\sum_{k=1}^{N}(y_k^2 + u_k^2 - \mathrm{E}_x[y_k^2 + u_k^2]) < t\right\} = F(t) \quad (14)$$

$$\lim_{N\to\infty} \mathrm{P}_x\left\{\frac{1}{\sqrt{N}}\sum_{k=1}^{N}(|\tilde\theta_k|^2 - \mathrm{E}_x[|\tilde\theta_k|^2]) < t\right\} = G(t) \quad (15)$$

where F and G are distribution functions of Gaussian random variables.

Consistency and Performance: Suppose that conditions A1 and A5 hold for the ϵ-parameterized process $\{\Phi^\epsilon : 0 \le \epsilon \le 1\}$, and let π^ϵ denote the invariant probability for Φ^ϵ. Then under the control law (4) we have

$$\lim_{\epsilon\to 0} \mathrm{E}_{\pi^\epsilon}[y_0^2 + u_0^2] = 0. \quad (16)$$

If the control (5) is applied and conditions A1, A4, and A5 hold then

$$\lim_{\epsilon\to 0} \mathrm{E}_{\pi^\epsilon}[y_0^2 + u_0^2] = (|\theta^0|^2 + 2)\sigma_e^2 \quad (17)$$

$$\lim_{\epsilon\to 0} \mathrm{E}_{\pi^\epsilon}[|\tilde\theta_0|^2] = 0. \quad (18)$$

The proof of Theorem 1 comprises the remainder of this paper.

III. Construction of a Lyapunov Function

Here we present an extension of the stability proof in [3] in order to construct a certain Lyapunov function on the process Φ.

In this section, we assume for simplicity that the control law (4) is applied. The control (5) may be considered in the same way as below, with only notational changes.

Proposition 3.1: Under conditions A1–A3 we have for all $n \ge 0$

$$|\varphi_n|^2 \le \xi_n + \sum_{i=0}^{n-1}\lambda^{n-i}\left(\prod_{j=i}^{n-1}(1 + 2K_1\lambda^{-1}\alpha_j)\right)\xi_i \quad (19)$$

where

$$\xi_i \triangleq 4K_1 d^2\left[\lambda^i|\varphi_0|^2 + \sum_{k=0}^{i}\lambda^{i-k}(v_{k+1}^2 + |\tilde\theta_k|^2 + 1)\right] \quad (20)$$

$\lambda \in (0, 1)$, $K_1 = p\lambda^{-(p-1)}$, and $\alpha_j = (|\varphi_j^T\tilde\theta_j|^2)/(d + |\varphi_j|^2)$.

Proof: We have for all $n \ge 1$

$$\begin{aligned}
|\varphi_n|^2 &\le \sum_{i=1}^{n}\lambda^{n-i}|\varphi_i|^2 \le \sum_{i=1}^{n}\lambda^{n-i}K_1 y_i^2 \\
&\le \sum_{i=1}^{n}\lambda^{n-i}2K_1(|\varphi_{i-1}^T\tilde\theta_{i-1}|^2 + v_i^2) \\
&\le \sum_{i=1}^{n}\lambda^{n-i}2K_1[(|\varphi_{i-1}|^2 + d)\alpha_{i-1} + v_i^2] \\
&\le \sum_{i=0}^{n-1}\lambda^{n-i-1}2K_1|\varphi_i|^2\alpha_i + \sum_{i=1}^{n}\lambda^{n-i}2K_1[d|\tilde\theta_i|^2 + v_i^2].
\end{aligned}$$

It follows that

$$\lambda^{-n}|\varphi_n|^2 \le \sum_{i=0}^{n-1}[(2K_1\lambda^{-1}\alpha_i)\lambda^{-i}|\varphi_i|^2] + 4K_1 d^2\sum_{i=0}^{n}\lambda^{-i}[|\theta_i|^2 + v_i^2 + 1] \quad (21)$$

Letting $x_i = \lambda^{-i}|\varphi_i|^2$, $h_i = 2K_1\lambda^{-1}\alpha_i$, and $f_i = 4K_1 d^2\sum_{i=0}^{n}\lambda^{-i}[|\theta_i|^2 + v_i^2 + 1]$ one obtains from (21)

$$x_n \le f_n + \sum_{i=0}^{n-1}h_i x_i$$

for each $n \ge 1$. To make this inequality valid for $n = 0$ set $f_0 \triangleq 4K_1 d^2[|\theta_0|^2 + v_0^2 + |\varphi_0|^2 + 1]$. Proposition 3.1 then follows from the Bellman–Gronwall lemma (see [2, p. 254]). □

Using Proposition 3.1 we now construct a stochastic process possessing a Lyapunov-like property. Throughout the rest of the note let $r \triangleq n_0 + 1$ and $\mathscr{G}_k = \mathscr{F}_{rk}$, $k \ge 0$, where $\mathscr{F}_k = \sigma\{\Phi_0, \cdots, \Phi_k\}$.

Proposition 3.2: Suppose that conditions A1–A3 hold, and that the initial condition Φ_0 satisfies $\mathrm{E}[|\Phi_0|^8] < \infty$. Then there exists an adapted sequence $\{V_k, G_k\}$ such that V_k is positive for all $k \ge 0$, and constants $\rho < 1$ and D_0, $D_1 < \infty$ which do not depend on Φ_0 such that for all $k \ge 1$

$$\mathrm{E}[V_k \mid \mathscr{G}_{k-1}] \le \rho V_{k-1} + D_0.$$

For all $k \ge 1$, $|\Phi_{rk}|^4 \le D_1 V_k$ and hence

$$\limsup_{k\to\infty} \mathrm{E}[y_k^4] \le \frac{D_0 D_1}{1-\rho}.$$

□

A major step in the proof of Proposition 3.2 is to obtain bounds on α_j and $\prod_{j=i}^{n-1}(1 + 2K_1\lambda^{-1}\alpha_j)$.

Lemma 3.1: Suppose that conditions A1–A3 hold, and let $\Delta_{k+1} \triangleq \theta_{k+1} - \theta_k$. For any $n > k \ge 0$

$$\begin{aligned}
\alpha_k \le\ & 2(|\tilde\theta_k|^2 - |\tilde\theta_{k+1}|^2) \\
&+ (8/d)v_{k+1}^2 \\
&+ 16(p^{1/2}L + |\Delta_{k+1}|)|\Delta_{k+1}| \\
&+ 12(p^{1/2}L + |\theta_k|)|\theta_k|\mathbf{1}_{\{\theta_k \notin D\}}
\end{aligned} \quad (22)$$

$$\begin{aligned}
\prod_{j=k}^{n-1}&(1 + 2K_1\lambda^{-1}\alpha_j)^4 \\
&\le \exp(16K_1\lambda^{-1}|\tilde\theta_k|^2)\prod_{j=k}^{n-1}\exp(8K_1\lambda^{-1}(8/d)v_{j+1}^2) \\
&\quad\times\prod_{j=k}^{n-1}\exp(8K_1\lambda^{-1}16(p^{1/2}L + |\Delta_{j+1}|)|\Delta_{j+1}|) \\
&\quad\times\prod_{j=k}^{n-1}\exp(8K_1\lambda^{-1}12(p^{1/2}L + |\theta_j|)|\theta_j|\mathbf{1}_{\{\theta_j \notin D\}}).
\end{aligned} \quad (23)$$

The first inequality follows from a long chain of estimates involving (1) (see [3]). The second result follows from the first by using the estimate $1 + x \le e^x$. □

Proof of Proposition 3.2: By (19) it follows that

$$|\varphi_n|^2 \le \sum_{i=0}^{n} \lambda^{n-i}\xi_i b_i$$

where $b_n = 1$, and for $0 \le i \le n-1$

$$b_i = \prod_{j=i}^{n-1}\left(1 + 2K_1\lambda^{-1}\alpha_j\right).$$

By the Cauchy-Schwartz inequality

$$\left(\sum_{i=0}^{n}\lambda^{n-i}\xi_i b_i\right)^2 \le \left(\sum_{i=0}^{n}\lambda^{n-i}\right)\left(\sum_{i=0}^{n}\lambda^{n-i}\xi_i^2 b_i^2\right) \le \frac{1}{2(1-\lambda)}\sum_{i=0}^{n}\lambda^{n-i}\left(\xi_i^4 + b_i^4\right).$$

Hence substituting the values of b into the equation above and recalling that $r = n_0 + 1$, we have by (23)

$$\begin{aligned} |\varphi_{rk}|^4 \le U_{rk} \triangleq & \frac{1}{2(1-\lambda)}\left[1 + \sum_{i=0}^{rk}\lambda^{k-i}\xi_i^4\right] \\ & + \frac{1}{2(1-\lambda)}\left[\sum_{i=0}^{rk}\lambda^{k-i}\exp\left(64K_1\lambda^{-1}(r+2)\right.\right. \\ & \left.\left.\cdot\left(|\psi_i|^2 + |\eta_{i+1}|^2 + |\theta^0|^2\right)\right)\right] \\ & + \frac{1}{2(1-\lambda)}\sum_{l=1}^{3}\sum_{m=1}^{r}\left[\sum_{i=0}^{rk}\lambda^{k-i}\prod_{j=\lfloor i/r\rfloor}^{k-1} Y_{rj+m}^{l}\right] \end{aligned} \quad (24)$$

where $\psi_i^{\mathrm{T}} = (\hat{\theta}_i^{\mathrm{T}}, \eta_i^{\mathrm{T}}, \cdots, \eta_{i-n_0+1}^{\mathrm{T}})$, and

$$Y_j^1 \triangleq \exp\left(r32K_1\lambda^{-1}(8/d)v_{j+1}^2\right) \quad (25)$$

$$Y_j^2 \triangleq \exp\left(r32K_1\lambda^{-1}\left(p^{1/2}L + |\Delta_{j+1}|\right)|\Delta_{j+1}|\right) \quad (26)$$

$$Y_j^3 \triangleq \exp\left(r32K_1\lambda^{-1}12\left(p^{1/2}L + |\theta_j|\right)|\theta_j|1_{\{\theta_j \notin D\}}\right). \quad (27)$$

We now define the stochastic process $\{V_k\}$ as

$$V_k \triangleq \mathrm{E}\left[U_{r(k+1)} \mid \mathscr{G}_k\right] \quad (28)$$

where $\mathscr{G}_k = \mathscr{F}_{rk}$. From the definition of U_{rk} in (24) it is easy to see that

$$|\varphi_{rk}|^4 \le U_{rk} \text{ and } |\psi_{rk}|^4 \le U_{rk}.$$

Hence,

$$(1/4)\lambda^r|\Phi_{rk}|^4 \le (1/2)\lambda^r\left(|\varphi_{rk}|^4 + |\psi_{rk}|^4\right) \le \lambda^r U_{rk} \le U_{r(k+1)}.$$

It follows that with $D_1 \triangleq 4\lambda^{-r}$ we have $|\Phi_{rk}|^4 \le D_1 V_{rk}$.

Under conditions A1–A3 the stochastic process $\{V_k\}$ satisfies the remaining conclusions of Proposition 3.2. To see this, consider

$$V_k^{(l,m)} = \mathrm{E}\left[\sum_{i=0}^{r(k+1)}\lambda^{r(k+1)-i}\prod_{j=\lfloor i/r\rfloor}^{k} Y_{rj+m}^{l} \mid \mathscr{G}_k\right]$$

for l and m fixed. The process $\{V_k\}$ may be written as a finite sum of random variables of this form, plus two remaining terms which may be dealt with in the same way as below.

For each l, m, and $k \ge 1$

$$\begin{aligned} & \mathrm{E}\left[V_k^{(l,m)} \mid \mathscr{G}_{k-1}\right] \\ &= \mathrm{E}\left[\mathrm{E}\left[\sum_{i=0}^{r(k+1)}\lambda^{r(k+1)-i}\prod_{j=\lfloor i/r\rfloor}^{k} Y_{rj+m}^{l} \mid \mathscr{F}_{r(k-1)+m}\right] \mid \mathscr{F}_{r(k-1)}\right] \\ &\le \mathrm{E}\left[\sum_{i=0}^{r(k+1)}\lambda^{k-i}\left(\lambda^r M^{(l,m)}\right)\prod_{j=\lfloor i/r\rfloor}^{k-1} Y_{rj+m}^{l} \mid \mathscr{F}_{r(k-1)}\right] \\ &\le \left(\lambda^r M^{(l,m)}\right)V_{k-1}^{(l,m)} + (1-\lambda)^{-1}M^{(l,m)} \end{aligned}$$

where the constant $M^{(l,m)} = \mathrm{E}[Y_{rk+m}^{l}]$ is less than λ^{-r} and may be defined in terms of the constants used in assumption A3. □

IV. Analysis of Φ

The stability proof above sets the stage for an analysis of the state process Φ.

Stochastic Controllability

To prove Theorem 1 we apply the techniques developed in [12], [10].

The following is the main result in this section.

Proposition 4.1: Suppose that conditions A1–A4 hold and that either control (4) or (5) is applied. Then there exists a unique invariant probability π. Furthermore, there exists a fixed probability μ on $\beta(X)$ with the following property: for each compact set $F \subset X$ there exists $\epsilon > 0$ and $N \ge 1$ such that

$$\sum_{k=1}^{N} P^k(x, B) \ge \epsilon\mu\{B\} \quad (30)$$

for all $x \in F$, and $B \in \beta(X)$.

A set F and measure μ satisfying (30) are called *petite* (see [10]) and Corollary 4.1 implies that the Markov chain Φ is *irreducible* (see [14], [13]).

Proof (sketch):

By Theorem 3.1 of [12], the proposition will hold for all initial conditions $\Phi_0 \in X$ if the matrix

$$C_{\Phi_0}^{M} \triangleq \left.\frac{\partial\Phi_M}{\partial(w_1, \cdots, w_m)}\right|_{(w_1, \cdots, w_M)} \quad (29)$$

has full-row rank for some sequence $(w_1, \cdots, w_M)$ which does not drive the state Φ_i, $1 \le i \le M$ to the boundary of X. Furthermore, the sequence $(w_1, \cdots, w_M)$ must evolve within the support of the distribution μ_w of w_i.

Uniqueness of the invariant probability must also be established; this follows by verifying that the nonlinear control system associated with Φ (with the disturbance w replaced by an input) is asymptotically controllable to some fixed state $\Phi^* \in X$.

Lengthy calculations show that this is indeed the case, and hence we have the proposition. □

Periodicity

One consequence of irreducibility is periodicity. Suppose that conditions A1–A3 hold, and let supp π denote the support of the invariant probability π — supp π is equal to the smallest closed subset of X which has π-measure one. It is shown in [12] (see also [14]) that the support of π may be written supp $\pi = \bigcup_{n=1}^{\lambda} D_n$ where the sets $\{D_1, \cdots, D_\lambda\}$ are closed and disjoint, and λ is the *period* of π.

It may be seen that π will be *aperiodic* (i.e., $\lambda = 1$) if the set supp π contains a fixed point. That is, a state $x^* \in$ supp π with the property that $\Phi_{k+1} = x^*$ when $\Phi_k = x^*$, and $w_{k+1} = 0$. It is possible to show that in this example a fixed point lying in supp π does exist, and has the form $x^* = (0, \hat{\theta}^*, 0)$. Hence we have the following lemma.

Lemma 4.1: Under the conditions of Proposition 4.1, the invariant probability π is aperiodic. □

Proof of Theorem 1

Exponential Stability: It follows from Corollary 4.1, Lemma 4.1, Proposition 4.2, and Theorem 6.3 of [10] that Φ is geometrically ergodic. Result (9) follows from this, and results (10) and (11) follow from Proposition 3.2 and Theorem 6.3 of [10].

Sample Path Convergence: Results (12) and (13) follow from Harris recurrence and Theorem 7.1 of [10]. Results (14) and (15) follow from Corollary 4.1, Proposition 4.2, and a result of [10].

Consistency and Performance: For the proof of this part of Theorem 1, we apply the general results of Appendix A.

First, by the proof of Proposition 3.1 and Proposition 3.2, there exists an extremely complex and nonlinear, but continuous function g of the constant L used in the estimation algorithm, and the constants $D_v^\epsilon \triangleq \mathrm{E}[\exp(\epsilon_0(v_1^\epsilon)^2)]$ and $D_\eta^\epsilon \triangleq \mathrm{E}[\exp(M_0|\eta_1^\epsilon|^2)]$ such that

$$\mathrm{E}_{\pi^\epsilon}\left[|\Phi_0^\epsilon|^4\right] \leq \limsup_{k\to\infty} \mathrm{E}_x\left[|\Phi_k^\epsilon|^4\right] \leq g\left(L, D_v^\epsilon, D_\eta^\epsilon\right). \quad (31)$$

By condition A5, the right-hand side of (31) is uniformly bounded in $\epsilon \in [0, 1]$. Since $|\cdot|^4$ is a moment on X it follows that $\{\pi^\epsilon : 0 \leq \epsilon \leq 1\}$ is tight.

Next, by condition A5 we have $\mu_w^\epsilon \overset{\text{weakly}}{\to} \mu_w^0 \triangleq \delta_0$, where δ_0 denotes the probability concentrated at the origin in $\mathbb{R}^{p+1}$. It follows by Proposition A.1 that $P_\epsilon \overset{\text{u on c}}{\to} P_0$ as $\epsilon \to 0$, where $\{P_\epsilon\}$ are the Markov transition functions corresponding to the parameterized family of Markov chains $\{\Phi^\epsilon\}$. Applying Proposition A.2 we have

$$\pi^\epsilon \overset{\text{weakly}}{\to} \mathscr{I}^0 \qquad \text{as } \epsilon \to 0$$

where $\mathscr{I}^0$ denotes the set of invariant probabilities for Φ^0. The results then follow from (31) and the following lemma. □

Lemma 4.2: Suppose that conditions A1 and A4 hold, that $w \equiv 0$, and let $\mathscr{I}^0$ denote the set of invariant probabilities for the resulting state process Φ^0

i) if the control (4) is applied then $y_k = u_k = 0$ a.s. $[\mathrm{P}_{\pi^0}]$ for any $\pi^0 \in \mathscr{I}^0$;

ii) if the control (5) is applied then $\mathscr{I}^0 = \{\pi^0\}$ is a singleton, where π^0 is defined so that for all $k \in \mathbb{Z}_+$

$$y_k = e_k \text{ and } \hat{\theta}_k = \theta_k = \theta^0 \qquad \text{a.s. } \mathrm{P}_{\pi^0}$$

Proof: When $w \equiv 0$ we obtain for any invariant probability $\theta_k = \theta^0$ a.s. $[\mathrm{P}_{\pi^0}]$, and by (3) it follows that

$$|\tilde{\theta}_{k+1}|^2 \leq |\tilde{\theta}_k|^2 - \frac{|\varphi_k^{\mathrm{T}}\tilde{\theta}_k|^2}{d + |\varphi_k|^2} \qquad \text{a.s. } [P_{\pi^0}].$$

Taking expectations of both sides with respect to P_{π^0} we obtain

$$\mathrm{E}_{\pi^0}\left[\frac{|\varphi_k^{\mathrm{T}}\tilde{\theta}_k|^2}{d + |\varphi_k|^2}\right] = 0, \text{ and hence } \varphi_k^{\mathrm{T}}\tilde{\theta}_k = 0 \qquad \text{a.s. } [\mathrm{P}_{\pi^0}]. \quad (32)$$

By (7) it immediately follows that $y_k = 0$ a.s. P_{π^0} when the control (4) is applied, and this proves i).

To prove ii), observe that (32) and (8) imply that for either control

$$\hat{\theta}_{k+1} = \hat{\theta}_k = \hat{\theta}_0 \qquad \text{a.s. } [\mathrm{P}_{\pi^0}]. \quad (33)$$

If the control (5) is applied then by (32) and (1) we have $\varphi_k = (e_k, \cdots, e_{k-n_0+1})^{\mathrm{T}}$, and it follows from (32) and (33) that

$$\begin{aligned}\sigma_e^2|\tilde{\theta}_0|^2 &= \tilde{\theta}_0^{\mathrm{T}}\mathrm{E}_{\pi^0}\left[\varphi_{n_0}\varphi_{n_0}^{\mathrm{T}}\right]\tilde{\theta}_0\\ &= \mathrm{E}_{\pi^0}\left[\left(\varphi_{n_0}\tilde{\theta}_0^{\mathrm{T}}\right)^2 |\tilde{\theta}_0\right]\\ &= \mathrm{E}_{\pi^0}\left[\left(\varphi_{n_0}\tilde{\theta}_{n_0}^{\mathrm{T}}\right)^2 |\tilde{\theta}_0\right]\\ &= 0 \qquad \text{a.s. } [\mathrm{P}_{\pi^0}].\end{aligned}$$

Hence $\theta_0 = \theta_0$ a.s. $[\mathrm{P}_{\pi^0}]$, and this is what was wanted. □

A Remark on the Consistency Proof

At present we do not see any way of establishing results (16)–(18) without the methods introduced in Appendix A.

It would seem that results (16)–(18) could be established directly from (31), or perhaps a tighter version of this bound. The problem with this direct approach is that the function g used in (31) is dependent on the constant L, and in particular does not converge to a useful limit for fixed L as $(D_v^\epsilon, D_\eta^\epsilon) \to (1, 1)$.

However, if a stronger bound is obtainable, then it may be possible to obtain "near consistency" without the introduction of the dither sequence $\boldsymbol{e}$. This approach is currently under investigation.

Appendix

In the appendix we prove some new general results for Markov chains on general state spaces.

Approximation of Markov Chains

Throughout this section we let $\{P_\epsilon : 0 \leq \epsilon \leq 1\}$ denote a family of Feller–Markov transition functions, and $\{\pi_\epsilon : 0 \leq \epsilon \leq 1\}$ a set of probabilities such that π_ϵ is invariant under P_ϵ for each $\epsilon \in [0, 1]$.

Our objective is to find general conditions on the Markov transition functions $\{P_\epsilon\}$ so that π_ϵ converges weakly to π_0 (denoted $\pi_\epsilon \overset{\text{weakly}}{\to} \pi_0$ as $\epsilon \to 0$). The results of this section are based on [15] and [18].

We say that the Markov transition functions $\{P_\epsilon : 0 < \epsilon \leq 1\}$ *converge* to the Markov transition function P_0 and write $P_\epsilon \to P_0$ as $\epsilon \to 0$, if for every $x \in X$ and $f \in C$ (the set of bounded and continuous functions on X),

$$\lim_{\epsilon\to 0} P_\epsilon f(x) = P_0 f(x).$$

In other words, for each $x \in X$, $P_\epsilon(x, \cdot) \overset{\text{weakly}}{\to} P_0(x, \cdot)$ as $\epsilon \to 0$. We say that $\{P_\epsilon : 0 < \epsilon \leq 1\}$ *converges uniformly on compact sets* to the Markov transition function P_0 if for every compact set $F \subset X$ and $f \in C$

$$\lim_{\epsilon\to 0} \sup_{x\in F} |P_\epsilon f(x) - P_0 f(x)| = 0.$$

This shall be denoted $P_\epsilon \overset{\text{u on c}}{\to} P_0$.

We list here some assumptions which we will need to refer to in this section.

R0: P_0 possesses at most one invariant probability.

R1: The set of probabilities $\{\pi_\epsilon : 0 \leq \epsilon \leq 1\}$ is tight.

R2: For each $f \in C$ the collection of functions $\{P_\epsilon f : 0 \leq \epsilon \leq 1\}$ is equicontinuous on compact subsets of X.

It will be shown under general conditions that condition R2 implies R3.

R3: Whenever a family of probabilities $\{\mu_\epsilon : 0 \leq \epsilon \leq 1\} \subset \mathscr{M}$ satisfy

$$\mu_\epsilon \overset{\text{weakly}}{\to} \mu_0 \qquad \text{as } \epsilon \to 0$$

it follows that

$$\mu_\epsilon P_\epsilon \overset{\text{weakly}}{\to} \mu_0 P_0 \qquad \text{as } \epsilon \to 0.$$

Remark A.1: It follows that Ascoli's theorem that $P_\epsilon \overset{\text{u on c}}{\to} P_0$ if and only if $P_\epsilon \to P_0$ as $\epsilon \to 0$ and condition R2 holds.

Remark A.2: If a moment V on X together with a constant $B < \infty$ exist such that

$$\limsup_{k\to\infty} P_\epsilon^k V(x) \leq B$$

for all $x \in X$ and all $\epsilon \in [0, 1]$, then it may be shown that condition R1 holds.

The first result below concerns perturbations of a disturbance distribution μ_w. A related result may be found in [9].

Suppose that the Markov chains $\{\Phi^\epsilon : 0 \le \epsilon \le 1\}$, have the form

$$\Phi_k^\epsilon = F(\Phi_{k-1}^\epsilon, w_k^\epsilon) \tag{34}$$

where $F: X \times \mathbb{R}^p \to X$ is continuous, and for each $\epsilon \in [0, 1]$, $w^\epsilon \triangleq \{w_k^\epsilon : k \in \mathbb{Z}_+\}$ is independent and identically distributed with $w_k^\epsilon \sim \mu_w^\epsilon$ for each $k \in \mathbb{Z}_+$, and $\epsilon \in [0, 1]$. Then the Markov transition functions P_ϵ, $\epsilon \in [0, 1]$, are defined for $g \in C$ by

$$P_\epsilon g(x) = \int_{\mathbb{R}^p} g(F(x, \lambda))\mu_w^\epsilon(d\lambda). \tag{35}$$

Proposition A.1: For the Markov transition functions P_ϵ, $\epsilon \in [0, 1]$, defined in (35), suppose that the function F is continuous and that $\mu^\epsilon \overset{\text{weakly}}{\to} \mu_w^0$ as $\epsilon \to 0$. Then $P_\epsilon \overset{\text{u on c}}{\to} P_0$ as $\epsilon \to 0$.

Proof: Fix $\delta > 0$, $g \in C$, and define the functions

$$\xi_x(t) = g(F(x, t)), \qquad x \in X, t \in \mathbb{R}^p.$$

Letting $G \subset X$ be compact we have

$$\sup_{x \in G} | P_\epsilon g(x) - P_0 g(x) |$$
$$= \sup_{x \in G} \left| \int \xi_x(t)\mu_w^\epsilon(dt) - \int \xi_x(t)\mu_w^0(dt) \right|.$$

The family of functions $\{\xi_x : x \in G\}$ is equicontinuous on compact subsets of $\mathbb{R}^p$, and it follows from weak convergence of $\{\mu_w^\epsilon\}$ and Theorem 6.8 of [15] that the right-hand side of the inequality above converges to zero as $\epsilon \to 0$. This implies that $P_\epsilon g$ converges to $P_0 g$ uniformly on compact subsets of X, which was what was wanted. □

The following result will be used to prove the "near optimality" result in Theorem 1 of this paper. Part ii) of Proposition A.2 is adapted from [5, Theorem 6 of Ch. 6, Section 4].

Proposition A.2:

i) If $P_\epsilon \overset{\text{u on c}}{\to} P_0$ as $\epsilon \to 0$ then condition R3 is satisfied.

ii) Suppose that assumptions R1 and R3 hold, and that $\pi_\epsilon P_\epsilon = \pi_\epsilon$ for each $\epsilon > 0$. Then

$$\pi_\epsilon \overset{\text{weakly}}{\to} \mathscr{I}_0 \qquad \text{as } \epsilon \to 0 \tag{36}$$

where $\mathscr{I}_0 \subset \mathscr{M}$ is the set of probabilities which are invariant under P_0.

Proof:

i) Let $\mu_\epsilon \overset{\text{weakly}}{\to} \mu$ as $\epsilon \to 0$, and fix $f \in C$. Then, letting $\langle \nu, f \rangle \triangleq \int f\, d\nu$

$$\langle \mu_\epsilon P_\epsilon, f \rangle = \langle \mu_\epsilon, (P_\epsilon - P_0) f \rangle + \langle \mu_\epsilon, P_0 f \rangle.$$

Since $P_\epsilon \overset{\text{u on c}}{\to} P_0$ as $\epsilon \to 0$ the first summand converges to zero as $\epsilon \to 0$. Hence, because $P_0 f \in C$

$$\lim_{\epsilon \to 0} \langle \mu_\epsilon P_\epsilon, f \rangle = \langle \mu_0, P_0 f \rangle = \langle \mu_0 P_0, f \rangle$$

which shows that $\mu_\epsilon P_\epsilon \overset{\text{weakly}}{\to} \mu_0 P_0$ as $\epsilon \to 0$. Hence condition R3 is satisfied, and this completes the proof of i).

ii) Let Υ be a limit point of $\{\pi_\epsilon : 0 < \epsilon \le 1\}$. Then for some sequence $\{\epsilon_i : i \in \mathbb{Z}_+\}$, converging to zero

$$\pi_{\epsilon_i} \overset{\text{weakly}}{\to} \Upsilon \qquad \text{as } i \to \infty.$$

Applying assumption R3 we find that $\pi_{\epsilon_i} P_{\epsilon_i} \overset{\text{weakly}}{\to} \Upsilon P_0$ as $i \to \infty$. Since $\pi_{\epsilon_i} P_{\epsilon_i} = \pi_{\epsilon_i}$ for all $i \in \mathbb{Z}_+$ we conclude that

$$\Upsilon P_0 = \Upsilon$$

and this completes the proof of the proposition. □

Here we give a sufficient condition to ensure the convergence of the invariant probabilities corresponding to a convergent sequence of Markov transition functions.

Corollary A.2: Suppose that $P_\epsilon \overset{\text{u on c}}{\to} P_0$ as $\epsilon \to 0$, and that conditions R0 and R1 hold. Then

$$\pi_\epsilon \overset{\text{weakly}}{\to} \pi_0 \qquad \text{as } \epsilon \to 0$$

where π_0 is invariant under P_0.

Proof: Follows immediately from Proposition A2. □

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