

## FURTHER RESULTS ON LEAST SQUARES BASED ADAPTIVE MINIMUM VARIANCE CONTROL\*

LEI GUO†

**Abstract.** Based on the recently established results on self-tuning regulators originally proposed by Åström and Wittenmark, this paper presents various novel and extended results on least squares based adaptive minimum variance control for linear stochastic systems. These results establish self-optimality, self-tuning property, and the best possible convergence rate of the control law in a variety of situations of interest.

**Key words.** stochastic adaptive control, self-tuning, least squares

**AMS subject classifications.** 93C40, 93E12, 93E10

### 1. Introduction.

**1.1. System description.** Consider the following SISO linear discrete-time stochastic system:

$$(1.1) \quad A(z)y_n = B(z)u_{n-1} + C(z)w_n, \quad n \geq 0,$$

where  $\{y_n\}$ ,  $\{u_n\}$  and  $\{w_n\}$  are the system output, input, and random disturbance sequences, respectively,  $y_n = u_n = w_n = 0$  for all  $n < 0$ , and  $A(z)$ ,  $B(z)$ , and  $C(z)$  are polynomials in backward-shift operator  $z$ :

$$\begin{aligned} A(z) &= 1 + a_1z + \cdots + a_pz^p, & p \geq 0, \\ B(z) &= b_1 + b_2z + \cdots + b_qz^{q-1}, & q \geq 1, \\ C(z) &= 1 + c_1z + \cdots + c_rz^r, & r \geq 0, \end{aligned}$$

with known upper bounds  $p$ ,  $q$ , and  $r$  for true orders and unknown coefficients  $a_i$ ,  $b_j$ , and  $c_k$ .

As usual, for the above model we adopt the following standard assumptions:

(A1)  $\{w_n, \mathcal{F}_n\}$  is a martingale difference sequence, i.e.,  $E[w_{n+1}|\mathcal{F}_n] = 0$ , and satisfies

$$(1.2) \quad \sup_n E[|w_{n+1}|^\beta | \mathcal{F}_n] < \infty, \text{ a.s. for some } \beta > 2,$$

$$(1.3) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n w_i^2 = \sigma^2 > 0 \quad \text{a.s.}$$

(A2) SPR condition:  $\max_{|z|=1} |C(z) - 1| < 1$ .

(A3) Minimum phase condition:  $B(z) \neq 0$ , for all  $z : |z| \leq 1$ .

Condition (A1) implies that the linear minimum variance predictor for  $y_{i+1}$  generated by (1.1) coincides with the minimum variance predictor  $E[y_{i+1}|\mathcal{F}_i]$  if  $\{u_i, \mathcal{F}_i\}$  is an adapted sequence. Condition (A2) is the usual SPR condition

$$\operatorname{Re} \left\{ \frac{1}{C(e^{j\lambda})} - \frac{1}{2} \right\} > 0 \quad \forall \lambda \in [0, 2\pi] \quad (j \triangleq \sqrt{-1}),$$

\* Received by the editors April 15, 1992; accepted for publication (in revised form) November 3, 1992. This work was supported by the National Natural Science Foundation of China.

† Institute of Systems Science, Chinese Academy of Sciences, Beijing 100080, People's Republic of China.

which implies that  $\sum_{i=1}^r c_i^2 < 1$ , and is implied by  $\sum_{i=1}^r |c_i| < 1$  (cf. Huang and Guo [1, pp. 1731, 1755]). This condition, together with the a priori knowledge about the orders  $p, q$ , and  $r$ , can be dispensed with for recursive identification of the linear model (1.1). We will not discuss that issue here and instead refer to Huang and Guo [1] for details. Condition (A3) is necessary for internal stability of minimum variance control systems even if the parameters in (1.1) are known (see, e.g., Kumar and Varaiya [2, p. 121]).

**1.2. Performance.** Our objective is to construct a control sequence  $\{u_n\}$  based on the past and current observations, such that the following averaged square tracking error is asymptotically minimized:

$$(1.4) \quad J_n \triangleq \frac{1}{n} \sum_{i=1}^n (y_i - y_i^*)^2,$$

where  $\{y_i^*\}$  is a reference sequence to be tracked, which is assumed to satisfy the following condition:

(A4)  $\{y_i^*\}$  is bounded almost surely and is independent of  $\{w_i\}$ .

For convenience of discussions, we may assume without loss of generality that  $\mathcal{F}_i = \sigma\{w_j, y_{j+1}^*, j \leq i\}$ . Then for any adapted input sequence  $\{u_i, \mathcal{F}_i\}$ ,  $y_i - y_i^* - w_i$  is  $\mathcal{F}_{i-1}$ -measurable for all  $i$ , and so by Chow's local convergence theorem for martingales (cf. [3]), it is easy to conclude that

$$(1.5) \quad J_n = \frac{1}{n} \sum_{i=1}^n w_i^2 + \begin{cases} R_n(1 + o(1)) & \text{on } [nR_n \xrightarrow[n \rightarrow \infty]{} \infty]; \\ O(\frac{1}{n}) & \text{on } [\lim_{n \rightarrow \infty} (nR_n) < \infty] \end{cases}$$

where  $R_n$  denotes the following "averaged regret":

$$(1.6) \quad R_n = \frac{1}{n} \sum_{i=1}^n (y_i - y_i^* - w_i)^2.$$

Consequently, by virtue of (1.3), we know that for any adapted sequence  $\{u_i, \mathcal{F}_i\}$  the asymptotic lower bound to  $J_n$  is  $\sigma^2$ , and that

$$(1.7) \quad J_n \xrightarrow[n \rightarrow \infty]{} \sigma^2 \text{ a.s.} \iff R_n \xrightarrow[n \rightarrow \infty]{} 0 \text{ a.s.}$$

which justifies the familiar concept "globally convergent" or "self-optimizing" for an adaptive controller that leads to  $R_n \xrightarrow[n \rightarrow \infty]{} 0$  a.s.. Moreover, from (1.5) it is apparent that  $R_n$  is of essential importance for the convergence rate of  $J_n$ , since it can be regarded as a second-order quantity (see also Wei [4, p. 1668]). It is also worth noting that once the self-optimality  $R_n \xrightarrow[n \rightarrow \infty]{} 0$  a.s. is proved, the global stability, i.e.,  $\sup_n (1/n) \sum_{i=1}^n (y_i^2 + u_i^2) < \infty$  a.s., can be derived trivially by using Assumptions (A1), (A3), and (A4).

**1.3. Estimation algorithms.** Let us denote the unknown parameters in (1.1) by

$$(1.8) \quad \theta = [-a_1 \cdots -a_p, b_1 \cdots b_q, c_1 \cdots c_r]^\tau.$$

Then the model (1.1) can be succinctly written in a regression form:

$$(1.9) \quad y_{n+1} = \theta^\tau \varphi_n^0 + w_{n+1}, \quad n \geq 0,$$

where  $\varphi_n^0$  is the regression vector defined by

$$(1.10) \quad \varphi_n^0 = [y_n \cdots y_{n-p+1}, u_n \cdots u_{n-q+1}, w_n \cdots w_{n-r+1}]^\tau.$$

The standard method for estimating  $\theta$  is the following recursive extended least squares (ELS) algorithm:

$$(1.11) \quad \theta_{n+1} = \theta_n + a_n P_n \varphi_n (y_{n+1} - \theta_n^\tau \varphi_n)$$

$$(1.12) \quad P_{n+1} = P_n - a_n P_n \varphi_n \varphi_n^\tau P_n, \quad a_n = (1 + \varphi_n^\tau P_n \varphi_n)^{-1},$$

$$(1.13) \quad \varphi_n = [y_n \cdots y_{n-p+1}, \quad u_n \cdots u_{n-q+1}, \widehat{w}_n \cdots \widehat{w}_{n-r+1}]^\tau,$$

$$(1.14) \quad \widehat{w}_n = y_n - \theta_n^\tau \varphi_{n-1}$$

with arbitrary initial values  $\theta_0, \varphi_0 \neq 0$  and  $P_0 \geq 0$ .

There is a vast literature on strong consistency of the above ELS algorithm (see, e.g., Caines [5], Chen and Guo [6], and the references therein). In a Bayesian framework assuming Gaussianity of both the noise  $\{w_n\}$  and the parameter  $\theta$ , it was shown by Sternby [7] that in the white noise case (i.e.,  $C(z) = 1$ ), the necessary and sufficient condition for strong consistency of the least squares (LS) estimate  $\theta_n$  is that

$$(1.15) \quad \lambda_{\min}(n) \xrightarrow{n \rightarrow \infty} \infty \text{ a.s.}$$

where  $\lambda_{\min}(n)$  denotes the minimum eigenvalue of  $P_{n+1}^{-1}$ , i.e.,

$$(1.16) \quad \lambda_{\min}(n) \triangleq \lambda_{\min} \left\{ \sum_{i=0}^n \varphi_i \varphi_i^\tau + P_0^{-1} \right\}.$$

In the non-Bayesian framework where  $\theta$  is an unknown constant vector as the case considered here, Lai and Wei [8] succeeded in showing that in the white noise case, strong consistency of the LS estimate still holds if (1.15) is strengthened into

$$(1.17) \quad \lambda_{\min}(n) \xrightarrow{n \rightarrow \infty} \infty, \quad \frac{\log \lambda_{\max}(n)}{\lambda_{\min}(n)} \xrightarrow{n \rightarrow \infty} 0 \text{ a.s.}$$

where  $\lambda_{\max}(n)$  denotes the maximum eigenvalue of  $P_{n+1}^{-1}$ . They also presented an example showing that relaxing the second part of (1.17) can result in a loss of strong consistency of the LS algorithm. The above consistency result can be easily generalized to colored noise and multivariable cases by resorting to the standard SPR condition (A2), and by using the standard recursions for the Lyapunov function studied earlier in (e.g., Ledwich and Moore [9], Solo [10], and Chen [11]) together with Chow’s local convergence theorem for martingales (see [12] and [13]).

Despite the celebrated convergence properties of the ELS algorithm, the basic stability issue of adaptive minimum variance control constructed by using the ELS algorithm has been a long-standing problem over the past two decades. The main difficulty is that we do not know if the condition (1.17) really holds for the closed-loop systems. In fact, over the past decade, most of the results in stochastic adaptive control theory have been established for adaptive control laws that are not based on ELS algorithm but based on a stochastic gradient (SG) algorithm (or its variant). This algorithm is formed by simply replacing the matrix gain  $\{a_n P_n\}$  in (1.11) by a scalar gain  $\{\mu/r_n\}$  with  $\mu > 0$ , where

$$(1.18) \quad r_n \triangleq 1 + \sum_{i=0}^n \|\varphi_i\|^2.$$

Goodwin, Ramadge, and Caines [14] obtained the first stability and optimality result on SG-based adaptive tracking algorithms, which stimulated considerable research efforts afterwards. However, as is observed in simulations, the SG algorithm exhibits much slow convergence rate as compared with the ELS algorithm. Chen and Guo [15], [16], [6] have given a comprehensive theoretical study for the convergence of SG algorithm and justified the convergence phenomena known by simulations. To be precise, for strong consistency of SG, the following condition was introduced by Chen and Guo [15]:

$$(1.19) \quad r_n \xrightarrow{n \rightarrow \infty} \infty, \quad \frac{\lambda_{\max}(n)}{\lambda_{\min}(n)} = O(\{\log r_n\}^\alpha) \text{ a.s.}, \quad \alpha \geq 0.$$

They showed that for the SG algorithm, if (1.19) holds with  $\alpha \leq \frac{1}{4}$ , then  $\theta_n \xrightarrow{n \rightarrow \infty} \theta$ , a.s. (see [15, Thm. 1], [16, Thm. 2], and [6, Thm. 4.5]). They also presented an example showing that in (1.19) the constant  $\alpha$  is not allowed to be greater than 1 for strong consistency of SG (see [6, pp. 124–129]).

Hence for strong consistency, the SG algorithm requires much more excitation than the LS algorithm does (note that (1.19) is much stronger than (1.17)). Moreover, in the white noise case under the condition (1.19) with  $\alpha = \frac{1}{4}$ , the guaranteed convergence rate for the SG algorithm is only of the order  $O(1/\log^\lambda r_n)$ , i.e.,

$$(1.20) \quad \|\theta_n - \theta\|^2 = O\left(\frac{1}{\log^\lambda r_n}\right) \text{ a.s. for some } \gamma > 0$$

(cf. [15, p. 141] or [6, p. 132]), while under the same conditions, the convergence rate for the LS algorithm is much faster:  $\|\theta_n - \theta\|^2 = O(\log^{\frac{5}{4}} r_n/r_n)$  a.s. (see, e.g., [6, p. 96] or [8, p. 155]).

**1.4. Background.** The standard adaptive minimum variance tracking control is constructed by simply identifying the adaptive predictor with the target value, i.e.,

$$(1.21) \quad \theta_n^T \varphi_n = y_{n+1}^*, \quad n \geq 1,$$

where  $\{\theta_n\}$  is generated by the ELS algorithm (1.11)–(1.14).

Åström and Wittenmark [17] were, apparently, the first to attempt an analysis of adaptive minimum variance control constructed by using LS-type estimates. They showed that if the LS parameter estimates should converge to some limit with no common factor, then the adaptive controller must necessarily be optimal. However, a difficult problem is whether these estimates are indeed convergent. To overcome this difficulty, Kumar [18] considered the case where the additive noise in (1.1) is *i.i.d.* and Gaussian. By using the technique of “Bayesian embedding,” he succeeded in showing that, outside an exceptional set of true parameter vectors of Lebesgue measure zero, the LS based self-tuning minimum variance control enjoys various important convergence properties.

Recently, Guo and Chen [19] solved the basic stability and optimality problem of ELS-based adaptive minimum variance control for the general system (1.1) under the standard conditions (A1)–(A3). The following was shown in [19]:

(i) If the “high frequency” gain  $b_1$  is known, then the standard ELS-based self-tuning tracker is globally stable and self-optimizing, with a rate of convergence for the regret:  $R_n = O(d_n/n^{1-\varepsilon})$  a.s. for all  $\varepsilon > 0$ , where  $\{d_n\}$  is a positive sequence satisfying  $d_n \leq d_{n+1}$ ,  $\sup_{n \geq 0} (d_{n+1}/d_n) < \infty$ , and

$$(1.22) \quad \|w_n\|^2 = O(d_n) \text{ a.s.}$$

(ii) If  $b_1$  is unknown, instead of using a fixed a priori estimate  $\hat{b}_1$  for  $b_1$  in designing the control law as in Åström and Wittenmark [17], a natural approach is to update this estimate with the current and past data. This was done in [19] by setting the on-line estimate (say  $\hat{b}_1(n)$ ) to be

$$(1.23) \quad \hat{b}_1(n) = \begin{cases} b_1(n) & \text{if } |b_1(n)| \geq \frac{1}{\sqrt{\log r_{n-1}}}, \\ b_1(n) + \operatorname{sgn}(b_1(n)) \frac{1}{\sqrt{\log r_{n-1}}} & \text{otherwise,} \end{cases}$$

where  $\operatorname{sgn}(\cdot)$  is the sign function,  $r_n$  is defined by (1.18) and  $b_1(n)$  is the  $(p+1)$ th component of  $\theta_n$  generated by the ELS algorithm (1.11)–(1.14). Then the resulting ELS-based adaptive control law is again shown to be stable and self-optimizing, with an implicitly established convergence rate  $R_n = O(1/\log n)$  a.s..

The purpose of this paper is to give further results on ELS-based adaptive minimum variance control, with emphases placed on the convergence rate of  $R_n$ . We will improve the convergence rate obtained in [19] and show that in some cases the limit of  $(n/\log n)R_n$  actually exists and is finite. We will also study the standard control law (1.21) (with no modifications on  $b_1(n)$ ) and address the consistency issue of parameter estimates.

**2. Preliminaries.** To begin with, consider the regulation problem where  $y_i^* \equiv 0$ . Let  $\lambda_{\min}(X)$  denote the minimum eigenvalue of a square matrix  $X$ . Then, from (1.9) it follows that

$$(2.1) \quad \lambda_{\min} \left( \sum_{i=0}^{n-1} \varphi_i^0 \varphi_i^{0\tau} \right) \|\theta\|^2 \leq \sum_{i=0}^{n-1} (\theta^\tau \varphi_i^0)^2 = \sum_{i=0}^{n-1} (y_{i+1} - w_{i+1})^2$$

and so by (1.6),

$$(2.2) \quad \lambda_{\min} \left( \frac{1}{n} \sum_{i=0}^{n-1} \varphi_i^0 \varphi_i^{0\tau} \right) \|\theta\|^2 \leq R_n,$$

which implies that the “self-optimality” and “persistency of excitation” cannot hold simultaneously in general for the closed-loop system resulting from regulation (see also [20, pp. 372–373] for a related discussion). Moreover, from (2.2) it is clear that the better the convergence rate of the regret  $R_n$ , the poorer the excitation of the regressor  $\varphi_i^0$  will have. This explains the familiar dilemma between estimation and control. From the following theorem, we will see which kind of excitation results we may have and how the degree of excitation of  $\{\varphi_i^0\}$  depends on  $\{y_i^*\}$  in a general asymptotically optimal tracking system.

For future reference, we list the following identifiability conditions.

(A5) The polynomials  $B(z)$  and  $A(z) - C(z)$  are coprime, and either  $\partial B(z) = q - 1$  or  $\partial(A(z) - C(z)) = \max(p, r)$ , where and hereafter  $\partial X(z)$  denotes the degree for a given polynomial  $X(z)$  in dummy variable  $z$ .

(A6) The polynomials  $A(z)$  and  $B(z)$  are coprime with  $|a_p| + |b_q| \neq 0$ .

The following theorem extends some related results in [22].

**THEOREM 2.1.** *Consider the linear model (1.1). Let the regret  $R_n$  be defined by (1.6), and the Assumptions (A1) and (A4) be satisfied. Suppose that  $\{\tau_n\}$  is a strictly increasing sequence of random integers such that  $R_{\tau_n+1} \xrightarrow[n \rightarrow \infty]{} 0$  holds on a set  $D$  of positive probability; then the following two conclusions hold:*

(i)

$$(2.3) \quad \liminf_{n \rightarrow \infty} \lambda_{\min} \left\{ \frac{1}{\tau_n} \sum_{i=0}^{\tau_n} \psi_i \psi_i^\tau \right\} > 0 \quad \text{a.s. on } D,$$

provided that (A5) holds, where

$$(2.4) \quad \psi_i = [y_i \cdots y_{i-p^*+1}, u_{i-1} \cdots u_{i-q+1}]^\tau, \quad p^* \triangleq \max(p, r).$$

(ii)

$$(2.5) \quad \liminf_{n \rightarrow \infty} \frac{\lambda_{\min} \left( \sum_{i=0}^{\tau_n} \varphi_i^0 \varphi_i^{0\tau} \right)}{\lambda_{\min}^*(\tau_n)} > 0 \quad \text{a.s. on } D,$$

provided that (A6) holds, and that

$$(2.6) \quad \sqrt{R_{\tau_n+1} + \frac{\log \log \tau_n}{\tau_n}} = o \left( \frac{\lambda_{\min}^*(\tau_n)}{\tau_n} \right) \quad \text{a.s. on } D,$$

where  $\varphi_i^0$  is defined by (1.10), and

$$(2.7) \quad \lambda_{\min}^*(n) = \lambda_{\min} \left( \sum_{i=1}^n Y_i^* Y_i^{*\tau} \right), \quad Y_i^* = [y_i^* y_{i-1}^* \cdots y_{i-p-q+1}^*]^\tau.$$

We remark that Theorem 2.1 holds irrespective of the control law structure and the minimum phase condition (A3). Following some proof ideas used in Chen and Guo [22], we preface the proof of the theorem by four simple facts, which are stated as lemmas since they will be frequently referenced in the sequel.

For any polynomial  $F(z)$ , denote its  $L_2$ -norm  $\|F(z)\|_2$  by

$$\|F(z)\|_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(e^{i\lambda})|^2 d\lambda.$$

In the sequel, we shall sometimes suppress the argument  $(z)$  for simplicity.

LEMMA 2.1. Let  $F(z)$  and  $G(z)$  be two coprime polynomials, and  $S_d$  be a set of polynomials  $(M(z), N(z))$ , defined by

$$S_d = \{(M(z), N(z)) : \|M(z)\|_2^2 + \|N(z)\|_2^2 = 1; \partial M + \partial N \leq d;$$

and either  $\partial M < \partial G$  or  $\partial N < \partial F\}$ .

Then for any integer  $d \geq 0$ ,  $\inf_{(M,N) \in S_d} \|FM + GN\|_2 > 0$ .

*Proof.* Suppose that the converse assertion were true; then it would necessarily imply that

$$(2.8) \quad FM + GN = 0$$

for some polynomial  $(M, N)$  in  $S_d$  and some integer  $d \geq 0$ . By the coprimeness of  $F$  and  $G$ , there exist polynomials  $L$  and  $H$  such that  $FL + GH = 1$ . If  $\partial M < \partial G$ , then  $G \neq 0$ , and we have by (2.8)

$$M = M(FL + GH) = L(-GN) + MGH = G(MH - LN).$$

From this it is easy to conclude that  $M = 0$ . By (2.8), we then have  $N = 0$  since  $G$  is a nonzero polynomial, and so  $\|M\|_2 + \|N\|_2 = 0$ . Similarly, if  $\partial N < \partial F$ , again we have  $\|M\|_2 + \|N\|_2 = 0$ . This contradicts with  $(M, N) \in S_d$ .  $\square$

LEMMA 2.2. *Let  $F$  and  $G$  be two coprime polynomials. For any integers  $m \geq 0, n \geq 0$ , and any sequence  $\{z_k\}$ , define for any  $k \geq 0$ ,*

$$Z_k = [F(z), zF(z) \cdots z^m F(z), G(z), zG(z) \cdots z^n G(z)]^\tau z_k.$$

If either  $m < \partial G$  or  $n < \partial F$ , then with  $c \triangleq \inf_{(M,N) \in S_{m+n}} \|FM + GN\|_2^2 > 0$ ,

$$\lambda_{\min} \left( \sum_{i=0}^k Z_i Z_i^\tau \right) \geq c \lambda_{\min} \left( \sum_{i=0}^k Z_i^0 Z_i^{0\tau} \right) \quad \forall k \geq 1,$$

where  $S_{m+n}$  is defined as in Lemma 2.1, and

$$Z_k^0 = [z_k, z_{k-1} \cdots z_{k-s}]^\tau, \quad s \triangleq \max\{m + \partial F, n + \partial G\}.$$

*Proof.* We first note that  $c > 0$  is guaranteed by Lemma 2.1. For any  $x \in \mathbb{R}^{n+m+2}$ ,  $\|x\| = 1$ , with  $x = [\alpha_0 \cdots \alpha_m, \beta_0 \cdots \beta_n]^\tau$ , set  $M(z) = \alpha_0 + \cdots + \alpha_m z^m$  and  $N(z) = \beta_0 + \cdots + \beta_n z^n$ . We have for all  $k \geq 1$ ,

$$\begin{aligned} \lambda_{\min} \left( \sum_{i=0}^k Z_i Z_i^\tau \right) &= \inf_{\|x\|=1} \sum_{i=0}^k (x^\tau Z_i)^2 \\ &= \inf_{\|x\|=1} \sum_{i=0}^k [(M(z)F(z) + N(z)G(z))z_i]^2 \\ &\geq \inf_{\|x\|=1} \|MF + NG\|_2^2 \lambda_{\min} \left( \sum_{i=0}^k Z_i^0 Z_i^{0\tau} \right) \\ &\geq \inf_{(M,N) \in S_{m+n}} \|MF + NG\|_2^2 \lambda_{\min} \left( \sum_{i=0}^k Z_i^0 Z_i^{0\tau} \right). \quad \square \end{aligned}$$

LEMMA 2.3. *Let  $x_k \in \mathbb{R}^d, (d > 0), k \geq 0$ , be a vector sequence,  $x_k = 0$ , for all  $k < 0$ , and  $F(z)$  be a polynomial with  $\|F(z)\|_2 \neq 0$ . Set  $\bar{x}_k = F(z)x_k$ . Then we have for all  $n > 0$ ,*

$$\lambda_{\min} \left( \sum_{k=0}^n x_k x_k^\tau \right) \geq \frac{1}{(\partial F + 1) \|F(z)\|_2^2} \lambda_{\min} \left( \sum_{k=0}^n \bar{x}_k \bar{x}_k^\tau \right).$$

*Proof.* Let the coefficients of  $F(z)$  be  $f_i, i = 0, \dots, \partial F$ . Then by the Schwarz inequality,

$$\begin{aligned} \lambda_{\min} \left( \sum_{k=0}^n \bar{x}_k \bar{x}_k^T \right) &= \inf_{\|x\|=1} \sum_{k=0}^n (x^T \bar{x}_k)^2 \\ &= \inf_{\|x\|=1} \sum_{k=0}^n [F(z)x^T x_k]^2 = \inf_{\|x\|=1} \sum_{k=0}^n \left[ \sum_{i=0}^{\partial F} f_i (x^T x_{k-i}) \right]^2 \\ &\leq \sum_{i=0}^{\partial F} f_i^2 \inf_{\|x\|=1} \sum_{k=0}^n \sum_{i=0}^{\partial F} (x^T x_{k-i})^2 \leq \|F(z)\|_2^2 (\partial F + 1) \lambda_{\min} \left( \sum_{k=0}^n x_k x_k^T \right). \quad \square \end{aligned}$$

We also need a simple corollary of the laws of the iterated logarithm for martingales established in Jain, Jogdeo, and Stout [21].

LEMMA 2.4. *Let  $\{w_i, \mathcal{F}_i\}$  satisfy condition (A1), and  $\{f_i, \mathcal{F}_i\}$  be an adapted sequence satisfying*

$$\sum_{i=1}^n f_i^2 = O(n) \quad \text{a.s.}, \quad f_n^2 = O(n^\delta) \quad \text{a.s.}, \quad \text{for some } \delta \in [0, 1).$$

Then as  $n \rightarrow \infty$ ,

$$\sum_{i=1}^n f_i w_{i+1} = O(\sqrt{n \log \log n}) \quad \text{a.s.}$$

*Proof.* We first consider the case  $|f_i| \geq 1$  a.s., for all  $i$ . By the martingale convergence theorem in [3] and the Kronecker lemma it follows that

$$\sum_{i=1}^n (E[w_{i+1}^2 | \mathcal{F}_i] - w_{i+1}^2) = o(n) \quad \text{a.s.}$$

So by (1.3)

$$\sum_{i=1}^n E[w_{i+1}^2 | \mathcal{F}_i] = (1 + o(1))\sigma^2 n \quad \text{a.s.}$$

Consequently, by noting  $|f_i| \geq 1$  a.s.,

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n f_i^2 E[w_{i+1}^2 | \mathcal{F}_i] \geq \sigma^2 > 0 \quad \text{a.s.}$$

Hence by applying Theorem 3.1 in [21] it is easy to see that the lemma is true.

In the general case, noting that  $f_i = [f_i + \text{sgn}(f_i)] - \text{sgn}(f_i)$ , and applying the just proved result to  $\sum_{i=1}^n [f_i + \text{sgn}(f_i)]w_{i+1}$  and  $\sum_{i=1}^n \text{sgn}(f_i)w_{i+1}$ , respectively, we see that the desired result is also true.  $\square$

We are now in a position to prove Theorem 2.1.

*Proof of Theorem 2.1.* Following Chen and Guo [22] or [16], set  $\xi_i = y_i - y_i^* - w_i$ ,  $z_i = \xi_i + y_i^*$ . Then by the assumption we have

$$(2.9) \quad \frac{1}{\tau_n} \sum_{i=0}^{\tau_n+1} \xi_i^2 \xrightarrow[n \rightarrow \infty]{} 0 \quad \text{on } D.$$

Note that

$$(2.10) \quad y_i = w_i + y_i^* + \xi_i = w_i + z_i,$$

and then by (1.1),

$$(2.11) \quad B(z)u_i = [A(z)y_{i+1} - C(z)w_{i+1}] = [A(z) - C(z)]w_{i+1} + A(z)z_{i+1}.$$

Part (i). By Lemma 2.3 we need only to consider

$$(2.12) \quad \bar{\psi}_i \triangleq B(z)\psi_i \triangleq \psi_i^w + \psi_i^z,$$

where, by (2.3), (2.9), and (2.10),

$$\begin{aligned} \psi_i^w &= [B(z), zB(z), \dots, z^{p^*-1}B(z), A(z) - C(z), \dots, z^{q-2}[A(z) - C(z)]]^\tau w_i, \\ \psi_i^z &= [B(z), zB(z), \dots, z^{p^*-1}B(z), A(z), \dots, z^{q-2}A(z)]^\tau z_i. \end{aligned}$$

By Lemma 2.2 we know that there exists  $c > 0$  such that

$$\lambda_{\min} \left\{ \sum_{i=0}^n \psi_i^w \psi_i^{w\tau} \right\} \geq c \lambda_{\min} \left\{ \sum_{i=0}^n [w_i \cdots w_{i-s}]^\tau [w_i \cdots w_{i-s}] \right\}$$

holds for all  $n > 0$ , where  $s = p^* + q - 2$ . Consequently, by (A1),

$$(2.13) \quad \liminf_{n \rightarrow \infty} \frac{1}{n} \lambda_{\min} \left( \sum_{i=0}^n \psi_i^w \psi_i^{w\tau} \right) > 0 \quad \text{a.s.}$$

Let  $\psi_i^{y^*}$  and  $\psi_i^\xi$  be defined in the same way as  $\psi_i^z$  (i.e., in the definition of  $\psi_i^z$  replace  $z_i$  by  $y_i^*$  and  $\xi_i$ , respectively). Then by the Schwarz inequality and (2.9), it is clear that

$$\frac{1}{\tau_n} \sum_{i=0}^{\tau_n} \psi_i^\xi \psi_i^{w\tau} \xrightarrow[n \rightarrow \infty]{} 0 \quad \text{on } D.$$

Also, by Assumptions (A1) and (A4),

$$\frac{1}{\tau_n} \sum_{i=0}^{\tau_n} \psi_i^{y^*} \psi_i^{w\tau} \xrightarrow[n \rightarrow \infty]{} 0 \quad \text{a.s.}$$

Hence we have

$$(2.14) \quad \frac{1}{\tau_n} \sum_{i=0}^{\tau_n} \psi_i^z \psi_i^{w\tau} \xrightarrow[n \rightarrow \infty]{} 0 \quad \text{a.s. on } D$$

Therefore, by (2.12)–(2.14) it is easy to see that

$$\liminf_{n \rightarrow \infty} \lambda_{\min} \left( \frac{1}{\tau_n} \sum_{i=0}^{\tau_n} \bar{\psi}_i \bar{\psi}_i^\tau \right) > 0 \quad \text{a.s. on } D.$$

From this and Lemma 2.3, the assertion (i) follows immediately.

Part (ii). Similarly, we consider the transformation  $\bar{\varphi}_i^0 = B(z)\varphi_i^0$ . By (2.10) and (2.11),  $\bar{\varphi}_i^0$  can be decomposed as  $\bar{\varphi}_i^0 = \varphi_i^w + \varphi_i^z$ , where

$$\begin{aligned} \varphi_i^w &= [zB(z) \cdots z^p B(z), A(z) - C(z) \cdots z^{q-1}[A(z) - C(z)], zB(z) \cdots z^r B(z)]^\tau w_{i+1}, \\ \varphi_i^z &= [zB(z), \dots, z^p B(z), A(z), \dots, z^{q-1} A(z), 0, \dots, 0]^\tau z_{i+1}. \end{aligned}$$

Let  $\varphi_i^{y^*}$  and  $\varphi_i^\xi$  be defined in the same way as for  $\varphi_i^z$ . For  $x = w, z, y^*$  and  $\xi$ , let  $\bar{\varphi}_i^x$  be the vector composed of the first  $(p + q)$  elements of  $\varphi_i^x$ . Then by (A1), (A4), and Lemma 2.4 it is easy to see that

$$(2.15) \quad \sum_{i=0}^{\tau_n} \varphi_i^w \bar{\varphi}_i^{y^* \tau} = O(\sqrt{\tau_n \log \log \tau_n}) \quad \text{a.s.}$$

Let  $x \in \mathbb{R}^{p+q+r}$  be any (random) vector,  $\|x\| = 1$ . Put  $x = (\alpha^\tau, \beta^\tau)^\tau, \alpha \in \mathbb{R}^{p+q}, \beta \in \mathbb{R}^r$ . Then by the Schwarz inequality, (2.15) and the fact that  $\bar{\varphi}_i^z = \bar{\varphi}_i^\xi + \bar{\varphi}_i^{y^*}$ ,

$$\begin{aligned} (2.16) \quad x^\tau \sum_{i=0}^{\tau_n} \bar{\varphi}_i^0 \bar{\varphi}_i^{0\tau} x &= \sum_{i=0}^{\tau_n} (x^\tau \varphi_i^w + x^\tau \varphi_i^z)^2 = \sum_{i=0}^{\tau_n} (x^\tau \varphi_i^w + \alpha^\tau \bar{\varphi}_i^z)^2 \\ &= \sum_{i=0}^{\tau_n} (x^\tau \varphi_i^w)^2 + O\left(\sqrt{\sum_{i=0}^{\tau_n} w_i^2 \sum_{i=0}^{\tau_n+1} \xi_i^2}\right) + O(\sqrt{\tau_n \log \log \tau_n}) \\ &\quad + \sum_{i=0}^{\tau_n} (\alpha^\tau \bar{\varphi}_i^z)^2 \\ &\geq \sum_{i=0}^{\tau_n} (x^\tau \varphi_i^w)^2 + O(\tau_n \sqrt{R_{\tau_n+1}}) + O(\sqrt{\tau_n \log \log \tau_n}) \\ &\quad + \sum_{i=0}^{\tau_n} (\alpha^\tau \bar{\varphi}_i^{y^*})^2 \\ &\geq \sum_{i=0}^{\tau_n} (x^\tau \varphi_i^w)^2 + (c\|\alpha\|^2 + o(1))\lambda_{\min}^*(\tau_n) \quad \text{a.s. on } D \end{aligned}$$

where for the last inequality we have used the assumption (2.6) and Lemma 2.2, and where the quantities  $c > 0$  and “ $o(1)$ ” are independent of the vector  $x$ .

Now, suppose that the converse assertion of (2.5) were true; then by Lemma 2.3 we know that there would be a set  $D_1 \subset D$  with  $P(D_1) > 0$  such that

$$\liminf_{n \rightarrow \infty} \frac{\lambda_{\min} \left( \sum_{i=0}^{\tau_n} \bar{\varphi}_i^0 \bar{\varphi}_i^{0\tau} \right)}{\lambda_{\min}^*(\tau_n)} = 0 \quad \text{on } D_1.$$

From this and (2.16) it is not difficult to find vectors  $x_n \in \mathbb{R}^{p+q+r}, \|x_n\| = 1, x_n = (\alpha_n^\tau, \beta_n^\tau)^\tau, \alpha_n \in \mathbb{R}^{p+q}$ , and a subsequence of  $\{\tau_n\}$ , which is also denoted by  $\{\tau_n\}$ , such that

$$(2.17) \quad \|\alpha_n\| \xrightarrow[n \rightarrow \infty]{} 0 \quad \text{a.s. on } D_1,$$

and that

$$\frac{1}{\tau_n} \sum_{i=0}^{\tau_n} (x_n^\tau \varphi_i^w)^2 = \frac{1}{\tau_n} \sum_{i=0}^{\tau_n} \{ \alpha_n^\tau \bar{\varphi}_i^w + \beta_n^\tau B(z) [w_i \cdots w_{i-r+1}]^\tau \}^2 \xrightarrow[n \rightarrow \infty]{} 0 \quad \text{a.s. on } D_1. \tag{2.18}$$

From (2.17) and (2.18), it is obvious that

$$\frac{1}{\tau_n} \sum_{i=0}^{\tau_n} \{ \beta_n^\tau B(z) [w_i \cdots w_{i-r+1}]^\tau \}^2 \xrightarrow[n \rightarrow \infty]{} 0 \quad \text{a.s. on } D_1.$$

Consequently, from this and (A1), it follows that

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \frac{1}{\tau_n} \sum_{i=0}^{\tau_n} \{ \beta_n^\tau B(z) [1, \dots, z^{r-1}]^\tau w_i \}^2 \\ &\geq \lim_{n \rightarrow \infty} \| \beta_n^\tau B(z) [1, \dots, z^{r-1}]^\tau \|_2^2 \lambda_{\min} \left\{ \frac{1}{\tau_n} \sum_{i=0}^{\tau_n} [w_i, \dots, w_{i-q-r+2}]^\tau [w_i, \dots, w_{i-q-r+2}] \right\} \\ &= \sigma^2 \lim_{n \rightarrow \infty} \| \beta_n^\tau B(z) [1, \dots, z^{r-1}]^\tau \|_2^2 \quad \text{a.s. on } D_1, \end{aligned}$$

which obviously implies that  $\beta_n \xrightarrow[n \rightarrow \infty]{} 0$  a.s. on  $D_1$ , and so by (2.17),  $\|x_n\| \xrightarrow[n \rightarrow \infty]{} 0$  a.s. on  $D_1$ . This contradicts with  $\|x_n\| = 1$ , and hence assertion (ii) is also true.  $\square$

Before concluding this section, we list some basic properties of the ELS algorithm here, which will be used frequently in later sections.

LEMMA 2.5. *For the system (1.1) and the ELS algorithm (1.11)–(1.14), if Conditions (A1) and (A2) hold and  $u_n$  is  $\mathcal{F}_n$ -measurable for  $n \geq 1$ , then*

- (i)  $\tilde{\theta}_{n+1}^\tau P_{n+1}^{-1} \tilde{\theta}_{n+1} = O(\log r_n) \quad \text{a.s.},$
- (ii)  $\sum_{i=1}^{n+1} \|\hat{w}_i - w_i\|^2 = O(\log r_n) \quad \text{a.s.},$
- (iii)  $\sum_{i=1}^n \frac{\|\tilde{\theta}_i^\tau \varphi_i\|^2}{1 + \varphi_i^\tau P_i \varphi_i} = O(\log r_n) \quad \text{a.s.},$

where  $\tilde{\theta}_n \triangleq \theta - \theta_n$ , and  $r_n$  is defined by (1.18).

Except (i), this lemma is the same as Lemma 1 in [19], but (i) is actually also established in the proof of that lemma.

COROLLARY 2.1. *Under the same conditions and notations as in Lemma 2.5, the following property holds:*

$$\|\tilde{\theta}_{n+1}\|^2 + \|\hat{w}_{n+1} - w_{n+1}\|^2 + \frac{\|\tilde{\theta}_n^\tau \varphi_n\|^2}{1 + \varphi_n^\tau P_n \varphi_n} = O(\log r_n) \quad \text{a.s.} \tag{2.19}$$

*Proof.* We need only to note that by (1.12) and the choice of the initial condition,  $P_{n+1}^{-1} \geq P_0^{-1} > 0$ , for all  $n \geq 0$ .  $\square$

**3. Adaptive minimum variance control (with  $b_1$  fixed).** Throughout this section we assume that the “high-frequency” gain  $b_1$  in the model (1.1) is known. The main consideration behind this is that results obtained in this case are relatively complete, which can indicate the greatest expected achievement in the general case.

Similar to (1.8)–(1.10), we rewrite (1.1) in the regression form

$$(3.1) \quad y_{n+1} - b_1 u_n = \theta^\tau \varphi_n^0 + w_{n+1}, \quad n \geq 0,$$

but here  $\theta$  and  $\varphi_n^0$  should be defined as

$$(3.2) \quad \theta = [-a_1 \dots -a_p \quad b_2 \dots b_q \quad c_1 \dots c_r]^\tau,$$

$$(3.3) \quad \varphi_n^0 = [y_n \dots y_{n-p+1}, u_{n-1} \dots u_{n-q+1}, w_n \dots w_{n-r+1}]^\tau.$$

The standard ELS algorithm for estimating  $\theta$  is as follows:

$$(3.4) \quad \theta_{n+1} = \theta_n + a_n P_n \varphi_n (y_{n+1} - b_1 u_n - \theta_n^\tau \varphi_n),$$

$$(3.5) \quad P_{n+1} = P_n - a_n P_n \varphi_n \varphi_n^\tau P_n, \quad a_n = (1 + \varphi_n^\tau P_n \varphi_n)^{-1},$$

$$(3.6) \quad \varphi_n = [y_n \dots y_{n-p+1}, u_{n-1} \dots u_{n-q+1}, \hat{w}_n \dots \hat{w}_{n-r+1}]^\tau,$$

$$(3.7) \quad \hat{w}_n = y_n - b_1 u_{n-1} - \theta_n^\tau \varphi_{n-1}, n \geq 0, \hat{w}_n = 0, n < 0,$$

with arbitrary initial values  $\theta_0, \varphi_0$ , and  $P_0 > 0$ .

We note that Lemma 2.5 and Corollary 2.1 also hold for the present algorithm, and in what follows we shall use them directly without explanations.

The “certainty equivalent” minimum variance adaptive control is defined by

$$(3.8) \quad u_n = b_1^{-1} (y_{n+1}^* - \theta_n^\tau \varphi_n).$$

We first treat the white noise case.

**THEOREM 3.1.** *Consider the system (1.1) with  $r = 0$  and  $E[w_{n+1}^2 | \mathcal{F}_n] = \sigma^2 > 0$ , a.s. for all  $n \geq 0$ . Suppose that (A1) and (A3)–(A5) hold. If the control law (3.4)–(3.8) is applied, then the closed-loop system has the following properties:*

$$(3.9) \quad \lim_{n \rightarrow \infty} \left( \frac{n}{\log n} \right) R_n = (p + q - 1) \sigma^2 \quad \text{a.s.},$$

and

$$(3.10) \quad \|\theta_n - \theta\|^2 = O\left(\frac{\log \log n}{n}\right) \quad \text{a.s.}, \quad \text{as } n \rightarrow \infty,$$

where  $R_n$  is defined by (1.6), and  $\theta$  is given by (3.2) with  $r = 0$ .

*Proof.* By Theorem 1 of Guo and Chen [19] we know that  $R_n \xrightarrow[n \rightarrow \infty]{} 0$  a.s., and that

$$(3.11) \quad \sum_{i=0}^n \|\varphi_i\|^2 = O(n) \quad \text{a.s.}$$

Hence by Theorem 2.1 (i), we have the following persistency of excitation property:

$$(3.12) \quad \liminf_{n \rightarrow \infty} \lambda_{\min} \left( \frac{1}{n} \sum_{i=0}^n \varphi_i \varphi_i^\tau \right) > 0 \quad \text{a.s.}$$

Also, by combining Lemma 2, (2.9), and Theorem 1 of Guo and Chen [19] we know that

$$(3.13) \quad \|\varphi_n\|^2 = O(n^\delta), \quad \text{a.s. } \forall \delta \in \left( \frac{2}{\beta}, 1 \right),$$

where  $\beta$  is defined in (1.2). Hence, by (3.12) and (3.13),

$$(3.14) \quad \varphi_n^\tau P_{n+1} \varphi_n \xrightarrow{n \rightarrow \infty} 0 \quad \text{a.s.}$$

By (3.11), (3.12), and (3.14) we know that Theorem 3 of Wei [4] is applicable (there is a slight difference between the LS estimates defined there and here due to initial conditions, but that is not essential since (3.12) has been established), and hence we have

$$(3.15) \quad \sum_{i=0}^{n-1} (\theta^\tau \varphi_i - \theta_i^\tau \varphi_i)^2 \sim \sigma^2 \log \det \left( \sum_{i=0}^{n-1} \varphi_i \varphi_i^\tau \right) \quad \text{a.s.}$$

But by (3.11) and (3.12) it is easy to verify that  $\log \det \left( \sum_{i=1}^{n-1} \varphi_i \varphi_i^\tau \right) \sim (p + q - 1) \log n$ . Hence, by combining (1.6), (3.1), (3.8), and (3.15) we see that (3.9) holds.

As for the second assertion of the theorem, by (3.4) and (3.5) we can express the estimation error as

$$(3.16) \quad \theta_n - \theta = P_n P_0^{-1} (\theta_0 - \theta) + P_n \sum_{i=0}^{n-1} \varphi_i w_{i+1}.$$

By (3.11), (3.13), and Lemma 2.4, we know that

$$(3.17) \quad \left\| \sum_{i=0}^{n-1} \varphi_i w_{i+1} \right\| = O(\sqrt{n \log \log n}) \quad \text{a.s.}$$

Finally, combining (3.12), (3.16), and (3.17) it is easy to see that (3.10) holds.  $\square$

*Remark 3.1.* The property (3.9) asserts, among other things, that  $O(\log n/n)$  is the best convergence rate for the regret  $R_n$  generated by LS-based adaptive control. The convergence rate  $O(\log \log n/n)$  in (3.10) is also obviously the best possible for the estimation error, since it is the same rate as that in the laws of the iterated logarithm. In a Bayesian framework, assuming that  $\{w_i\}$  is *i.i.d.* with a Gaussian distribution  $N(0, \sigma^2)$  and that  $\theta$  has a certain truncated Gaussian prior distribution  $\pi$ , Lai [23] showed that under some stability conditions on the system and some regularity conditions on the input sequence  $\{u_n\}$ , the order  $(p + q - 1)\sigma^2(1 + o(1)) \log n/n$  is a *lower bound* to the expected regret  $E_\pi[R_n]$  in the regulation problem. According to Lai's definition in [23, p. 37], the control law of Theorem 3.1 is "asymptotically efficient". It is also interesting to note that when the system orders  $p$  and  $q$  are increasing with the time (or data size)  $n$ , similar results as (3.15) are also obtainable (see, Huang and Guo [1]).

Next, we consider the general colored noise case  $r > 0$ . Let us write  $\theta_n$  defined by (3.4)–(3.7) in its component form:

$$(3.18) \quad \theta_n = [-a_{1n}, \dots, -a_{pn}, b_{2n}, \dots, b_{qn}, c_{1n}, \dots, c_{rn}]^\tau,$$

and set

$$(3.19) \quad \theta_n^* = [c_{1n} - a_{1n}, \dots, c_{p^*n} - a_{p^*n}, b_{2n}, \dots, b_{qn}]^T, \quad p^* = \max(p, r),$$

where by definition  $c_{in} = a_{jn} = 0$ , for  $i > r, j > p$ .

Similarly, denote  $(c_i = a_j = 0$ , for all  $i > r, j > p)$ ,

$$(3.20) \quad \theta^* = [c_1 - a_1, \dots, c_{p^*} - a_{p^*}, b_2, \dots, b_q]^T.$$

It is easy to see that (cf. [2, p. 122]) for the regulation problem  $y_i^* \equiv 0$  with  $b_1$  known, to construct the nonadaptive (asymptotically) optimal control law, it is sufficient to know only  $\theta^*$ , and hence  $\theta^*$  may be regarded as the “true parameter.”

**THEOREM 3.2.** *Let (A1)–(A4) hold, and let the adaptive control law (3.4)–(3.8) be applied to the system (1.1).*

(i) *For the regulation problem  $y_i^* \equiv 0$ , if (A5) holds, then*

$$(3.21) \quad \|\theta_n^* - \theta^*\|^2 + R_n = O\left(\frac{d_n}{n^{1-\varepsilon}}\right) \quad a.s. \forall \varepsilon > 0,$$

where  $d_n$  is defined as in (1.22), and  $\theta_n^*$  and  $\theta^*$  are respectively defined by (3.19) and (3.20).

(ii) *For the general tracking problem, if (A6) holds and  $\{y_i^*\}$  satisfies*

$$(3.22) \quad n^{\frac{1+\delta}{2}} \sqrt{d_n} = O(\lambda_{\min}^*(n)) \quad a.s. \text{ for some } \delta > 0,$$

where  $d_n$  and  $\lambda_{\min}^*(n)$  are defined in (1.22) and (2.7), respectively, then as  $n \rightarrow \infty$ ,

$$(3.23) \quad R_n = O\left(\frac{\log n}{n}\right) a.s., \quad \|\theta_n - \theta\|^2 = O\left(\frac{\log n}{\lambda_{\min}^*(n)}\right) a.s.,$$

where  $\theta_n$  and  $\theta$  are respectively given by (3.4) and (3.2).

*Proof.* (i) By Theorem 1 in [19] we know that  $R_n = O(d_n/n^{1-\varepsilon})$ , a.s., for all  $\varepsilon > 0$ . So for (3.21) we need only to consider the convergence rate of the estimation error. By Lemma 2.5 (i) we know that

$$(3.24) \quad \tilde{\theta}_{n+1}^T P_{n+1}^{-1} \tilde{\theta}_{n+1} = O(\log r_n) = O(\log n), \text{ a.s.}$$

where  $\tilde{\theta}_{n+1} = \theta_{n+1} - \theta$ . By (3.19) and (3.20) and the fact that  $P_{n+1}^{-1} = \sum_{i=0}^n \varphi_i \varphi_i^T + P_0^{-1}$ , we can rewrite (3.24) as

$$\sum_{i=0}^n \left[ \psi_i^T \tilde{\theta}_{n+1}^* + \sum_{j=1}^{p^*} (c_{jn+1} - c_j)(\hat{w}_{i-j+1} - y_{i-j+1}) \right]^2 + \tilde{\theta}_{n+1}^T P_0^{-1} \tilde{\theta}_{n+1} = O(\log n), \text{ a.s.,}$$

where  $\tilde{\theta}_{n+1}^* = \theta_{n+1}^* - \theta^*$ , and  $\psi_i$  and  $p^*$  are defined by (2.4). By Lemma 2.5(ii), Corollary 2.1 and the fact that

$$\sum_{i=0}^n (y_i - w_i)^2 = O(n^\varepsilon d_n) \quad a.s., \forall \varepsilon > 0,$$

it is easy to see that

$$\sum_{i=0}^n \left[ \sum_{j=1}^{p^*} (c_{jn+1} - c_j)(\hat{w}_{i-j+1} - y_{i-j+1}) \right]^2 = O(n^\varepsilon d_n) \quad a.s. \forall \varepsilon > 0.$$

Therefore, we have

$$\sum_{i=0}^n (\psi_i^\tau \tilde{\theta}_{n+1}^*)^2 = O(n^\epsilon d_n) \quad \text{a.s., } \forall \epsilon > 0.$$

From this and Theorem 2.1 (i), we obtain  $\|\tilde{\theta}_{n+1}^*\|^2 = O(d_n/n^{1-\epsilon})$ , a.s.. This proves assertion (i).

(ii). Again, by [19],  $R_n = O(\frac{d_n}{n^{1-\epsilon}})$ , a.s.,  $\forall \epsilon > 0$ . Hence, by (3.22) we know that Theorem 2.1 (ii) is applicable, and so we have

$$\liminf_{n \rightarrow \infty} \frac{\lambda_{\min} \left( \sum_{i=0}^n \varphi_i^0 \varphi_i^{0\tau} \right)}{\lambda_{\min}^*(n)} > 0 \quad \text{a.s.}$$

Consequently, by Lemma 2.5 (ii) and the fact that

$$\lambda_{\min} \left( \sum_{i=0}^n \varphi_i^0 \varphi_i^{0\tau} \right) \leq 2\lambda_{\min} \left( \sum_{i=0}^n \varphi_i \varphi_i^\tau \right) + 2 \sum_{i=0}^n \|\varphi_i^0 - \varphi_i\|^2,$$

we have

$$(3.25) \quad \liminf_{n \rightarrow \infty} \frac{\lambda_{\min} \left( \sum_{i=0}^n \varphi_i \varphi_i^\tau \right)}{\lambda_{\min}^*(n)} > 0 \quad \text{a.s.,}$$

which in conjunction with (3.24) yields the second assertion in (3.23). By (3.22), (3.25) and Lemma 2 in [19] it is not difficult to see that  $\varphi_n^\tau P_n \varphi_n \xrightarrow[n \rightarrow \infty]{} 0$ . Therefore, by Lemma

2.5(iii),  $\sum_{i=0}^n \|\tilde{\theta}_i^\tau \varphi_i\|^2 = O(\log n)$ , and hence the first assertion in (3.23) is also true.  $\square$

*Remark 3.2.* For the regulation problem, the one degree of freedom identifiability problem as shown in Becker, Kumar, and Wei [24] does not occur in Theorem 3.2, since  $b_1$  is not estimated. For the general tracking problem, it is clear that in Theorem 3.2,  $\{y_i^*\}$  is not necessarily required to be “sufficiently rich” or “persistently exciting.” Condition (3.22) is considerably weaker than the corresponding nonpersistence of excitation condition used in [22] and [16] for the SG-based algorithm. It would be of interest to establish similar results for a lower-dimensional ELS-based adaptive controller when  $\{y_i^*\}$  is generated by a homogeneous finite-order linear difference equation  $H(z)y_i^* = 0$ , as was done by Kumar and Praly [25] for the SG-based algorithm.

**4. Adaptive minimum variance control (the general case).** In the general case where  $b_1$  is not available, the analysis becomes much more complicated. Throughout this section, we assume that  $\{\theta_n\}$  is generated by the ELS algorithm (1.11)–(1.14).

First, the minimum variance adaptive control law defined from (1.21) can be explicitly written as

$$(4.1) \quad u_n = \frac{1}{b_1(n)} \{y_{n+1}^* + (b_1(n)u_n - \theta_n^\tau \varphi_n)\},$$

provided that  $b_1(n) \neq 0$  a.s., where  $b_1(n)$  is the ELS estimate for  $b_1$  given by  $\theta_n$ .

When (4.1) is applied, the first problem is that the set  $\{b_1(n) = 0\}$  may have a positive probability, which is known as the zero divisor problem in stochastic adaptive control (cf. Meyn and Caines [26]). There are at least three ways to deal with this problem.

(a) Guarantee  $P\{b_1(n) = 0\} = 0$  by assuming that all finite-dimensional distributions of  $\{w_n\}$  are absolutely continuous with respect to Lebesgue measure (see, [26] or [5, pp. 778–782]). The absolute continuity assumption can be weakened to continuity only if  $\{w_n\}$  is an independent sequence (cf. [16]).

(b) Guarantee  $P\{b_1(n) = 0\} = 0$  by adding an independent random sequence with continuous distributions to the input signal. Such a sequence is preferably decaying with the time so that it does not upset the control performance (cf. [22]).

(c) Replace  $b_1(n)$  appearing in the denominator of (4.1) by a quantity (say  $\hat{b}_1(n)$ ), which is close to  $b_1(n)$  but does not equal to zero (see, e.g., (1.23) or [19]).

In the sequel, whenever the control law (4.1) is concerned we always assume that  $P\{b_1(n) = 0\} = 0$ . The following lemma plays a key role in this section.

LEMMA 4.1. *For the system (1.1) assume that (A1)–(A4) are satisfied. At each time instant  $n$ , let the control law  $u_n$  be defined from the following equation:*

$$(4.2) \quad y_{n+1}^* = \theta_n^\tau \varphi_n + \Delta \hat{b}_{1n} u_n,$$

where  $\{\theta_n\}$  is given by the ELS algorithm (1.11)–(1.14), and  $\Delta \hat{b}_{1n} \in \mathcal{F}_n$  is such that either  $\Delta \hat{b}_{1n} \equiv 0, \forall n$  or  $\Delta \hat{b}_{1n} \xrightarrow[n \rightarrow \infty]{} 0$  a.s. Then for any strictly increasing random sequence  $\{\tau_n\}$  satisfying

$$(4.3) \quad \inf_n |b_1(\tau_n + 1) - b_1| > 0 \quad \text{a.s. on } D,$$

with  $P(D) > 0$ , and with  $b_1(n)$  being the component of  $\theta_n$  estimating  $b_1$ , the following properties hold as  $n \rightarrow \infty$ :

$$(4.4) \quad \sup_{k \leq \tau_n} \|\varphi_k\|^2 = O(\tau_n^\varepsilon d_{\tau_n}) \quad \text{a.s. on } D, \forall \varepsilon > 0,$$

and

$$(4.5) \quad r_{\tau_n} = O(\tau_n) \quad \text{a.s. on } D,$$

where  $r_n$  and  $d_n$  are defined by (1.18) and (1.22), respectively.

*Proof.* Before starting the proof, we remark that the case  $\Delta \hat{b}_{1n} \equiv 0$  corresponds to the control law (4.1), while the case  $\Delta \hat{b}_{1n} \neq 0$  corresponds to a (slight) modification of  $b_1(n)$ .

We first prove (4.4). By (1.9) and (4.2) we have with  $\tilde{\theta}_k \triangleq \theta - \theta_k$ ,

$$(4.6) \quad \begin{aligned} y_{k+1} &= \theta^\tau \varphi_k + \theta^\tau (\varphi_k^0 - \varphi_k) + w_{k+1} \\ &= \tilde{\theta}_k^\tau \varphi_k + y_{k+1}^* - \Delta \hat{b}_{1k} u_k + \theta^\tau (\varphi_k^0 - \varphi_k) + w_{k+1}. \end{aligned}$$

Following Guo and Chen [19], denoting  $\delta_k = \text{tr}(P_k - P_{k+1})$ ,  $\alpha_k = \|\tilde{\theta}_k^\tau \varphi_k\|^2 / (1 + \varphi_k^\tau P_k \varphi_k)$ , and using Corollary 2.1 and the fact that  $\varphi_k^\tau P_{k+1} \varphi_k \leq 1$ , we have by (4.6)

$$(4.7) \quad \begin{aligned} y_{k+1}^2 &\leq 3\|\tilde{\theta}_k^\tau \varphi_k\|^2 + 3(\Delta \hat{b}_{1k})^2 u_k^2 + O(\log r_k + d_k) \\ &\leq 3\alpha_k \{1 + \varphi_k^\tau P_{k+1} \varphi_k + \varphi_k^\tau (P_k - P_{k+1}) \varphi_k\} \\ &\quad + 3(\Delta \hat{b}_{1k})^2 u_k^2 + O(\log r_k + d_k) \\ &\leq 3\alpha_k \{2 + \delta_k \|\varphi_k\|^2\} + 3(\Delta \hat{b}_{1k})^2 u_k^2 + O(\log r_k + d_k) \\ &= 3\alpha_k \delta_k \|\varphi_k\|^2 + 3(\Delta \hat{b}_{1k})^2 u_k^2 + O(\log r_k + d_k). \end{aligned}$$

By the stability of  $B(z)$  and (1.1) there exists a constant  $\lambda \in (0, 1)$  such that

$$(4.8) \quad u_k^2 = O\left(\sum_{i=0}^{k+1} \lambda^{k-i} y_i^2\right) + O\left(\sum_{i=0}^{k+1} \lambda^{k-i} w_i^2\right).$$

Consequently,

$$(4.9) \quad \begin{aligned} [\|\varphi_k\|^2 - u_k^2] &= O\left(\sum_{i=0}^k \lambda^{k-i} y_i^2\right) + O\left(\sum_{i=0}^r (\hat{w}_{k-i})^2\right) + O\left(\sum_{i=0}^k \lambda^{k-i} w_i^2\right) \\ &= O\left(\sum_{i=0}^k \lambda^{k-i} y_i^2\right) + O(\log r_k + d_k) \quad \text{a.s.}, \end{aligned}$$

where for the last relationship we have used Lemma 2.5 (ii).

Note that  $P_{n+1}^{-1} = \sum_{i=0}^n \varphi_i \varphi_i^T + P_0^{-1}$ , and we have by Lemma 2.5 (i),

$$(4.10) \quad \sum_{i=0}^n \|\tilde{\theta}_{n+1}^T \varphi_i\|^2 = O(\log r_n) \quad \text{a.s.},$$

and consequently,

$$(4.11) \quad \max_{i \leq n} \|\tilde{\theta}_{n+1}^T \varphi_i\|^2 = O(\log r_n) \quad \text{a.s.}$$

For simplicity of statements, we shall omit the phrase ‘‘a.s. on  $D$ ’’ in the remainder of the proof, and unless otherwise stated all relationships hold on  $D$  with a possible exception set of probability zero. Denote  $\tilde{b}_1(\tau_n + 1) = b_1 - b_1(\tau_n + 1)$ , we have by (4.3),  $\inf_n |\tilde{b}_1(\tau_n + 1)| > 0$ . Consequently, by (4.11) and the fact that  $\|\tilde{\theta}_{n+1}\|^2 = O(\log r_n)$  a.s., we have for all  $k \leq \tau_n$  and all  $n \geq 1$ ,

$$(4.12) \quad \begin{aligned} u_k^2 &= \frac{1}{(\tilde{b}_1(\tau_n + 1))^2} \{\tilde{b}_1(\tau_n + 1) u_k\}^2 \\ &= \frac{1}{(\tilde{b}_1(\tau_n + 1))^2} \{[\tilde{\theta}_{\tau_n+1}^T \varphi_k - \tilde{b}_1(\tau_n + 1) u_k] - \tilde{\theta}_{\tau_n+1}^T \varphi_k\}^2 \\ &\leq \frac{2}{(\tilde{b}_1(\tau_n + 1))^2} \{\|\tilde{\theta}_{\tau_n+1}^T \varphi_k - \tilde{b}_1(\tau_n + 1) u_k\|^2 + \|\tilde{\theta}_{\tau_n+1}^T \varphi_k\|^2\} \\ &= O((\log r_{\tau_n}) [\|\varphi_k\|^2 - u_k^2]) + O(\log r_{\tau_n}) \\ &= O((\log r_{\tau_n}) \sum_{i=0}^k \lambda^{k-i} y_i^2) + O(\log^2 r_{\tau_n} + d_{\tau_n} \log r_{\tau_n}), \end{aligned}$$

where for the last relationship we have used (4.9), and where and hereafter the ‘‘ $O$ ’’ constant depends neither on  $k$  nor on  $n$ .

Combining (4.9) and (4.12) we get for all  $k \leq \tau_n$ ,

$$(4.13) \quad \|\varphi_k\|^2 = O([\log r_{\tau_n}] \sum_{i=0}^k \lambda^{k-i} y_i^2) + O(\log^2 r_{\tau_n} + d_{\tau_n} \log r_{\tau_n}).$$

Substituting (4.8) and (4.13) into (4.7) and noticing  $\Delta \hat{b}_{1n} \xrightarrow[n \rightarrow \infty]{} 0$ , it is easy to see that for all  $k \leq \tau_n$ , and all large  $n$ ,

$$(4.14) \quad y_{k+1}^2 = O\left(\alpha_k \delta_k \log r_{\tau_n} \sum_{i=0}^k \lambda^{k-i} y_i^2\right) + o\left(\sum_{i=0}^k \lambda^{k-i} y_i^2\right) + O(\log^3 r_{\tau_n} + d_{\tau_n} \log^2 r_{\tau_n}).$$

Now, following [19] we set  $L_k = \sum_{i=0}^k \lambda^{k-i} y_i^2$ . Then by (4.14) there are constants  $\delta > 0$  and  $c > 0$  such that  $(1 + \delta)\lambda < 1$  and that

$$y_{k+1}^2 \leq \lambda[(1 + \delta)c\alpha_k \delta_k (\log r_{\tau_n}) + \delta]L_k + O(\log^3 r_{\tau_n} + d_{\tau_n} \log^2 r_{\tau_n})$$

holds for all suitably large  $n$  and all  $k \leq \tau_n$ . Consequently, by denoting  $\gamma \triangleq (1 + \delta)\lambda < 1$ , we obtain for large  $n$  and for all  $k \leq \tau_n$ ,

$$L_{k+1} = \lambda L_k + y_{k+1}^2 \leq \gamma(1 + c\alpha_k \delta_k \log r_{\tau_n})L_k + O(\log^3 r_{\tau_n} + d_{\tau_n} \log^2 r_{\tau_n}).$$

Hence, iterating this inequality  $k$  times we get for all large  $n$  and for all  $k \leq \tau_n$ ,

$$(4.15) \quad \begin{aligned} L_{k+1} &\leq \gamma^{k+1} \prod_{i=0}^k (1 + c\alpha_i \delta_i \log r_{\tau_n}) L_0 \\ &+ O\left(\sum_{i=0}^k \gamma^{k-i} \prod_{j=i+1}^k (1 + c\alpha_j \delta_j \log r_{\tau_n}) [\log^3 r_{\tau_n} + d_{\tau_n} \log^2 r_{\tau_n}]\right). \end{aligned}$$

By Lemma 2.5 (iii) and the convergency of the series  $\sum_{i=1}^{\infty} \delta_i$ , we know that for any small  $\varepsilon > 0$ , there exists  $i > 0$  large enough such that

$$\varepsilon^2 \sum_{j=i+1}^k \alpha_j \leq \frac{\varepsilon}{2} \log r_k, \quad \varepsilon^{-2} c \sum_{j=i+1}^{\infty} \delta_j < \frac{\varepsilon}{2}.$$

Hence we have for all  $i \leq k \leq \tau_n$ ,

$$(4.16) \quad \begin{aligned} \prod_{j=i+1}^k (1 + c\alpha_j \delta_j \log r_{\tau_n}) &\leq \prod_{j=i+1}^k (1 + \varepsilon^2 \alpha_j) \prod_{j=i+1}^k (1 + c\varepsilon^{-2} \delta_j \log r_{\tau_n}) \\ &\leq \exp\left\{\varepsilon^2 \sum_{j=i+1}^k \alpha_j + \sum_{j=i+1}^k c\varepsilon^{-2} \delta_j \log r_{\tau_n}\right\} \leq r_{\tau_n}^{\varepsilon}. \end{aligned}$$

Substituting this into (4.15), it is easy to conclude that for large  $n$ ,

$$(4.17) \quad L_{k+1} = O(r_{\tau_n}^{\varepsilon} d_{\tau_n}), \quad \forall k \leq \tau_n, \quad \forall \varepsilon > 0.$$

Substituting this into (4.13) we know that  $\sup_{k \leq \tau_n} \|\varphi_k\|^2 = O(r_{\tau_n}^{\varepsilon} d_{\tau_n})$  for all  $\varepsilon > 0$ , and hence (4.4) will be true if (4.5) is proved.

We now prove (4.5). By (4.17) and the assumption  $\Delta \hat{b}_{1n} \xrightarrow[n \rightarrow \infty]{} 0$ , it follows from (4.6) that

$$\begin{aligned} \sum_{k=1}^{\tau_n} y_{k+1}^2 &= O\left(\sum_{k=1}^{\tau_n} \alpha_k (1 + \varphi_k^T P_k \varphi_k)\right) + o\left(\sum_{k=1}^{\tau_n} u_k^2\right) + O(\log r_{\tau_n} + \tau_n) \\ &= O\left(r_{\tau_n}^{\varepsilon} d_{\tau_n} \sum_{k=1}^{\tau_n} \alpha_k\right) + o(r_{\tau_n}) + O(\log r_{\tau_n} + \tau_n), \quad \forall \varepsilon > 0. \end{aligned}$$

From this, Lemma 2.5, and (4.8), it is easy to see that

$$(4.18) \quad r_{\tau_n} = O(r_{\tau_n}^{\varepsilon} d_{\tau_n} \log r_{\tau_n}) + o(r_{\tau_n}) + O(\tau_n), \quad \forall \varepsilon > 0$$

But as noted in ([19, p. 804]),  $d_k$  can be taken as  $d_k = k^\delta$  for all  $\delta \in (\frac{2}{\beta}, 1)$ . Hence, from (4.18) it is easy to conclude (4.5), and hence the proof is complete.  $\square$

Let  $D_1$  be a set defined by

$$(4.19) \quad D_1 = \left\{ w : \liminf_{n \rightarrow \infty} |b_1(n)| \neq 0 \right\},$$

where  $b_1(n)$  denotes the component of  $\theta_n$  estimating  $b_1$ .

For any constant  $a \in (0, |b_1|)$ , define a sequence  $\{\tau_n\}$  recursively by

$$(4.20) \quad \tau_n = \inf\{k > \tau_{n-1} : |b_1(k+1)| < a\}, \quad \tau_0 = 0, \quad n \geq 1.$$

Note that (A3) implies  $b_1 \neq 0$ , and so the interval  $(0, |b_1|)$  is not empty.

On the complement set of  $D_1, D_1^c$ , it is obvious that  $\tau_n < \infty$  for all  $n \geq 1$ . Hence, if we set

$$(4.21) \quad \sigma_n = \begin{cases} n, & w \in D_1, \\ \tau_n, & w \in D_1^c, \end{cases}$$

then  $\sigma_n < \infty$  a.s. for all  $n$ , and  $\sigma_n \xrightarrow[n \rightarrow \infty]{} \infty$  a.s..

**THEOREM 4.1.** *For the system (1.1) assume that (A1)–(A4) are satisfied, and that the control law defined by (4.1) is applied. Then the following hold:*

(i) *For the sequence  $\{\sigma_n\}$  defined by (4.19)–(4.21), as  $n \rightarrow \infty$*

$$r_{\sigma_n} = O(\sigma_n) \text{ a.s. and } R_{\sigma_{n+1}} = O\left(\frac{d_{\sigma_n}}{\sigma_n^{1-\varepsilon}}\right) \text{ a.s., } \forall \varepsilon > 0,$$

where  $R_n, r_n$  and  $d_n$  are defined by (1.6), (1.18), and (1.22), respectively.

(ii)

$$R_n = O\left(\frac{1}{n^{1-\delta}}\right) \text{ a.s. on } D, \quad \forall \delta \in \left(\frac{2}{\beta}, 1\right),$$

where  $\beta$  is defined in (1.2) and  $D = D_1 \cup D_2$  with  $D_1$  defined by (4.19) and  $D_2$  defined by

$$D_2 = \left\{ w \in D_1^c : \sup_n \frac{\tau_{n+1}}{\tau_n} < \infty \right\};$$

here  $\{\tau_n\}$  is defined by (4.20).

*Proof.* (i). On the set  $D_1$ , by a completely similar argument as that used for Theorem 2 in [19], it is known that  $R_n = O(d_n/n^{1-\varepsilon})$  a.s. on  $D_1$ , for all  $\varepsilon > 0$ . So we need only to consider the complement set  $D_1^c$ . By the definition of  $\tau_n$  we have

$$\inf_n |b_1(\tau_n + 1) - b_1| \geq |b_1| - a > 0 \quad \text{on } D_1^c.$$

Therefore, by Lemma 4.1 we know that  $r_{\tau_n} = O(\tau_n)$  and  $\sup_{i \leq \tau_n} \|\varphi_i\|^2 = O(\tau_n^\varepsilon d_{\tau_n})$  a.s. on  $D_1^c$ . Hence, by (4.6) (with  $\Delta \hat{b}_{1n} \equiv 0$ ) and Lemma 2.5, we have

$$(4.22) \quad \begin{aligned} R_{\tau_{n+1}} &= \frac{1}{\tau_n + 1} \sum_{i=0}^{\tau_n} (y_{i+1} - y_{i+1}^* - w_{i+1})^2 \\ &= O\left(\frac{1}{\tau_n} \sum_{i=0}^{\tau_n} \|\tilde{\theta}_i^\tau \varphi_i\|^2\right) + O\left(\frac{1}{\tau_n} \sum_{i=0}^{\tau_n} \|\varphi_i - \varphi_i^0\|^2\right) \\ &= O\left(\frac{1}{\tau_n} \sum_{i=0}^{\tau_n} \frac{\|\tilde{\theta}_i^\tau \varphi_i\|^2}{1 + \varphi_i^\tau P_i \varphi_i} (1 + \|\varphi_i\|^2)\right) + O\left(\frac{\log r_{\tau_n}}{\tau_n}\right) \\ &= O\left(\frac{d_{\tau_n} \log \tau_n}{\tau_n^{1-\varepsilon}}\right) \text{ a.s. on } D_1^c, \forall \varepsilon > 0. \end{aligned}$$

Hence the conclusion (i) holds.

(ii) As is just mentioned above (ii) holds on  $D_1$ , since  $d_n$  can be taken as  $n^\delta$  for all  $\delta \in (2/\beta, 1)$ . Hence we need only to consider the set  $D_2$ . By the definition of  $\{\tau_n\}$ , we know that on  $D_2, \tau_n \rightarrow \infty, \sup_n(\tau_{n+1}/\tau_n) < \infty$ , and by (4.22),  $R_{\tau_n} = O(1/\tau_n^{1-\delta})$  a.s. on  $D_2$  for all  $\delta \in (2/\beta, 1)$ . Consequently,

$$\begin{aligned} \sup_n [n^{1-\delta} R_n] &= \sup_k \sup_{n \in [\tau_k, \tau_{k+1}]} [n^{1-\delta} R_n] \\ &\leq \sup_k \frac{\tau_{k+1}}{\tau_k} [\tau_{k+1}^{1-\delta} R_{\tau_{k+1}}] < \infty \quad \text{a.s. on } D_2. \end{aligned}$$

Hence assertion (ii) is also true.  $\square$

*Remark 4.1.* From Guo and Chen [19], we know that under conditions of Theorem 4.1, if  $\liminf_{n \rightarrow \infty} |b_1(n)| \neq 0$  a.s., then  $R_n \rightarrow 0$  a.s. Theorem 4.1 asserts, among other things, that if  $\lim_{n \rightarrow \infty} |b_1(n)| = 0$  a.s. does hold, then since  $P(D_2) = 1$  and again we have  $R_n \rightarrow 0$  a.s.. This result is rather interesting since  $b_1(n)$  appears as the divisor in the control law (4.1), and small  $b_1(n)$  seems to yield large input signal  $u_n$  (but actually does not). The key idea behind the proof of Theorem 4.1 (or Lemma 4.1) is as follows: if  $\lim_{n \rightarrow \infty} |b_1(n)| = 0$ , then  $|\tilde{b}_1(n+1)| \geq |b_1|/2 > 0$  for all suitably large  $n$ . Thus, for each fixed large  $n$ , and for all  $i \leq n, u_i$  will have a significant contribution to  $\tilde{\theta}_{n+1}^\tau \varphi_i$  if it is not small. But by (4.11) we know that  $\|\tilde{\theta}_{n+1}^\tau \varphi_i\|^2 = O(\log r_n)$  a.s. for all  $i \leq n$ . Hence, for all  $i \leq n, u_i^2$  will be dominated by a ‘‘linear combination’’ of  $\{y_i^2, \dots, y_{i-p+1}^2, u_{i-1}^2 \dots u_{i-q+1}^2, \hat{w}_i^2 \dots \hat{w}_{i-r+1}^2\}$ , and thus we can successfully sidestep the difficult ‘‘small divisor’’ problem in the analysis. Certainly, in this approach, it would be of considerable interest to preclude the case where the sequence  $\{b_1(n)\}$  visits the interval  $(-a, a)$  with  $0 < a < |b_1|$  in a very scattering way (i.e.,  $P(D^c) > 0$ ).

We now consider the case where the set  $D$  defined in Theorem 4.1 does have probability one.

**THEOREM 4.2.** *Consider the system (1.1), the ELS algorithm (1.11)–(1.14), and the control law (4.1). If (A1)–(A4) and (A6) hold, and in addition, the reference signal  $\{y_i^*\}$  satisfies*

$$(4.23) \quad n^{\frac{1+\varepsilon}{2}} \sqrt{d_n} = O(\lambda_{\min}^*(n)) \quad \text{a.s., for some } \varepsilon > 0,$$

where  $d_n$  and  $\lambda_{\min}^*(n)$  are defined in (1.22) and (2.7), respectively, then as  $n \rightarrow \infty$

$$(4.24) \quad R_n = O\left(\frac{\log n}{n}\right) \text{ a.s.,} \quad \|\theta_n - \theta\|^2 = O\left(\frac{\log n}{\lambda_{\min}^*(n)}\right) \text{ a.s.,}$$

where  $R_n$  is the regret defined by (1.6). Furthermore, if (4.23) is replaced by  $n = O(\lambda_{\min}^*(n))$  a.s., and  $E[w_{n+1}^2 | \mathcal{F}_n] = \sigma^2$  a.s., then for the white noise case ( $r = 0$ ), (4.24) can be strengthened into

$$\lim_{n \rightarrow \infty} \left(\frac{n}{\log n}\right) R_n = (p+q)\sigma^2 \quad \text{a.s.,} \quad \|\theta_n - \theta\|^2 = O\left(\frac{\log \log n}{n}\right) \text{ a.s.}$$

*Proof.* By Theorem 4.1 (i) and (4.23) we know that Theorem 2.1 (ii) is applicable to the sequence  $\{\sigma_n\}$ , and hence we have

$$\liminf_{n \rightarrow \infty} \frac{\lambda_{\min} \left( \sum_{i=0}^{\sigma_n} \varphi_i^0 \varphi_i^{0\tau} \right)}{\lambda_{\min}^*(\sigma_n)} > 0 \quad \text{a.s.}$$

Consequently, similar to the proof of (3.25), it is easy to see that

$$(4.25) \quad \liminf_{n \rightarrow \infty} \frac{\lambda_{\min} \left( \sum_{i=0}^{\sigma_n} \varphi_i \varphi_i^\tau \right)}{\lambda_{\min}^*(\sigma_n)} > 0 \quad \text{a.s.}$$

By this and Lemma 2.5 (i) it is easy to see that

$$(4.26) \quad \|\tilde{\theta}_{\sigma_n+1}\|^2 = O \left( \frac{\log r_{\sigma_n}}{\lambda_{\min}^*(\sigma_n)} \right) \quad \text{a.s.}$$

By Theorem 4.1 (i), we know that  $\log r_{\sigma_n} = O(\log \sigma_n)$  a.s., and so by (4.23) and (4.26) we conclude that  $\tilde{\theta}_{\sigma_n+1} \rightarrow 0$  a.s., and in particular,

$$(4.27) \quad b_1(\sigma_n + 1) \xrightarrow[n \rightarrow \infty]{} b_1 \quad \text{a.s.}$$

We now prove that  $P(D_1) = 1$  where  $D_1$  is defined by (4.19). Otherwise, we would have  $P(D_1^c) > 0$ , and on  $D_1^c$  by the definition of  $\sigma_n$  we know that  $\sigma_n < \infty$  for all  $n$ , and that

$$(4.28) \quad |b_1(\sigma_n + 1)| < a \quad \forall n \geq 1 \quad \text{on } D_1^c,$$

which clearly contradicts with (4.27) since  $a < |b_1|$ . Hence  $P(D_1) = 1$  and we have  $\liminf_{n \rightarrow \infty} |b_1(n)| \neq 0$  a.s. Therefore, by a similar means as in the proof of Theorem 2 in Guo and Chen [19], we obtain  $R_n = O(d_n/n^{1-\varepsilon})$  a.s. and  $\|\varphi_n\|^2 = O(n^\varepsilon d_n)$  a.s. for all  $\varepsilon > 0$ . Using this and a similar argument as for (4.25) and (4.26), we know that (4.25) and (4.26) also hold with  $\{\sigma_n\}$  replaced by  $\{n\}$ . Hence we have proved the second assertion in (4.24). Since (4.25) holds with  $\{\sigma_n\}$  replaced by  $\{n\}$  and  $\|\varphi_n\|^2 = O(n^{\delta/2} \sqrt{d_n})$  a.s., for all  $\delta \in (2/\beta, 1)$ , we know that  $\varphi_n^\tau P_n \varphi_n \xrightarrow[n \rightarrow \infty]{} 0$  a.s. By this and Lemma 2.5 it follows

from (4.6) (with  $\Delta \hat{b}_{1n} \equiv 0$ ) that the first assertion in (4.24) is also true. Finally, the last two assertions of the theorem can be proved in exactly the same way as in Theorem 3.1, and the details are not repeated.  $\square$

*Remark 4.2.* (i) Again, the best possible convergence rate  $O(\log n/n)$  is established for the regret  $R_n$ . It is worth noting that this result is established without introducing any modifications to the standard minimum variance control law (4.1). This fact makes Theorem 4.2 differ essentially from the existing results including those in the recent work [19].

(ii) The (nonpersistent) excitation condition (4.23) on the reference signal  $\{y_n^*\}$  can be easily verified for a large class of deterministic and/or stochastic signals. In principle, we can always make this condition satisfied by use of the ‘‘continually disturbed demand method’’ of Caines and Lafortune [27] or the ‘‘diminishing excitation technique’’ in Chen and Guo [16]. To be precise, for any desired trajectory  $\{y_{nd}^*\}$  that is bounded and independent of  $\{w_n\}$ , we may take the reference signal in (4.1) to be

$$(4.29) \quad y_n^* = y_{nd}^* + v_n,$$

where  $\{v_n\}$  is a zero mean independent bounded random sequence which is independent of  $\{w_n, y_{nd}^*\}$ . Then with some suitable moment conditions on  $\{v_n\}$  it is easy to see that (4.23) holds. In order that the ‘‘dither’’ does not influence the self-optimality the variance of  $\{v_n\}$  must be chosen to satisfy  $E v_n^2 \xrightarrow[n \rightarrow \infty]{} 0$ . This is possible since the excitation requirement

(4.23) is not necessarily persistent. The disadvantage of adding the “dither”  $\{v_n\}$  in such a way is that it may influence the convergence rate of tracking.

Next, we consider the case where (4.23) fails. As a typical example, we shall only consider the regulation problem ( $y_n^* \equiv 0$ ). Similar to (1.23), we set for  $n \geq 1$ ,

$$(4.30) \quad \hat{b}_1(n) = \begin{cases} b_1(n) & \text{if } |b_1(n)| \geq \frac{1}{\sqrt{n \log(n+1)}}; \\ b_1(n) + \frac{\text{sgn}(b_1(n))}{\sqrt{n \log(n+1)}} & \text{otherwise.} \end{cases}$$

Instead of (4.1), we define the control  $u_n$  by

$$(4.31) \quad u_n = \frac{1}{\hat{b}_1(n)} \{b_1(n)u_n - \theta_n^T \varphi_n\}, \quad n \geq 1,$$

which obviously has the form of (4.2):

$$(4.32) \quad \theta_n^T \varphi_n + \Delta \hat{b}_{1n} u_n = 0,$$

where  $\Delta \hat{b}_{1n} \triangleq \hat{b}_1(n) - b_1(n)$ . By (4.30) it is clear that

$$(4.33) \quad |\hat{b}_1(n)|^2 \geq \frac{1}{n \log(n+1)}, \quad |\Delta \hat{b}_{1n}|^2 \leq \frac{1}{n \log(n+1)} \xrightarrow{n \rightarrow \infty} 0.$$

**THEOREM 4.3.** *Consider the model (1.1) with white additive noise (i.e.,  $r = 0$ ). Assume that (A1), (A3), and (A5) are satisfied, and that in (1.22),  $d_n = O(n^\varepsilon)$  for all  $\varepsilon > 0$ . If the control law defined by (4.30) and (4.31) is applied, then as  $n \rightarrow \infty$ ,*

$$(4.34) \quad \sum_{i=1}^n (y_i - w_i)^2 = O(n^\varepsilon) \quad \text{a.s.,} \quad \forall \varepsilon > 0,$$

*Proof.* Let  $D_1$  and  $\{\tau_n\}$  be defined by (4.19) and (4.20), respectively. As explained earlier, (4.34) holds on  $D_1$ , and so we need only to consider  $D_1^c$ . In the remainder of the proof all relationships are established on  $D_1^c$  with a possible exception set of probability zero, and we shall omit the phrase “a.s. on  $D_1^c$ ” for simplicity.

By (4.20) we know that on  $D_1^c$ ,  $\tau_n < \infty$ , for all  $n \geq 1$ ,  $\lim_{n \rightarrow \infty} \tau_n = \infty$ , and  $\inf_n |b_1(\tau_n + 1) - b_1| \geq |b_1| - a > 0$ . Hence, by Lemma 4.1 we know that

$$(4.35) \quad r_{\tau_n} = O(\tau_n) \quad \text{and} \quad \sup_{k \leq \tau_n} \|\varphi_k\|^2 = O(\tau_n^\varepsilon d_{\tau_n}).$$

Consequently, by (4.6) and (4.33),

$$(4.36) \quad \begin{aligned} \sum_{i=1}^{\tau_n} (y_{i+1} - w_{i+1})^2 &= O\left(\sum_{i=1}^{\tau_n} \|\tilde{\theta}_i^T \varphi_i\|^2\right) + O\left(\sum_{i=1}^{\tau_n} (\Delta \hat{b}_{1i} u_i)^2\right) \\ &= O\left(\tau_n^\varepsilon d_{\tau_n} \sum_{i=1}^{\tau_n} \frac{\|\tilde{\theta}_i^T \varphi_i\|^2}{1 + \varphi_i^T P_i \varphi_i}\right) + O\left(\sum_{i=1}^{\tau_n} \frac{1}{i \log(i+1)} u_i^2\right) \\ &= O(\tau_n^\varepsilon d_{\tau_n} \log \tau_n) + O\left(\tau_n^\varepsilon d_{\tau_n} \sum_{i=1}^{\tau_n} \frac{1}{i \log(i+1)}\right) \quad \forall \varepsilon > 0, \\ &= O(\tau_n^\varepsilon) \quad \forall \varepsilon > 0. \end{aligned}$$

Hence Theorem 2.1 (i) is applicable, and we then have

$$(4.37) \quad \liminf_{n \rightarrow \infty} \lambda_{\min} \left( \frac{1}{\tau_n} \sum_{i=1}^{\tau_n} \psi_i \psi_i^\tau \right) > 0,$$

where  $\{\psi_i\}$  is defined by (2.4) with  $p^* = p$ .

Let  $a_i(n), b_j(n)$  be the estimates for  $a_i, b_j, 1 \leq i \leq p, 1 \leq j \leq q$ , given by  $\theta_n$ . Set

$$(4.38) \quad \bar{\theta}_n = [-a_1(n), \dots, -a_p(n), b_2(n), \dots, b_q(n)]^\tau.$$

Then by (4.31) we have

$$(4.39) \quad u_n = -\frac{1}{\hat{b}_1(n)} \bar{\theta}_n^\tau \psi_n$$

Now, we prove that

$$(4.40) \quad \left\| \frac{1}{\hat{b}_1(\tau_n + 1)} \bar{\theta}_{\tau_n + 1} \right\|^2 = O(\tau_n^\varepsilon) \quad \forall \varepsilon > 0.$$

By (4.36) we know that  $\sum_{i=1}^{\tau_n} \|\theta^\tau \varphi_i\|^2 = O(\tau_n^\varepsilon)$ , which in conjunction with (4.10) and (4.35) gives

$$\sum_{i=1}^{\tau_n} (\theta_{\tau_n + 1}^\tau \varphi_i)^2 \leq 2 \sum_{i=1}^{\tau_n} (\tilde{\theta}_{\tau_n + 1}^\tau \varphi_i)^2 + 2 \sum_{i=1}^{\tau_n} (\theta^\tau \varphi_i)^2 = O(\tau_n^\varepsilon).$$

From this, by noting that  $\theta_{\tau_n + 1}^\tau \varphi_i = \bar{\theta}_{\tau_n + 1}^\tau \psi_i + b_1(\tau_n + 1)u_i$ , we have for all  $\varepsilon > 0$ ,

$$\begin{aligned} \sum_{i=1}^{\tau_n} (\bar{\theta}_{\tau_n + 1}^\tau \psi_i)^2 &\leq 2 \sum_{i=1}^{\tau_n} (\theta_{\tau_n + 1}^\tau \varphi_i)^2 + 2b_1^2(\tau_n + 1) \sum_{i=1}^{\tau_n} u_i^2 \\ &\leq O(\tau_n^\varepsilon) + 4\{[\hat{b}_1(\tau_n + 1)]^2 + [\Delta \hat{b}_1(\tau_n + 1)]^2\} \sum_{i=1}^{\tau_n} u_i^2. \end{aligned}$$

Multiplying  $1/[\hat{b}_1(\tau_n + 1)]^2$  from both sides of this inequality, and noticing (4.33) and (4.35) we get

$$\frac{1}{[\hat{b}_1(\tau_n + 1)]^2} \sum_{i=1}^{\tau_n} (\bar{\theta}_{\tau_n + 1}^\tau \psi_i)^2 = O\left(\frac{\tau_n^\varepsilon}{[\hat{b}_1(\tau_n + 1)]^2}\right) + O\left(\sum_{i=1}^{\tau_n} u_i^2\right) = O(\tau_n^{1+\varepsilon} \log(\tau_n + 1) + \tau_n).$$

From this and (4.37) we see that for all suitably large  $n$ ,

$$\begin{aligned} \left\| \frac{1}{\hat{b}_1(\tau_n + 1)} \bar{\theta}_{\tau_n + 1} \right\|^2 &\leq O\left(\left\| \frac{1}{\hat{b}_1(\tau_n + 1)} \bar{\theta}_{\tau_n + 1} \right\|^2 \lambda_{\min} \left( \frac{1}{\tau_n} \sum_{i=1}^{\tau_n} \psi_i \psi_i^\tau \right)\right) \\ &\leq O\left(\frac{1}{\tau_n [\hat{b}_1(\tau_n + 1)]^2} \sum_{i=1}^{\tau_n} (\bar{\theta}_{\tau_n + 1}^\tau \psi_i)^2\right) = O(\tau_n^\varepsilon \log(\tau_n + 1) + 1) \quad \forall \varepsilon > 0. \end{aligned}$$

Hence (4.40) holds.

Next, we prove that

$$(4.41) \quad \|\varphi_n\|^2 = O(nr_n)^\varepsilon d_n \quad \forall \varepsilon > 0.$$

Note that by (4.20), we know that on  $D_1^c$ ,

$$(4.42) \quad |b_1(k+1)| \geq a > 0, \quad \forall k \in [\tau_n + 1, \tau_{n+1} - 1], \quad \forall n \geq 1.$$

Hence from (4.39), (4.40), (4.42), and the fact that  $\|\bar{\theta}_{k+1}\|^2 = O(\log r_k)$ , it follows that on  $D_1^c$

$$(4.43) \quad u_{k+1}^2 = \begin{cases} O(\tau_n^\varepsilon \|\psi_{\tau_n+1}\|^2), & \forall \varepsilon > 0, \quad k = \tau_n; \\ O((\log r_k) \|\psi_{k+1}\|^2), & k \in [\tau_n + 1, \tau_{n+1} - 1]. \end{cases}$$

Similar to (4.9) it is easy to see by (A3) that

$$\|\psi_k\|^2 = O\left(\sum_{i=0}^k \lambda^{k-i} y_i^2\right) + O(d_k) \quad \forall k \geq 1.$$

From this and (4.43) we have, for all  $k \in [\tau_n + 2, \tau_{n+1}]$ ,

$$(4.44) \quad \begin{aligned} \|\varphi_k\|^2 &= \|\psi_k\|^2 + u_k^2 \\ &= O\left([\log r_k] \sum_{i=0}^k \lambda^{k-i} y_i^2\right) + O(d_k \log r_k). \end{aligned}$$

Substituting this together with (4.8) into (4.7) and noting that  $\Delta \hat{b}_{1n} \xrightarrow{n \rightarrow \infty} 0$ , we get for all  $k \in [\tau_n + 2, \tau_{n+1}]$ , and all suitably large  $n$ ,

$$(4.45) \quad y_{k+1}^2 = O\left(\alpha_k \delta_k (\log r_k) \sum_{i=0}^k \lambda^{k-i} y_i^2\right) + o\left(\sum_{i=0}^k \lambda^{k-i} y_i^2\right) + O(d_k \log^2 r_k).$$

Set  $L_k = \sum_{i=0}^k \lambda^{k-i} y_i^2$ . Similar to the proof of (4.15), from (4.45) we have for some  $\gamma \in (0, 1)$ ,  $\forall k \in [\tau_n + 2, \tau_{n+1}]$ , and all large  $n$ ,

$$(4.46) \quad \begin{aligned} L_{k+1} &\leq \gamma^{k-\tau_n-1} \prod_{i=\tau_n+2}^k (1 + c\alpha_i \delta_i \log r_i) L_{\tau_n+2} \\ &+ O\left(\sum_{i=\tau_n+2}^k \gamma^{k-i} \prod_{j=i+1}^k (1 + c\alpha_j \delta_j \log r_j) d_i \log^2 r_i\right), \end{aligned}$$

where  $c > 0$  is a constant. Similar to the proof of (4.16) we know that for all small  $\varepsilon > 0$  and all  $k \geq i$ , with  $i$  suitably large,  $\prod_{j=i+1}^k (1 + c\alpha_j \delta_j \log r_j) \leq r_k^\varepsilon$ . Substituting this into (4.46) yields for large  $n$ ,

$$(4.47) \quad L_{k+1} = O(r_k^\varepsilon L_{\tau_n+2}) + O(r_k^\varepsilon d_k), \quad \forall k \in [\tau_n + 2, \tau_{n+1}], \quad \forall \varepsilon > 0.$$

By (4.35), (1.1), and (A3), it is easy to see that

$$(4.48) \quad L_{\tau_n+1} + \|\psi_{\tau_n+1}\|^2 = O(\tau_n^\varepsilon d_{\tau_n}), \quad \forall \varepsilon > 0.$$

Consequently by (4.43),  $u_{\tau_n+1}^2 = O(\tau_n^\varepsilon d_{\tau_n})$ , for all  $\varepsilon > 0$ . From this, (1.1), and (4.35) again, we obtain  $L_{\tau_n+2} = O(\tau_n^\varepsilon d_{\tau_n})$  for all  $\varepsilon > 0$ . This in conjunction with (4.47) and (4.48) yields

$$(4.49) \quad L_{k+1} = O([kr_k]^\varepsilon d_k), \quad \forall k \in [\tau_n, \tau_{n+1}], \quad \forall \varepsilon > 0.$$

From this it is easy to convince oneself that

$$\|\varphi_k\|^2 = O([kr_k]^\varepsilon d_k), \quad \forall k \in [\tau_n, \tau_{n+1}], \quad \forall \varepsilon > 0,$$

holds for all suitably large  $n$ . This implies (4.41), since  $\tau_n \xrightarrow{n \rightarrow \infty} \infty$ .

By (4.41) and a similar proof as for (4.18) we get  $r_n = O([nr_n]^\varepsilon d_n \log r_n) + o(r_n) + O(n)$  for all  $\varepsilon > 0$ . Hence it follows that  $r_n = O(n)$ . Then, by (4.41) and the assumption that  $d_n = O(n^\varepsilon)$  for all  $\varepsilon > 0$ , we obtain  $\|\varphi_n\|^2 = O(n^\varepsilon)$  for all  $\varepsilon > 0$ . Therefore, similar to the proof of (4.36), we get  $\sum_{i=1}^n (y_{i+1} - w_{i+1})^2 = O(n^\varepsilon)$  for all  $\varepsilon > 0$  a.s. on  $D_1^c$ . This completes the proof.  $\square$

*Remark 4.3.* The advantage of the modification (4.30) over (1.23) as used in [19] is clear. When (1.23) is used, the cumulated square errors resulting from the modification of  $b_1(n)$  is of the order  $O(n/\log n)$ , i.e.,

$$\sum_{i=1}^n (\Delta \hat{b}_{1i})^2 = O\left(\sum_{i=1}^n \frac{1}{\log r_i}\right) = O\left(\frac{n}{\log n}\right).$$

Hence in Theorem 2 of Guo and Chen [19], the guaranteed convergence rate for the averaged regret  $R_n$  is only of the order  $O(1/\log n)$ , which is clearly much slower than the rate  $R_n = O(1/n^{1-\varepsilon})$  a.s. for all  $\varepsilon > 0$ , obtained in Theorem 4.3. Of course, it would be of interest to generalize Theorem 4.3 to the colored noise case and to show that the left-hand side of (4.34) is of the order  $O(\log n)$ .

**5. Concluding remarks.** The convergence rate of least-squares-based adaptive algorithm has been observed in practice to be superior to any other type of implementable on-line recursive algorithms including the extensively studied stochastic gradient algorithm. In this paper, we have obtained various new results on the standard ELS-based adaptive minimum variance control for SISO ARMAX systems, and improved on the recent work [19] in many aspects. In particular, we have obtained the best possible convergence rate  $O(\log n/n)$  for the averaged regret of tracking in several situations of interest. This rate is not believed to be achievable, for example, for the stochastic gradient based adaptive algorithm. For further study, it is desirable to generalize the result  $R_n = O(\log n/n)$  to general tracking problems with arbitrarily bounded reference signal  $\{y_n^*\}$ , using (preferably) the control law (4.1).

#### REFERENCES

- [1] D. W. HUANG AND L. GUO, *Estimation of nonstationary ARMAX models based on the Hannan-Rissanen method*, Ann. Statist., 18 (1990), pp. 1729–1756.
- [2] P. R. KUMAR AND P. VARAIYA, *Stochastic Systems: Estimation, Identification and Adaptive Control*, Prentice-Hall, Englewood Cliffs, NJ, 1986.
- [3] Y. S. CHOW, *Local convergence of martingales and the law of large numbers*, Ann. Math. Statist., 36 (1965), pp. 552–558.
- [4] C. Z. WEI, *Adaptive prediction by least squares predictors in stochastic regression models with applications to time series*, Ann. Statist., 15 (1987), pp. 1667–1682.
- [5] P. E. CAINES, *Linear Stochastic Systems*, John Wiley, New York, 1988.

- [6] H. F. CHEN AND L. GUO, *Identification and Stochastic Adaptive Control*, Birkhäuser, Boston, 1991.
- [7] J. STERNBY, *On consistency of least squares method using martingale theory*, IEEE Trans. Automat. Control, 22 (1977), pp. 346–352.
- [8] T. L. LAI AND C. Z. WEI, *Least squares estimates in stochastic regression models with applications to identification and control of dynamic systems*, Ann. Statist., 10 (1982), pp. 154–166.
- [9] G. LEDWICH AND J. B. MOORE, *Multivariable self-tuning filters*, in Differential Games and Control Theory, II. Lecture Notes in Pure and Appl. Math., Marcel-Dekker, New York, 1977, pp. 345–376.
- [10] V. SOLO, *The convergence of AML*, IEEE Trans. Automat. Control, 24 (1979), pp. 958–962.
- [11] H. F. CHEN, *Strong consistency and convergence rate of least squares identification*, Scientia Sinica (Ser. A), 25 (1982), pp. 771–784.
- [12] H. F. CHEN AND L. GUO, *Convergence rate of least-squares identification and adaptive control for stochastic systems*, Internat. J. Control, 44 (1986), pp. 1459–1476.
- [13] T. L. LAI AND C. Z. WEI, *Extended least squares and their applications to adaptive control and prediction in linear systems*, IEEE Trans. Automat. Control, 31 (1986), pp. 898–906.
- [14] G. C. GOODWIN, P. RAMADGE AND P. E. CAINES, *Discrete time stochastic adaptive control*, SIAM J. Control Optimiz., 19 (1981), pp. 829–853.
- [15] H. F. CHEN AND L. GUO, *Strong consistency of recursive identification by no use of persistent excitation condition*, Acta. Math. Appl. Sinica, 2 (1985), pp. 133–145.
- [16] ———, *Asymptotically optimal adaptive control with consistent parameter estimates*, SIAM J. Control Optimiz., 25 (1987), pp. 558–575.
- [17] K. J. ÅSTRÖM AND B. WITTENMARK, *On self-tuning regulators*, Automatica, 9 (1973), pp. 195–199.
- [18] P. R. KUMAR, *Convergence of adaptive control schemes using least-squares parameter estimates*, IEEE Trans. Automat. Control, 35 (1990), pp. 416–423.
- [19] L. GUO AND H. F. CHEN, *The Åström–Wittenmark self-tuning regulator revisited and ELS-based adaptive trackers*, IEEE Trans. Automat. Control, 36 (1991), pp. 802–812.
- [20] P. R. KUMAR, *A survey of some recent results in stochastic adaptive control*, SIAM J. Control Optimiz., 23 (1985), pp. 329–380.
- [21] N. C. JAIN, K. JOGDEO AND W. F. STOUT, *Upper and lower functions for martingales and mixing processes*, Ann. Probability, 3 (1975), pp. 119–145.
- [22] H. F. CHEN AND L. GUO, *Strong consistency of parameter estimates in optimal stochastic adaptive tracking systems*, Sci. Sinica (Ser. A), 29 (1986), pp. 1145–1156.
- [23] T. L. LAI, *Asymptotically efficient adaptive control in stochastic regression models*, Adv. Appl. Math., 7 (1986), pp. 23–45.
- [24] A. BECKER, P. R. KUMAR AND C. Z. WEI, *Adaptive control with the stochastic approximation algorithm: Geometry and convergence*, IEEE Trans. Automat. Control, 30 (1985), pp. 154–166.
- [25] P. R. KUMAR AND L. PRALY, *Self-tuning trackers*, SIAM J. Control Optimiz., 25 (1987), pp. 1053–1071.
- [26] S. P. MEYN AND P. E. CAINES, *The zero divisor problem of multivariable stochastic adaptive control*, System Control Lett., 6 (1985), pp. 235–238.
- [27] P. E. CAINES AND S. LA FORTUNE, *Adaptive control with recursive identification for stochastic linear systems*, IEEE Trans. Automat. Control, 29 (1984), pp. 312–321.