On Critical Stability of Discrete-Time Adaptive Nonlinear Control

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***Abstract*—In this paper, we examine the global stability and instability problems for a class of discrete-time adaptive nonlinear stochastic control. The systems to be controlled may exhibit chaotic behavior and are assumed to be linear in unknown parameters but nonlinear in output dynamics, which are characterized by a nonlinear function [say, $f(x)$]. It is found and proved that in the scalar parameter case there is a critical stability phenomenon for least squares (LS)-based adaptive control systems. To be specific, let the growth rate of $f(x)$ be $f(x) = O(\|x\|^b)$ with $b \geq 0$, then it is found that $b = 4$ is a critical value for global stability, i.e., the closed-loop adaptive system is globally stable if $b < 4$ and is unstable in general if $b \geq 4$. As a consequence, we find an interesting phenomenon that the linear case does not have: for some LS-based certainty equivalence adaptive controls, even if the LS parameter estimates are strongly consistent, the closed-loop systems may still be unstable. This paper also indicates that adaptive nonlinear stochastic control that is designed based on, e.g., Taylor expansion (or Weierstrass approximation) for nonlinear models, may not be feasible in general.**

***Index Terms*—Adaptive control, discrete-time, global stability, instability, least squares, nonlinear system, random noises.**

I. Introduction

THE discrete-time single-input/single-output nonlinear stochastic model that can be conveniently used in adaptive control may be described as follows:

$$y_{t+1} = \theta^\tau f_t(x_t) + w_{t+1}, \qquad t \geq 0 \tag{1}$$

$$x_t = (y_t \cdots y_{t-p+1}, u_t \cdots u_{t-q+1})^\tau \tag{2}$$

where y_t, u_t, and w_t are the system output, input, and noise sequences, respectively, $f_t(\cdot)$: $\mathbb{R}^{p+q} \to \mathbb{R}^d$ is a known nonlinear function of x_t, and $\theta \in \mathbb{R}^d$ is an unknown parameter.

Though this model seems to be special, it can be justified from both black-box and grey-box modeling viewpoints. On the one hand, (1) may be regarded as an approximation to the familiar nonlinear auto-regressive with exogenous inputs (NARX) model as discussed in, e.g., [1], since the nonlinear term may be viewed as a finite sum of a certain basis function expansion of an unknown nonparametric function of x_t. Various basis functions may be used in this expansion (cf., [2]), and a typical and classical case is the Taylor series or polynomial expansion where the basis function consists of multinomials in the components of x_t. On the other hand, many practical systems may be modeled as (1) by use of the basic principles in physics, chemistry, or biology. Three typical situations of (1) may be illustrated as follows.

1) *Bilinear models*: Such models arise naturally in many chemical and biological processes (cf., [3]–[5]), where the nonlinearity is characterized by some linear products of the input and output variables. The simplest bilinear model appears to be the following one (cf., [18]):

$$y_{t+1} = ay_t + bu_t + cy_t u_t + w_{t+1}, \qquad c \neq 0 \tag{3}$$

and we shall return to this model shortly.

2) *Hammerstein models*: This kind of model is linear in output process but nonlinear in input process with nonlinearity typically characterized by a polynomial. Some practical extremum control problems may be investigated based on such models (cf., [6]).

3) *Models with output nonlinearity*: This case is perhaps more realistic and more interesting. A typical situation may be described as follows:

$$y_{t+1} = a_0 + a_1 y_t + a_2 y_t^2 + a_3 y_t^3 + b_1 y_{t-1} + u_t + w_{t+1}. \tag{4}$$

It is worth noting that if the system is uncontrolled ($u_t \equiv 0$) and undisturbed ($w_t \equiv 0$), then (4) may be reduced to several standard chaotic models extensively studied in chaos dynamical systems. The celebrated Logistic map [7], the two-dimensional Hénon map [8], and the cubic map [9], are such examples. These examples suggest that complex behavior may not necessarily require complex mathematical models. We shall in this paper focus on models that have output nonlinearities.

Over the past three decades, extensive study has been made on the adaptive control of linear stochastic models where in (1) the function $f_t(\cdot)$ takes the special form $f_t(x_t) \equiv x_t, \forall t \geq 0$. Almost all of the theoretical progress has been concerned with minimum phase linear stochastic systems, and the most recent advances can be found in, for example, [10]–[15]. As far as adaptive control of nonminimum phase stochastic systems is concerned, the stability analysis has been hampered mainly by the fact that the estimated models may not be uniformly controllable. This difficulty has recently been circumvented in [16] by using the self-convergence property of a class of weighted least squares (WLS) together with a method of random regularization. Consequently, a simple and complete solution can be given to both the stochastic adaptive pole-placement and linear-quadratic-Gaussian (LQG) control problems [16].

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One may naturally expect that the existing results on linear stochastic systems can at least be extended to the nonlinear system (1) with $f_t(\cdot)$ satisfying the following growth condition for some $b \geq 1$:

$$\|f_t(x)\| \leq k_1 + k_2\|x\|^b, \qquad \forall t \in \mathbb{R}^+, \forall x \in R^{p+q} \tag{5}$$

where $k_1 \geq 0$ and $k_2 \geq 0$ are constants. Unfortunately, even for minimum phase nonlinear stochastic systems, the only case that can be dealt with by the existing methods is the linear growth case, i.e., $b = 1$ in (5) (cf., [17]). Indeed, to the best of the author's knowledge, there are as yet no concrete stability results on stochastic adaptive control of (1) when the function $f_t(\cdot)$ has a growth rate faster than linear.

Notwithstanding this, a great deal of effort has been made in the literature. In particular, Cho and Marcus [18] showed that bilinear stochastic models are in general nonminimum phase (in the sense that bounded output does not imply bounded input), and hence minimum variance control may not be feasible for such class of systems. Based on this observation, they considered a weighted one-step-ahead certainty equivalence adaptive control for the first-order model (3) by using a stochastic gradient (SG) estimation algorithm. The stability proof of the closed-loop adaptive systems in [18], however, requires a certain condition on the on-line parameter estimate. A more recent work on adaptive bilinear control may be found in [19], where stability analysis was carried out under a certain condition on the output process. Neither of the conditions in [18] and [19] is as yet known to be verifiable. In Appendix A, however, we shall show that by using a regularized WLS estimate similar to that used in [15] and [16], the condition imposed on the parameter estimate in [18] can be dispensed with, giving a complete stability result on adaptive control of (3).

Adaptive control of (4) is, apparently, closely related to the problem of controlling chaos, which has attracted much research interest in recent years. Various approaches have been suggested in the literature (see, e.g., [20]–[22]), yet complete and rigorous theoretic results seem to be hard to find.

In the deterministic framework, significant progress has been made in adaptive nonlinear control in the past several years (cf., e.g., [23]–[26]), and most of the results are concerned with continuous-time systems. Unfortunately, it has been found that the existing continuous-time methods are hardly applicable to the discrete-time case, due to some inherent difficulties in discrete-time models, as detailed in an interesting paper [27]. As a matter of fact, even in the noise-free case, there are as yet no general discrete-time adaptive nonlinear control results that allow the nonlinear function to grow at a rate faster than linear. In recent work [28], the standard LS-based adaptive regulation control was analyzed for a class of discrete-time deterministic nonlinear systems.

We shall in this paper be concerned with the more realistic (and more complicated) stochastic case, dealing with adaptive control of discrete-time stochastic systems possessing output nonlinearities [see (6)]. The main contribution is to establish that in the scalar parameter case, the value $b = 4$ in (5) is a critical case for global stability of a class of adaptive nonlinear stochastic tracking control systems. As a consequence, we find that in general it is impossible to have global stability results for stochastic adaptive control systems when the nonlinear function $f_t(\cdot)$ is a high-order polynomial of its variables. This means that adaptive control based on the method of Weierstrass approximation (i.e., to approximate nonlinear functions by polynomials) may not be feasible in general. Also, we shall conclude an interesting fact that the linear case does not have: for some LS certainty equivalence adaptive controls, even if the on-line LS parameter estimates are strongly consistent, the closed-loop systems may still blow up.

The remainder of the paper is organized as follows: Section II presents the main results on both instability and stability of adaptive systems, together with some remarks and discussions; Sections III and IV give the proof for the main theorems; and some concluding remarks are made in Section V. Appendix A also contains a stability result on adaptive bilinear stochastic control.

II. The Main Results

Throughout the sequel, we shall focus on stochastic systems with output nonlinearities and consider the following special case of (1) and (2):

$$y_{t+1} = \theta^\tau f_t(y_t, y_{t-1}, \cdots, y_{t-p+1}) + u_t + w_{t+1}, \quad t \geq 0 \tag{6}$$

where $\theta \in \mathbb{R}^d$ is a random or deterministic unknown parameter.

The standard LS estimate θ_t for θ can be recursively defined by

$$\theta_{t+1} = \theta_t + P_{t+1}\phi_t(y_{t+1} - u_t - \phi_t^\tau \theta_t) \tag{7}$$

$$P_{t+1} = P_t - \frac{P_t\phi_t\phi_t^\tau P_t}{1 + \phi_t^\tau P_t \phi_t}, \qquad P_0 > 0 \tag{8}$$

$$\phi_t \triangleq f_t(y_t \cdots y_{t-p+1}), \qquad t \geq 0 \tag{9}$$

where (θ_0, P_0) are the deterministic initial conditions of the algorithm, and ϕ_0 is possibly a random initial value of the system.

Let $\{y_t^*\}$ be a known bounded deterministic reference signal to be tracked. The certainty equivalence adaptive tracking control is defined by

$$u_t = -\theta_t^\tau \phi_t + y_{t+1}^*, \qquad t \geq 0 \tag{10}$$

substituting this into (6), we have the following closed-loop equation:

$$y_{t+1} = \tilde{\theta}_t^\tau \phi_t + y_{t+1}^* + w_{t+1}, \qquad t \geq 0 \tag{11}$$

where $\tilde{\theta}_t \triangleq \theta - \theta_t$.

To facilitate the analysis of the above closed-loop control system, we need the following definitions.

Definition 1: The closed-loop control system (6)–(10) is said to be globally stable, if for any initial conditions (θ_0, ϕ_0, P_0), the averaged input and output signals are bounded almost surely, i.e.,

$$\limsup_{t\to\infty} \frac{1}{t}\sum_{i=1}^{t}(y_i^2 + u_i^2) < \infty \quad \text{a.s.} \tag{12}$$

Definition 2: The nonlinear system (6) [or the nonlinear function $f_t(\cdot)$] is said to belong to a class C_b with $b \geq 1$, if there exist two constants $k_1 \geq 0$ and $k_2 \geq 0$ such that

$$\|f_t(x)\| \leq k_1 + k_2\|x\|^b, \qquad \forall\, x \in \mathbb{R}^p,\ t \geq 0 \tag{13}$$

where $\|\cdot\|$ denotes the Euclidean norm.

We shall first show that if the nonlinear stochastic system (6) belongs to C_b with $b \geq 4$, then the closed-loop adaptive system (6)–(10) may not have global stability in general.

A. An Instability Theorem

For the above purpose, we need only consider the following typical situation:

$$y_{t+1} = \theta y_t^b + u_t + w_{t+1}, \qquad t \geq 0 \tag{14}$$

$$u_t = -\theta_t \phi_t, \qquad \phi_t = y_t^b \tag{15}$$

where $\theta \in \mathbb{R}^1$ is a random (or deterministic) unknown parameter, and θ_t is the LS estimate generated by (7) and (8).

Let us denote for $b \geq 1$

$$D_1 = \left\{ \tfrac{1}{2} \leq |\tilde{\theta}_0| \leq 1,\ |y_0| \geq 1 + (10)^{4/b},\ P_0 = 1 \right\} \tag{16}$$

$$D_2 = \bigcap_{i=1}^{\infty} \{(i+1)^{-5/4} \leq |w_i| \leq (i+1)^{1/b}\}. \tag{17}$$

Note that the first set is related to the initial conditions, and the second is related to the noise distributions. More discussions on D_2 will be given in Remark 2.1.

Theorem 2.1: Consider the closed-loop adaptive control system described by (14) and (15) with $b \geq 4$. Then on the set $D \triangleq D_1 \cap D_2$, the long run average

$$\frac{1}{n}\sum_{t=1}^{n} y_t^2 \to \infty, \quad \text{as } n \to \infty$$

at a rate faster than exponential.

The proof of this theorem is given in Section III. This result actually is concerned with the pointwise property of $\{y_t\}$ on D. Several remarks explaining Theorem 2.1 are now in order.

Remark 2.1—Discussions on D_2: It is worth noting that deterministic disturbances are not excluded in Theorem 2.1. In particular, if $|w_i| = 1, \forall\, i$ (e.g., the Bernoulli distribution), then it is obvious that $P(D_2) = 1$. In general, if $\{w_t\}$ is an i.i.d. sequence with distribution function Lipschitz continuous at the origin, and with $P(|w_1| \leq 1) \neq 0$, and $E|w_1|^b < \infty$ for some $b \geq 1$, then it is easy to show that $P(D_2) > 0$ always holds.

Theorem 2.1 concerns the instability of the LS-based adaptive control (15). One may naturally ask: can we find another controller that can stabilize (14) in the case of $b \geq 4$? The following remark gives a negative answer to this question.

Remark 2.2—Nonstabilizability of (14) When $b \geq 4$: Consider (14). Let $\{w_i\}$ be a white noise (i.i.d.) sequence with standard Gaussian distribution $N(0, 1)$, and let the unknown parameter θ be a random variable with Gaussian distribution $N(\theta_0, 1)$. Assume that the initial value y_0 is either deterministic, satisfying $|y_0| \geq 1+(10)^{4/b}$, or random with the tail of its distribution nonzero (e.g., the Gaussian distribution). Assume also that $\{\theta, y_0, w_t, t \geq 0\}$ are independent. Then, it can be shown (see Appendix B) that no almost surely stabilizing controller exists for (14) with $b \geq 4$, i.e., for any feedback control sequence $\{u_t\}$, there exists a set D_0 with positive probability such that

$$\frac{1}{n}\sum_{t=1}^{n} y_t^2 \to \infty, \quad \text{as } n \to \infty, \text{ on } D_0.$$

Remark 2.3—Consistency of LS: Let (θ, y_0) be deterministic and the initial values (θ_0, y_0, P_0) satisfy the conditions required in D_1 defined by (16). Also let the noise $\{w_i\}$ be i.i.d. with symmetric Bernoulli distribution. Then we have $P(D) = P(D_1) \cdot P(D_2) = 1$.

Furthermore, by (7) and (8), it can be derived that

$$\tilde{\theta}_t = \frac{1}{r_{t-1}} \left\{ P_0^{-1}\tilde{\theta}_0 - \sum_{i=0}^{t-1} \phi_i w_{i+1} \right\} \tag{18}$$

where $\tilde{\theta}_t = \theta - \theta_t$ and

$$r_t = P_{t+1}^{-1} = P_0^{-1} + \sum_{i=0}^{t} \phi_i^2. \tag{19}$$

By Theorem 2.1, there is a constant $M > 0$ such that for all large n

$$(Mn)^b \leq \left(\sum_{i=1}^{n} y_i^2 \right)^b \leq n^{b-1} \sum_{i=1}^{n} y_i^{2b} \leq n^{b-1} r_n. \tag{20}$$

Hence $r_n \to \infty$ a.s. By applying first the martingale convergence theorem and then the Kronecker lemma to the series $\sum_{i=1}^{\infty} (\phi_i / r_i) w_{i+1}$, it follows that $1/r_t \sum_{i=0}^{t} \phi_i w_{i+1} \to 0$ a.s. Consequently, by (18) we know that $\theta_t \to \theta$, a.s. This fact together with Theorem 2.1 shows that in the control law (15) even if the LS parameter estimate θ_t converges to the true parameter θ almost surely, the closed-loop system (14) and (15) is still unstable almost surely. □

Remark 2.4: Theorem 2.1 can be generalized to systems more general than (14). For instance, consider the following system:

$$y_{t+1} = \theta f(y_t) + u_t + w_{t+1}$$

where the nonlinear function $f(\cdot)$ satisfies $|x|^{2b} \leq 1 + |f(x)|^2$, $x \in \mathbb{R}^1$, $b \geq 4$. Then the result of Theorem 2.1 is still true, and the proof is completely similar.

B. Global Stability

Now, to show that $b = 4$ is really a critical value for global stability, we have to prove that the closed-loop control system (14) and (15) is indeed globally stable whenever $b < 4$. This will be proved in a somewhat more general setting, namely, (6) with $\theta \in \mathbb{R}^1$ and $f_t(\cdot) \in C_b$ with $b < 4$. We need the following noise conditions.

A1) $\{w_t, \mathcal{F}_t\}$ is a martingale difference sequence where $\{\mathcal{F}_t\}$ is a nondecreasing sequence of σ-algebras with

$(\theta, \phi_0) \in \mathcal{F}_0$. Also, assume that

$$\sup_t E[|w_{t+1}|^\beta | \mathcal{F}_t] < \infty \quad \text{a.s.} \quad \text{for some } \beta > 2 \tag{21}$$

$$\sum_{i=1}^{t} |w_i|^{2b} = O(t), \quad \text{a.s.} \tag{22}$$

and

$$|w_{t+1}|^{2b} = O(1) + o(r_t) \quad \text{a.s.} \quad \text{as } t \to \infty \tag{23}$$

where b is defined as in (13), and r_t is defined by (19) with ϕ_t defined by (9).

Theorem 2.2: Consider the stochastic control system (6) with $\theta \in \mathbb{R}^1$. Assume that the nonlinear function $f_t(\cdot)$ satisfies condition (13) with $b < 4$. Assume also that the noise condition A1) is satisfied. Then, the closed-loop adaptive control system described by (6)–(10) is globally stable and the tracking error $\{y_i - y_i^*\}$ satisfies

$$\sum_{i=1}^{n} (y_i - y_i^* - w_i)^2 = O(\log n) \quad \text{a.s.} \tag{24}$$

The proof is given in Section IV.

Remark 2.5: The phenomenon that $b = 4$ is a critical case for global stabilizability is mainly determined by: 1) the inherent nonlinear structure of the systems to be controlled; 2) the uncertainties of the system parameter; and 3) the random noises involved. It does not depend on the performance index to be minimized. It does not even depend on the particular LS algorithm used in the analysis (see Remark 2.2). From the analysis in the next two sections, it can be seen that the asymptotic behavior of the closed-loop control system essentially hinges upon a two-dimensional linear map whose behavior is determined mainly by the roots of the quadratic equation: $z^2 - bz + b = 0$, $b > 0$. Note that $b = 4$ is precisely the critical case for this equation to have real or complex roots.

Remark 2.6: In the noise-free case where in (6) $w_t \equiv 0$, it has been shown in [28] that the critical case for global stability of the closed-loop system (6)–(10) with $\theta \in \mathbb{R}^1$ is $b = 8$, where b is defined in (13). Consequently, by Theorem 2.1 we find that in the case where $b \in [4, 8)$, although the LS-based control algorithm (7)–(10) can stabilize the (deterministic) system (6) with $w_t \equiv 0$, it cannot stabilize the actual stochastic system (6) in general. This instability phenomenon is different from those already known in the literature of robust adaptive control (cf., e.g., [30]), since here we are concerned with the standard LS-based control algorithms together with *white noise* disturbances.

Remark 2.7: Condition (23) can be verified also for a large class of systems with possibly unbounded noises. In fact, if $\liminf_{t\to\infty} 1/t \sum_{i=1}^t w_i^2 \neq 0$ a.s., then by using the martingale convergence techniques as used in, e.g., [15, p. 448], it can be shown that for any $u_t \in \mathcal{F}_t$, (6) gives $\liminf_{t\to\infty} 1/t \sum_{i=1}^t y_{i+1}^2 \neq 0$ a.s. This can then be used in guaranteeing (23). For example, for the case of (14) we have $\phi_t = y_t^b$. Hence, similar to (20) we know that $t = O(r_t)$ a.s. Consequently, (23) is implied by $|w_{t+1}|^{2b} = o(t)$, a.s., which in turn is implied by, for example, the assumption that $\{w_i\}$ is i.i.d. with $E|w_1|^{2b} < \infty$.

III. The Proof of Instability

The proof of Theorem 2.1 is divided into several lemmas. Throughout this section, the system and controller are defined by (14) and (15), and the sets D_1 and D_2 are defined by (16) and (17) with $b \geq 4$. To facilitate the instability analysis, we need to establish a chain of inequalities satisfied by $\{r_i\}$, and this is the content of our first lemma.

Lemma 3.1: On the set $D_3 \triangleq D_2 \cap \{2^b(i+1)^2 \leq r_i, i \leq t+1\}$ we have

$$r_{i+1} \geq \left(\frac{\alpha_i}{3}\right)^b \left(\frac{r_i}{r_{i-1}}\right)^b, \qquad 0 \leq i \leq t$$

where $\alpha_i \triangleq (\phi_i \tilde{\theta}_i)^2/(1 + \phi_i^2 P_i)$ and r_i is defined by (19).

Proof: First, note that $y_t^* \equiv 0$, so by (11)

$$y_{i+1} = \phi_i \tilde{\theta}_i + w_{i+1}, \qquad \forall i \geq 0. \tag{25}$$

Note also that by (19)

$$1 + \phi_i^2 P_i = \frac{r_i}{r_{i-1}}, \qquad i \geq 0. \tag{26}$$

It follows from (25) and (26) that

$$(y_{i+1} - w_{i+1})^2 = \alpha_i \frac{r_i}{r_{i-1}}, \qquad \forall i \geq 0. \tag{27}$$

But, by the fact that $y_i^{2b} = \phi_i^2 \leq r_i$, we have on D_3

$$\begin{aligned}(y_{i+1} - w_{i+1})^2 &\leq 2y_{i+1}^2 + 2w_{i+1}^2 \leq 2[y_{i+1}^2 + (i+2)^{2/b}] \\ &\leq 2\left[y_{i+1}^2 + \tfrac{1}{2} r_{i+1}^{1/b}\right] \leq 2\left[r_{i+1}^{1/b} + \tfrac{1}{2} r_{i+1}^{1/b}\right] \\ &= 3r_{i+1}^{1/b}, \qquad 0 \leq i \leq t.\end{aligned}$$

From this and (27) it follows that $3r_{i+1}^{1/b} \geq \alpha_i (r_i/r_{i-1})$, $0 \leq i \leq t$. This completes the proof. □

To analyze the chain of inequalities in Lemma 3.1, we need to find a lower bound to $\{\alpha_i\}$, and this is done in the following lemma.

Lemma 3.2: If for some $t \geq 0$

$$\frac{r_i}{r_{i-1}} \geq 4(i+2)^4 + 1, \qquad 0 \leq i \leq t$$

then on $D = D_1 \cap D_2$, we must have

$$\alpha_i \geq \frac{3}{13(i+1)^{5/2}}, \qquad 0 \leq i \leq t$$

where α_i is defined in Lemma 3.1.

Proof: Since $|\tilde{\theta}_0| \leq 1$, and on D_2, $\max_{1 \leq j \leq i-1} |w_j|^2 \leq i^{2/b} \leq \sqrt{i}$, we have by the Schwarz inequality $(1 \leq i \leq t)$

$$\begin{aligned}\left\{\tilde{\theta}_0 - \sum_{j=0}^{i-2} \phi_j w_{j+1}\right\}^2 &\leq i\left\{(\tilde{\theta}_0)^2 + \sum_{j=0}^{i-2} \phi_j^2 w_{j+1}^2\right\} \\ &\leq i\left\{1 + \sqrt{i} \sum_{j=0}^{i-2} \phi_j^2\right\} \leq i^{3/2} r_{i-2}.\end{aligned}$$

Consequently, by $|w_i|^2 \geq (i+1)^{-(5/2)}$ we have

$$\begin{aligned}(i+1)^{-(5/2)}\phi_{i-1}^2 &\leq (\phi_{i-1}w_i)^2\\ &\leq 2\left\{\tilde{\theta}_0 - \sum_{j=0}^{i-1}\phi_j w_{j+1}\right\}^2 + 2\left\{\tilde{\theta}_0 - \sum_{j=0}^{i-2}\phi_j w_{j+1}\right\}^2\\ &\leq 2\left\{\tilde{\theta}_0 - \sum_{j=0}^{i-1}\phi_j w_{j+1}\right\}^2 + 2i^{3/2}r_{i-2}, \qquad (1 \leq i \leq t)\end{aligned}$$

or

$$\left\{\tilde{\theta}_0 - \sum_{j=0}^{i-1}\phi_j w_{j+1}\right\}^2 \geq \frac{\phi_{i-1}^2}{2(i+1)^{5/2}} - i^{3/2}r_{i-2}, \qquad (1 \leq i \leq t). \tag{28}$$

Now, by the assumption we have

$$\frac{r_{i-1}+\phi_i^2}{r_{i-1}} \geq 4(i+2)^4+1, \qquad 0 \leq i \leq t$$

or

$$\phi_i^2 \geq 4(i+2)^4 r_{i-1}, \qquad 0 \leq i \leq t. \tag{29}$$

Hence, by (28) and (29) we obtain

$$\begin{aligned}\left\{\tilde{\theta}_0 - \sum_{j=0}^{i-1}\phi_j w_{j+1}\right\}^2 &\geq \frac{\phi_{i-1}^2}{4(i+1)^{5/2}} + \left[\frac{\phi_{i-1}^2}{4(i+1)^{5/2}} - i^{3/2}r_{i-2}\right]\\ &\geq \frac{\phi_{i-1}^2}{4(i+1)^{5/2}}, \qquad 1 \leq i \leq t.\end{aligned}$$

Consequently, by (18) and $P_0 = 1$, we obtain

$$(\tilde{\theta}_i)^2 \geq \frac{\phi_{i-1}^2}{4r_{i-1}^2(i+1)^{5/2}}, \qquad 1 \leq i \leq t.$$

From this, by the definition of α_i and (26) we have for $1 \leq i \leq t$

$$\alpha_i = (\phi_i\tilde{\theta}_i)^2\frac{r_{i-1}}{r_i} \geq \frac{\phi_i^2\phi_{i-1}^2}{4r_i r_{i-1}(i+1)^{5/2}}. \tag{30}$$

Now, by (29)

$$\begin{aligned}1 = \frac{\phi_i^2}{r_i}\left(1+\frac{r_{i-1}}{\phi_i^2}\right) &\leq \frac{\phi_i^2}{r_i}\left(1+\frac{1}{4(i+2)^4}\right)\\ &\leq \frac{\phi_i^2}{r_i}\left(1+\frac{1}{4\times 2^4}\right) = \frac{\phi_i^2}{r_i}\cdot\frac{1+4^3}{4^3}.\end{aligned}$$

So

$$\frac{\phi_i^2}{r_i} \geq \frac{4^3}{1+4^3}, \qquad 0 \leq i \leq t.$$

Substituting this into (30) we obtain

$$\begin{aligned}\alpha_i &\geq \frac{1}{4(i+1)^{5/2}}\left(\frac{4^3}{1+4^3}\right)^2\\ &> \frac{3}{13(i+1)^{5/2}}, \qquad 1 \leq i \leq t.\end{aligned} \tag{31}$$

Also, since for $i = 0$, $|y_0| > (10)^{4/b}$, $|\tilde{\theta}_0| \geq \frac{1}{2}$, we have

$$\alpha_0 = \frac{(\tilde{\theta}_0)^2 y_0^{2b}}{1+y_0^{2b}} \geq \frac{(\tilde{\theta}_0)^2 10^8}{1+10^8} > \frac{\left(\frac{1}{2}\right)^2\times 12}{1+12} = \frac{3}{13}.$$

So (31) is also true for $i = 0$, and hence the proof is completed. □

The following lemma is the key leading to the instability proof of the adaptive systems.

Lemma 3.3: On the set D we have

$$\frac{r_i}{r_{i-1}} \geq 4(i+2)^4+1, \qquad \forall i \geq 0.$$

Proof: We use induction. For $i = 0$

$$\frac{r_0}{r_{-1}} = r_0 = 1+y_0^{2b} > 1+(10)^8 > 1+2^6$$

and the assertion holds for $i = 0$. Now, assume that on D

$$\frac{r_i}{r_{i-1}} \geq 4(i+2)^4+1, \qquad \forall i{:}\, 0 \leq i \leq t \tag{32}$$

holds for some $t \geq 0$. We need to prove that this inequality also holds for $i = t+1$.

Let us first show that for D_3 defined in Lemma 3.1 we have $D_3 = D_2$ or

$$2^b(i+1)^2 \leq r_i, \qquad 0 \leq i \leq t+1. \tag{33}$$

For $i = 0, 1$, we have by $|y_0| \geq 2$ and $b \geq 4$

$$r_1 \geq r_0 = 1+y_0^{2b} \geq 1+2^{2b} \geq 2^b(i+1)^2.$$

Now, for $1 \leq i \leq t$, by (32)

$$r_{i+1} \geq r_i \geq r_{i-1}[4(i+2)^4+1] > r_0(i+2)^2 > 2^b(i+2)^2.$$

Hence, (33) is true.

Therefore, by Lemmas 3.1 and 3.2 we have on D

$$\begin{aligned}r_{i+1} &\geq \left(\frac{\alpha_i}{3}\right)^b\left(\frac{r_i}{r_{i-1}}\right)^b\\ &\geq \left(\frac{1}{13(i+1)^{5/2}}\right)^b\left(\frac{r_i}{r_{i-1}}\right)^b, \qquad 0 \leq i \leq t.\end{aligned} \tag{34}$$

In order to analyze (34), we introduce the following notations:

$$\lambda_1 = \tfrac{1}{2}\left(b+\sqrt{b^2-4b}\right) \tag{35}$$

$$\lambda_2 = \tfrac{1}{2}\left(b-\sqrt{b^2-4b}\right) \tag{36}$$

$$z_i = x_i - \lambda_2 x_{i-1}, \qquad x_i = \log r_i. \tag{37}$$

Then, by taking logarithm on both sides of (34) we have

$$x_{i+1} \geq b(x_i - x_{i-1}) - b\log 13 - \frac{5b}{2}\log(i+1), \qquad 0 \leq i \leq t.$$

From this and the fact that $\lambda_1+\lambda_2 = \lambda_1\lambda_2 = b$, it follows that

$$z_{i+1} \geq \lambda_1 z_i - b\log 13 - \frac{5b}{2}\log(i+1), \qquad 0 \leq i \leq t \tag{38}$$

where the initial condition is

$$z_0 = x_0 - \lambda_2 x_{-1} = \log(1+y_0^{2b}) - \lambda_2 \log 1 = \log(1+y_0^{2b}). \tag{39}$$

By iterating (38), we have

$$z_{t+1} \ge \lambda_1^{t+1} z_0 - b\sum_{i=0}^{t} \lambda_1^{(t-i)}\left[\log 13 + \frac{5}{2}\log(i+1)\right] = \lambda_1^{t+1}\left\{z_0 - b\sum_{i=0}^{t}\frac{1}{\lambda_1^{i+1}}\left[\log 13 + \frac{5}{2}\log(i+1)\right]\right\}. \tag{40}$$

Now, we proceed to analyze the right-hand side of (40). First, it is easy to verify that

$$\sum_{i=0}^{\infty}\frac{1}{\lambda_1^{i+1}} = \frac{1}{\lambda_1 - 1}, \quad \sum_{i=0}^{\infty}\frac{i+1}{\lambda_1^{i}} = \left(\frac{\lambda_1}{\lambda_1-1}\right)^2. \tag{41}$$

So, by the convexity of e^x

$$\exp\left\{\sum_{i=0}^{\infty}\frac{\lambda_1-1}{\lambda_1^{i+1}}\log(i+1)\right\} \le \sum_{i=0}^{\infty}\frac{\lambda_1-1}{\lambda_1^{i+1}}\exp\{\log(i+1)\} = \frac{\lambda_1-1}{\lambda_1}\sum_{i=0}^{\infty}\frac{i+1}{\lambda_1^{i}} = \frac{\lambda_1}{\lambda_1-1}$$

which implies that

$$\sum_{i=0}^{\infty}\frac{\log(i+1)}{\lambda_1^{i+1}} \le \frac{1}{\lambda_1-1}\log\left(\frac{\lambda_1}{\lambda_1-1}\right). \tag{42}$$

Also, note that for any $b \ge 4$, $4(\lambda_1 - 1) \ge b$ and $\lambda_1/(\lambda_1-1) \le 2$. We then have by (41) and (42)

$$b\sum_{i=0}^{\infty}\frac{1}{\lambda_1^{i+1}}\left[\log 13 + \frac{5}{2}\log(i+1)\right] \le \frac{b}{\lambda_1-1}\left\{\log 13 + \frac{5}{2}\log\left(\frac{\lambda_1}{\lambda_1-1}\right)\right\} \le 4\log(13\times 2^{5/2}) = \log(13^4\times 2^{10}). \tag{43}$$

Next, by (39)

$$z_0 - \log(13^4\times 2^{10}) > \log\left(\frac{y_0^{2b}}{13^4\times 2^{10}}\right) = 2\log\left(\frac{|y_0|^b}{13^2\times 2^5}\right) \ge 2\log\left(\frac{(10^{4/b}+1)^b}{13^2\times 2^5}\right) \ge 2\log\left(\frac{11^4}{13^2\times 2^5}\right) \ge \frac{3}{2} \tag{44}$$

where for the second to the last inequality we have used the fact that $(10^{4/b}+1)^b$ is an increasing function of b and $b \ge 4$.

Substituting (43) and (44) into (40), we get $z_{t+1} \ge \frac{3}{2}\lambda_1^{t+1}$. Hence, by the definition of z_t and x_t in (37), we have $x_{t+1} - \lambda_2 x_t \ge \frac{3}{2}\lambda_1^{t+1}$ or $\log r_{t+1} - \lambda_2 \log r_t \ge \frac{3}{2}\lambda_1^{t+1}$. From this it follows that

$$\frac{r_{t+1}}{r_t} \ge r_t^{\lambda_2 - 1}\exp\left\{\frac{3}{2}\lambda_1^{t+1}\right\}. \tag{45}$$

Now, since $\lambda_2 > 1$

$$r_t^{\lambda_2-1} \ge r_0^{\lambda_2-1} > (y_0^{2b})^{\lambda_2-1} \ge (10^8)^{\lambda_2-1} > e^{18(\lambda_2-1)}.$$

Hence, by (45)

$$\frac{r_{t+1}}{r_t} \ge \exp\left\{\frac{3}{2}\lambda_1^{t+1} + 18(\lambda_2-1)\right\}.$$

Consequently, the proof of Lemma 3.3 will be completed by directly applying the following lemma.

Lemma 3.4: Let λ_1 and λ_2 be defined by (35) and (36) with $b \ge 4$. Then

$$\exp\left\{\tfrac{3}{2}\lambda_1^{t+1} + 18(\lambda_2-1)\right\} \ge 4(t+3)^4 + 1, \qquad \forall t \ge 0.$$

Proof: Note that $\lambda_1 + \lambda_2 = b$ and $\lambda_2 > 1$, so we have

$$\begin{aligned}&\tfrac{1}{2}\lambda_1 + 18(\lambda_2 - 1)\\ &\quad > \tfrac{1}{2}(\lambda_1+\lambda_2) - \tfrac{1}{2} + 17(\lambda_2-1)\\ &\quad = \frac{b}{2} - \frac{1}{2} + 17\frac{4b}{(b+\sqrt{b^2-4b})^2}\\ &\quad \ge \frac{b}{2} - \frac{1}{2} + \frac{17}{b} \ge 5.\end{aligned}$$

From this it follows that

$$\begin{aligned}&\exp\left\{\tfrac{3}{2}\lambda_1^{t+1} + 18(\lambda_2-1)\right\}\\ &\quad \ge \exp\left\{\tfrac{1}{2}\lambda_1 + 18(\lambda_2-1)\right\}\exp\left\{\lambda_1^{t+1}\right\}\\ &\quad \ge e^5\cdot\exp\left\{\lambda_1^{t+1}\right\}.\end{aligned}$$

Note that $\lambda_1 \ge 2$, so we only need to prove that

$$e^5\exp\{2^{t+1}\} \ge 4(t+3)^4 + 1, \qquad \forall t \ge 0.$$

For $t = 0$, this can be verified easily. For $t \ge 1$, by the Taylor expansion we have

$$e^5\exp\{2^{t+1}\} \ge e^5\exp(t+3) > 1 + 4(t+3)^4.$$

Hence, the proof is completed. □

Proof of Theorem 2.1: By Lemma 3.3 we have

$$r_t^{1/4} > (t+2)r_{t-1}^{1/4} > \prod_{i=0}^{t}(i+2), \qquad \forall t \ge 0.$$

From this, we have by the Stirling's formula

$$1 + \sum_{i=0}^{t} y_i^2 \ge \left(1 + \sum_{i=0}^{t} y_i^{2b}\right)^{1/b} = r_t^{1/b} > \left\{\prod_{i=0}^{t}(i+2)\right\}^{4/b} \sim \left\{\sqrt{2\pi(t+2)}\left(\frac{t+2}{e}\right)^{t+2}\right\}^{4/b}.$$

Hence, the proof of Theorem 2.1 is completed. □

IV. The Proof of Global Stability

The proof of Theorem 2.2 is prefaced with three lemmas. A crucial ingredient in this proof is the following useful technical lemma on growth rate of nonlinear recursions.

Lemma 4.1: Let $\{S_t\}$ and $\{C_t\}$ be two positive nondecreasing sequences such that

$$S_{t+1} \le C_t + \sum_{i=1}^{t} \delta_i \frac{S_i^L}{S_{i-1}^M}, \qquad \forall t \ge 0 \tag{46}$$

and that $C_t = O(S_t)$, where $L > 0$, $\delta_i \ge 0$, and $\sum_{i=1}^{\infty} \delta_i < \infty$. If $L^2 < 4M$, then $\{S_{t+1}/S_t\}$ is bounded and $S_t = O(C_t)$.

The proof is given in Appendix C.

Remark 4.1: If $C_t \equiv C$ (a constant), then the conclusion of Lemma 4.1 implies that $\{S_t\}$ is bounded. This case was considered in [28], where it was also shown that the condition $L^2 < 4M$ cannot be relaxed in general.

Lemma 4.2: Under the conditions of Theorem 2.2, if $r_t \to \infty$ but $\liminf_{t\to\infty} r_t/t = 0$, then

$$\sup_t \frac{r_{t+1}}{r_t} < \infty$$

where r_t is defined by (19) with ϕ_i defined by (9).

Proof: First of all, we prove that there exists a subsequence $\{\tau_n\}$ such that

$$|\phi_{\tau_n}|^2 + |\phi_{\tau_n+1}|^2 + \cdots + |\phi_{\tau_n+p-1}|^2 \underset{n\to\infty}{\longrightarrow} 0. \tag{47}$$

Let $[x]$ denote the integer part of a real number $x > 0$, and let

$$L_i \triangleq |\phi_{ip}|^2 + \cdots + |\phi_{(i+1)p-1}|^2, \qquad \forall i \ge 0.$$

Then we have

$$\sum_{i=0}^{[t/p]-1} L_i \le \sum_{i=0}^{t} |\phi_i|^2, \qquad \forall t \ge p.$$

Hence by $\liminf_{t\to\infty}(r_t/t) = 0$, we have $\liminf_{t\to\infty} \sum_{i=0}^{[t/p]-1} L_i/[t/p] = 0$. This implies that $\liminf_{i\to\infty} L_i = 0$, and hence (47) is proved.

By (47), we can take n large enough such that

$$\frac{|\phi_{\tau_n+i}|^2}{r_{\tau_n+i-1}} \le 1, \qquad i = 0, 1, \cdots, p-1. \tag{48}$$

Now, let us denote $\alpha_i = (\phi_i^\tau \tilde{\theta}_i)^2/(1 + \phi_i^\tau P_i \phi_i)$, then by [15, Corollary 3.1], we see that $\alpha_i = O(\log r_i)$ a.s. $\forall i \ge 0$. By this, (23), and the property $r_t \to \infty$, it follows that there is an integer n large enough such that

$$\begin{aligned} &2k_1^2 + 2k_2^2 \left\{ \sum_{j=\tau_n+i-p}^{\tau_n+i-1} [4\alpha_j + 2(y_{j+1}^* + w_{j+1})^2] \right\}^b \\ &\quad \le r_{\tau_n+i-1}, \qquad \forall i \ge 1 \end{aligned} \tag{49}$$

where k_1 and k_2 are the constants defined in (13) (this inequality is obvious since when both sides of (49) are divided by r_{τ_n+i-1}, the left-hand side will tend to zero as $n \to \infty$).

Now, we take n large enough such that both (48) and (49) hold. For this fixed n, we proceed to prove that

$$\begin{aligned} y_{\tau_n+i+1}^2 &\le 4\alpha_{\tau_n+i} + 2(y_{\tau_n+i+1}^* + w_{\tau_n+i+1})^2 \\ &\forall i \ge 0. \end{aligned} \tag{50}$$

First, we prove (50) for $0 \le i \le p-1$.

By (11) and the definition of α_i we have

$$\begin{aligned} y_{\tau_n+i+1}^2 &\le 2\alpha_{\tau_n+i} \frac{r_{\tau_n+i}}{r_{\tau_n+i-1}} + 2(y_{\tau_n+i+1}^* + w_{\tau_n+i+1})^2, \\ &\forall i \ge 0. \end{aligned} \tag{51}$$

For $0 \le i \le p-1$, by (48) we know that

$$\frac{r_{\tau_n+i}}{r_{\tau_n+i-1}} = 1 + \frac{|\phi_{\tau_n+i}|^2}{r_{\tau_n+i-1}} \le 2$$

substituting this into (51), we then get (50) for $0 \le i \le p-1$.

Next, we complete the proof of (50) by induction. Suppose that for some $t \ge p-1$ (50) holds for all $i \le t$. Then by (13) and (49) we have

$$\begin{aligned} \frac{|\phi_{\tau_n+t+1}|^2}{r_{\tau_n+t}} &\le \frac{1}{r_{\tau_n+t}} \left\{ 2k_1^2 + 2k_2^2 \left[\sum_{j=\tau_n+t-p+2}^{\tau_n+t+1} y_j^2 \right]^b \right\} \\ &\le \frac{1}{r_{\tau_n+t}} \left\{ 2k_1^2 + 2k_2^2 \left[\sum_{j=\tau_n+t-p+2}^{\tau_n+t+1} \right.\right. \\ &\qquad \left.\left. \cdot\ (4\alpha_{j-1} + 2(y_j^* + w_j)^2) \right]^b \right\} \\ &\le 1 \end{aligned} \tag{52}$$

and so $(r_{\tau_n+t+1})/(r_{\tau_n+t}) \le 2$. Consequently, substituting this into (51) we have

$$y_{\tau_n+t+2}^2 \le 4\alpha_{\tau_n+t+1} + 2(y_{\tau_n+t+2}^* + w_{\tau_n+t+2})^2.$$

Thus, (50) also holds for $i = t+1$. This completes the induction argument for (50). By (50) we know that (52) actually holds for all $t \ge p-1$ and consequently

$$\frac{r_{\tau_n+t+1}}{r_{\tau_n+t}} \le 2, \qquad \text{for all } t \ge p-1$$

which certainly means that $\sup_t (r_{t+1})/r_t < \infty$. □

Lemma 4.3: Consider (6) and the LS algorithm (7)–(9). Assume that the noise condition A1) is satisfied [(22) and (23) are not necessary here]. Then we have

$$\sum_{i=1}^{\infty} \frac{\alpha_i^r}{r_i^\epsilon} < \infty, \qquad \text{a.s. } \forall r \ge 1,\ \forall \epsilon > 0$$

where α_i and r_i are defined, respectively, by

$$\alpha_i = \frac{(\phi_i^\tau \tilde{\theta}_i)^2}{1 + \phi_i^\tau P_i \phi_i} \tag{53}$$

and

$$r_i = \|P_0^{-1}\| + \sum_{j=0}^{i} \|\phi_j\|^2. \tag{54}$$

Proof: Consider the Lyapunov function $W_i = V_i/r_i^\epsilon$ where $V_i = \tilde{\theta}_i^\tau P_i^{-1}\tilde{\theta}_i$. By (6)–(9) we have the following standard relationship (cf., e.g., [10, p. 808]):

$$V_{i+1} = V_i - \alpha_i - 2\,\frac{\phi_i^\tau \tilde{\theta}_i}{1+\phi_i^\tau P_i \phi_i}\, w_{i+1} + \phi_i^\tau P_{i+1}\phi_i w_{i+1}^2.$$

Hence, we have

$$\begin{aligned} W_{i+1} \le\, & W_i - \frac{\alpha_i}{r_i^\epsilon} - \left(\frac{2}{r_i^\epsilon}\right)\cdot \frac{\phi_i^\tau \tilde{\theta}_i}{1+\phi_i^\tau P_i\phi_i}\, w_{i+1} \\ & + \frac{\phi_i^\tau P_{i+1}\phi_i}{r_i^\epsilon}\, w_{i+1}^2. \end{aligned} \tag{55}$$

Now, it can readily be shown that $\sum_{i=1}^\infty (\phi_i^\tau P_{i+1}\phi_i)/r_i^\epsilon < \infty, \forall\, \epsilon > 0$. Hence, similar to the proof of [15, Corollary 3.1 (i)], it can be derived from (55) that

$$\sum_{i=1}^\infty \frac{\alpha_i}{r_i^\epsilon} < \infty \quad \text{a.s.} \quad \forall \epsilon > 0. \tag{56}$$

Next, by the fact that (cf., [15, p. 438]) $\alpha_i = O(\log r_i)$, we know that

$$\alpha_i^{r-1} = O(r_i^\epsilon), \qquad \forall\, r \ge 1, \forall\, \epsilon > 0. \tag{57}$$

Hence by the arbitrariness of ϵ in (56) and (57) we get

$$\sum_{i=1}^\infty \frac{\alpha_i^r}{r_i^\epsilon} = \sum_{i=1}^\infty \left(\frac{\alpha_i}{r_i^{\epsilon/2}}\right)\cdot\left(\frac{\alpha_i^{r-1}}{r_i^{\epsilon/2}}\right) < \infty$$

and this completes the proof. □

Proof of Theorem 2.2: First of all, we show that if

$$\sup_t \frac{r_{t+1}}{r_t} < \infty \tag{58}$$

then Theorem 2.2 will follow immediately. By the fact that (cf., [15, p. 438])

$$\sum_{i=0}^t \alpha_i = O(\log r_t), \qquad \alpha_i = \frac{(\phi_i^\tau \tilde{\theta}_i)^2}{1+\phi_i^\tau P_i \phi_i} \tag{59}$$

we have from (11) and (58)

$$\begin{aligned} \sum_{t=0}^n (y_{t+1} - y_{t+1}^* - w_{t+1})^2 &= \sum_{t=0}^n \alpha_t\, \frac{r_t}{r_{t-1}} \\ &= O(\log r_n). \end{aligned} \tag{60}$$

By (13) we see that

$$\begin{aligned} \log r_n &= \log\left(P_0^{-1} + \sum_{i=1}^n \phi_i^2\right) \\ &\le \log\left(n + \sum_{i=1}^n \|[y_i \cdots y_{i-p+1}]\|^{2b}\right) + O(1) \\ &= O\left(\log\left(1+\sum_{i=0}^n y_i^2\right)\right) + O(\log n). \end{aligned} \tag{61}$$

Substituting this into (60) we are easily convinced of the fact that $\sum_{i=0}^n y_i^2 = O(n)$. Consequently, by (60) and (61) we get the desired result (24).

Next, we prove (58) by considering the following three cases separately.

Case i): $\lim_{t\to\infty} r_t < \infty$. In this case, (58) holds trivially since $\{r_t\}$ is nondecreasing.

Case ii): $\lim_{t\to\infty} r_t = \infty$, and $\liminf_{t\to\infty} r_t/t = 0$. By Lemma 4.2, (58) holds again in this case.

Case iii): $\liminf_{t\to\infty} r_t/t \ne 0$. In this case, $t = O(r_t)$, as $t \to \infty$. Similar to (51) we have

$$y_{i+1}^2 \le 2\alpha_i\, \frac{r_i}{r_{i-1}} + 2(y_{i+1}^* + w_{i+1})^2, \qquad \forall\, i \ge 0.$$

So, by conditions (13) and (22) we have

$$\begin{aligned} r_{t+1} &= P_0^{-1} + \sum_{i=0}^{t+1} \phi_i^2 \\ &\le P_0^{-1} + 2k_1^2(t+2) + 2k_2^2 \sum_{i=0}^{t+1} \|[y_i \cdots y_{i-p+1}]\|^{2b} \\ &= O(t) + O\left(\sum_{i=0}^t y_{i+1}^{2b}\right) \\ &= O(t) + O\left(\sum_{i=1}^t \left[\alpha_i\left(\frac{r_i}{r_{i-1}}\right) + (y_{i+1}^* + w_{i+1})^2\right]^b\right) \\ &= O(t) + O\left(\sum_{i=1}^t \alpha_i^b \left(\frac{r_i}{r_{i-1}}\right)^b\right) \\ &= O(t) + O\left(\sum_{i=1}^t \delta_i \left(\frac{r_i^{b+\epsilon}}{r_{i-1}^b}\right)\right), \qquad \forall\, \epsilon > 0 \end{aligned} \tag{62}$$

where $\delta_i = \alpha_i^b/r_i^\epsilon$ (without loss of generality we may assume that $b \ge 1$).

Now, applying Lemma 4.3, we know that $\sum_{i=1}^\infty \delta_i < \infty$ a.s. $\forall\, \epsilon > 0$. Also, since $0 < b < 4$, we can take ϵ small enough such that $0 < \epsilon < \sqrt{b}(2 - \sqrt{b})$. Therefore, applying Lemma 4.1 to (62) we know that (58) is also true. This completes the proof of Theorem 2.2. □

V. Concluding Remarks

We have in this contribution found and proved the critical stability of a class of discrete-time adaptive nonlinear stochastic control systems. The implications of our results include the following: 1) in the nonlinear case, strongly consistent LS estimates may not ensure global stability of the certainty equivalence adaptive control; 2) adaptive control laws that are designed based on the Weierstrass approximation (or Taylor expansion) of nonlinear stochastic systems may not be feasible in general; 3) control schemes (including the LS-based ones) that have been proved to be stable in the noise-free case may indeed lose their stability in the presence of zero mean bounded white noises; and 4) some chaotic dynamical systems can be adaptively controlled to follow a desired orbit under certain noisy environments. The results of this paper also indicate what may be done and what cannot be done for more general nonlinear stochastic systems.

Both the new results and the analytical methods provided in this paper may be regarded as a start toward a more comprehensive investigation and understanding of adaptive nonlinear stochastic control systems. Many interesting problems remain open even for the seemingly simple nonlinear model (1) and

(2); for example, it would be of interest to generalize Theorem 2.2 to the vector parameter case. Clearly, more efforts are needed in this challenging field.

APPENDIX A
ADAPTIVE CONTROL OF (3)

Denote

$$\theta = [a, b, c]^\tau \quad \text{and} \quad \phi_t = [y_t, u_t, y_t u_t]^\tau$$

then (3) can be written as a standard linear regression

$$y_{t+1} = \theta^\tau \phi_t + w_{t+1}.$$

Now, let $\theta_t \triangleq [a_t, b_t, c_t]^\tau$ be the WLS estimate for θ, which is recursively defined by [16, p. 80, eqs. (9)–(11)] (but with the present regressors). Similar to (64) and (65) in [15, pp. 443–444], we consider the following regularized WLS estimate for θ:

$$\hat{\theta}_t = \theta_t + P_t^{1/2} e_{i_t} \tag{63}$$

with

$$i_t = \underset{0 \le i \le 3}{\operatorname{argmax}} \, |c_t + e_3^\tau P_t^{1/2} e_i|$$

where P_t is the information matrix associated with WLS, $e_0 = 0$, and $\{e_i, i = 1, 2, 3\}$ is the natural orthogonal basis of $\mathbb{R}^3$, i.e., $[e_1, e_2, e_3] = I_{3\times 3}$.

Then we have the following result.

Lemma A.1: Consider the bilinear model (3). Let the noise process $\{w_t, \mathcal{F}_t\}$ be a martingale difference sequence with the conditional variance process almost surely bounded. Also let the input process $\{u_t\}$ be adapted to $\{\mathcal{F}_t\}$. Then the regularized WLS estimate defined by (63) has the following properties:

1) $\limsup_{t\to\infty} \|\hat{\theta}_t\| < \infty$ a.s.;

2) $\sum_{i=0}^{t} \|\tilde{\theta}_i^\tau \phi_i\|^2 = o(r_t) + O(1)$ a.s.;

3) $\inf_t |\hat{c}_t| \ge \epsilon$ a.s. for some $\epsilon > 0$;

where $\tilde{\theta}_t \triangleq \hat{\theta}_t - \theta$, $r_t = 1 + \sum_{i=0}^t \|\phi_i\|^2$ and $\hat{c}_t$ is the estimate for c given by $\hat{\theta}_t$.

Proof: The first two assertions follow directly from [16, Lemma 2], since that lemma actually holds for general linear regression models. So, we need only to prove 3). But, by using [16, Lemma 1 (i)] and the fact that $c \neq 0$, we can prove 3) by using the same technique as that used in the proof of [15, p. 444, Th. 6.3]. This completes the proof. □

Remark: In comparison with the SG estimate used in [18], the regularized WLS estimate not only has a faster convergence rate, but also can guarantee the Property 3) of Lemma A.1, which has previously been assumed as a condition in [18].

Now, following [18] we consider the control performance

$$J(u) = E[y_{t+1}^2 + \lambda u_t^2 | \mathcal{F}_t], \qquad \lambda > 0.$$

The certainty equivalence control is (cf., [18])

$$u_t = -\frac{\hat{a}_t y_t [\hat{c}_t y_t + \hat{b}_t]}{(\hat{c}_t y_t + \hat{b}_t)^2 + \lambda} \tag{64}$$

but here the estimates $\hat{a}_t$, $\hat{b}_t$, and $\hat{c}_t$ are defined by the regularized WLS estimate (63).

Note that the three properties of parameter estimates as listed in Lemma A.1 are sufficient for the stability analysis in [18] to carry through. Hence, following the proof of [18], we can obtain the following theorem.

Theorem A.1: Consider the bilinear system (3) where the noise process is a martingale difference sequence satisfying (21). Then the certainty equivalence control (64) defined by using the regularized WLS estimate (63) is stabilizing, i.e.,

$$\limsup_{t\to\infty} \frac{1}{t} \sum_{i=0}^{t} (y_i^2 + u_i^2) < \infty \quad \text{a.s.}$$

APPENDIX B
PROOF OF REMARK 2.2

Let $\{\theta, y_0, w_t, t \ge 0\}$ satisfy conditions of Remark 2.2. Then it is well known that the LS algorithm (7) and (8) with $P_0 = 1$ coincides with the standard Kalman filter, which generates the minimum variance estimate for θ and produces the conditional variance of the estimate, i.e.,

$$\theta_t = E[\theta | \mathcal{F}_t^y], \; P_t = E[(\tilde{\theta}_t)^2 | \mathcal{F}_t^y], \qquad t \ge 0$$

where $\mathcal{F}_t^y \triangleq \sigma\{y_i, i \le t\}$.

Now, let $u_t \in \mathcal{F}_t^y$ be any measurable feedback control. Then it is known (cf., [29]) that given $\mathcal{F}_t^y$, the conditional distribution of y_{t+1} is Gaussian with conditional mean and variance respectively given by

$$m_t \triangleq E[y_{t+1} | \mathcal{F}_t^y] = u_t + \theta_t \phi_t$$

and

$$\sigma_t^2 \triangleq \operatorname{Var}(y_{t+1} | \mathcal{F}_t^y) = \phi_t P_t \phi_t + 1.$$

Consequently, we have

$$\begin{aligned} E[y_{t+1}^2 | \mathcal{F}_t^y] &= m_t^2 + \sigma_t^2 \\ &= (u_t + \theta_t \phi_t)^2 + \phi_t P_t \phi_t + 1 \\ &\ge \phi_t P_t \phi_t + 1 \\ &= \frac{\phi_t^2}{r_{t-1}} + 1 = \frac{r_t}{r_{t-1}}, \qquad \forall t \ge 0, \quad \text{a.s.} \end{aligned} \tag{65}$$

where $r_t = 1 + \sum_{i=0}^t \phi_i^2$.

The idea behind the proof of nonstabilizability is as follows: if starting with (65) we can show that on a set with positive probability the relationship (34) appearing in the proof of Lemma 3.3 holds in the current situation, then similar to the proof of Theorem 2.1 we can show that $r_t \to \infty$ at a rate faster than exponential, resulting in the desired nonstabilizability. To this end, we need to get an appropriate upper bound on $E[y_{t+1}^2 | \mathcal{F}_t^y]$ first.

Denote

$$A_0 = \{\omega: |y_0| \geq 1 + (10)^{4/b}\},$$
$$A_t = \{\omega: E[y_t^2|\mathcal{F}_{t-1}^y] \leq 13t^{5/2}r_t^{1/b}\}, \qquad t \geq 1$$
$$D_0 = \bigcap_{t=0}^{\infty} A_t.$$

We now proceed to show that $P(D_0) > 0$.

Set

$$L_t(x) = \frac{\sqrt{1+x^2}}{\sqrt{13}(t+1)^{5/4}}, \qquad t \geq 0, x \in \mathbb{R}^1.$$

Since $r_{t+1} \geq y_{t+1}^{2b}$ and y_{t+1} is conditional Gaussian (with the conditional mean and variance denoted by m_t and σ_t^2, respectively), we have for any $t \geq 0$

$$\begin{aligned} &P(A_{t+1}^c|\mathcal{F}_t^y)\\ &\quad= P(E[y_{t+1}^2|\mathcal{F}_t^y] \geq 13(t+1)^{5/2}r_{t+1}^{1/b}|\mathcal{F}_t^y)\\ &\quad\leq P(\sigma_t^2 + m_t^2 \geq 13(t+1)^{5/2}y_{t+1}^2|\mathcal{F}_t^y)\\ &\quad= P\left(\left|\frac{y_{t+1}}{\sigma_t}\right| \leq L_t\left(\frac{m_t}{\sigma_t}\right)\middle|\mathcal{F}_t^y\right)\\ &\quad= \frac{1}{\sqrt{2\pi}}\int_{-L_t(x)-x}^{L_t(x)-x} e^{-(\lambda^2/2)}\,d\lambda\bigg|_{x=m_t/\sigma_t}\\ &\quad\leq \frac{1}{\sqrt{2\pi}}\sup_{x\in R^1}\int_{-L_t(x)-x}^{L_t(x)-x} e^{-(\lambda^2/2)}\,d\lambda \triangleq a_t. \end{aligned} \tag{66}$$

Now, by the mean value formula, for any $t \geq 0$ and $x \in \mathbb{R}^1$ there exists $\lambda_t(x) \in [-L_t(x) - x, L_t(x) - x]$ such that

$$a_t = \frac{1}{\sqrt{2\pi}}\sup_{x\in R^1}[e^{-(\lambda_t^2(x)/2)}2L_t(x)] \leq \frac{\sqrt{2}c}{\sqrt{13\pi}(t+1)^{5/4}}$$

where

$$c = \sup_{\substack{t\geq 0\\ x\in R^1}}(e^{-(\lambda_t^2(x)/2)}\sqrt{x^2+1})$$

which can easily be shown to be finite. Consequently, we have

$$\sum_{t=0}^{\infty} a_t < \infty. \tag{67}$$

Next, let $I(\cdot)$ be the indicator function of a set, then by the measurability $A_t \in \mathcal{F}_t^y$, $\forall t \geq 0$, we have by (66) and (67)

$$\begin{aligned} P\left(\bigcap_{i=0}^{t+1} A_i\right) &= E\prod_{i=0}^{t+1} I(A_i)\\ &= E\left\{E[I(A_{t+1})|\mathcal{F}_t^y]\prod_{i=0}^{t} I(A_i)\right\}\\ &= E\left\{[1 - P(A_{t+1}^c|\mathcal{F}_t^y)]\prod_{i=0}^{t} I(A_i)\right\}\\ &\geq (1-a_t)E\left\{\prod_{i=0}^{t} I(A_i)\right\} \geq \cdots\\ &\geq P(A_0)\prod_{i=0}^{t}(1-a_i)\\ &\geq P(A_0)\prod_{i=0}^{\infty}(1-a_i) > 0. \end{aligned}$$

Hence, we conclude that

$$P(D_0) = \lim_{t\to\infty} P\left(\bigcap_{i=0}^{t+1} A_i\right) > 0.$$

Finally by (65), we know that on D_0

$$r_{t+1} \geq \left(\frac{1}{13(t+1)^{5/2}}\right)^b \left(\frac{r_t}{r_{t-1}}\right)^b, \qquad \forall t \geq 0$$

which is similar to (34). Hence by the proofs of Lemma 3.3 and Theorem 2.1, we conclude that $r_t \to \infty$ on D_0 at a rate faster than exponential, and hence the desired nonstabilizability is proved.

Appendix C
Proof of Lemma 4.1

We first prove several auxiliary results.

Lemma C.1: Let $\{x_t\}$ be a nonnegative sequence satisfying

$$x_{t+1} \leq c + \sum_{i=0}^{t} \delta_i x_i, \qquad \forall t \geq 0$$

where $c > 0$ is a constant, $\delta_i \geq 0$, and $\sum_{i=0}^{\infty}\delta_i < \infty$. Then $\{x_t\}$ is bounded.

Proof: Denote $x_t^* = \max_{1\leq i\leq t} x_i$ and take $t_0 > 1$ large enough such that $\gamma \triangleq \sum_{i=t_0}^{\infty}\delta_i < 1$. Then we have

$$x_{t+1}^* \leq c + \sum_{i=0}^{t_0-1}\delta_i x_i + \gamma x_t^*, \qquad \forall t \geq t_0$$

and this implies that $\{x_t^*\}$ is bounded. □

Lemma C.2: Let $L > 2$ and recursively define a sequence $\{\lambda_i\}$ as follows:

$$\lambda_{i+1} = \frac{(L-1)\lambda_i - M}{\lambda_i - 1}, \qquad \lambda_0 = L-1,\, i \geq 0. \tag{68}$$

If $L^2 < 4M$, then there exists a finite integer i_0 such that $\lambda_{i_0} \leq (M-1)/(L-2)$.

Proof: First of all, $L^2 < 4M$ implies that

$$L < 2\sqrt{M} \leq 1 + M \tag{69}$$

which in turn implies that $(M-1)/(L-2) > M/(L-1)$. So we only need to prove that there exists a finite integer i_0 such that

$$\lambda_{i_0} \leq \frac{M}{L-1}. \tag{70}$$

Now, consider the following function:

$$f(\lambda) = \frac{(L-1)\lambda - M}{\lambda - 1}.$$

It is easy to see that $f(\lambda)$ is strictly increasing in $(M/(L-1), \infty)$ and $f(\infty) = L-1$.

Suppose that (70) were not true. Then we would have $\lambda_i > M/(L-1)$, $\forall i \geq 0$. Thus, by the increasing property of $f(\lambda)$ we would have

$$\begin{aligned} \lambda_1 &= f(\lambda_0) < f(\infty) = \lambda_0\\ \lambda_2 &= f(\lambda_1) < f(\lambda_0) = \lambda_1\\ \cdots &\quad \cdots. \end{aligned}$$

Continuing this argument, we would know that $\{\lambda_i\}$ is monotonically decreasing and has a lower bound $M/(L-1)$.

Thus, there must be a real number $\lambda^* \in [M/(L-1), \infty)$ such that $\lambda_i \longrightarrow_{i\to\infty} \lambda^*$, and by (68) λ^* must satisfy

$$\lambda^*(\lambda^* - 1) = (L-1)\lambda^* - M$$

or λ^* is the real root of the quadratic equation $\lambda^2 - L\lambda + M = 0$. However, the inequality $L^2 < 4M$ means that there is no real root for this equation, and we thus have a contradiction. Hence, the lemma is proved. □

Lemma C.3: Let $\{S_t\}$ and $\{\lambda_i\}$ be recursively defined in Lemma 4.1 and Lemma C.2 with $L > 2$. If $\lambda_{i+1} \in [1, L-1]$ for some $i \geq 0$, then boundedness of $\{S_{t+1}/S_t^{\lambda_i}\}$ implies boundedness of $\{S_{t+1}/S_t^{\lambda_{i+1}}\}$.

Proof: First of all, by (68)

$$\lambda_i = \frac{M - \lambda_{i+1}}{L - 1 - \lambda_{i+1}}.$$

By $S_j^{\lambda_{i+1}} \leq S_t^{\lambda_{i+1}}, \forall j \leq t$, and the inequality (46) for S_{t+1} we have

$$\begin{aligned}\frac{S_{t+1}}{S_t^{\lambda_{i+1}}} &\leq \frac{C_t}{S_t^{\lambda_{i+1}}} + \frac{1}{S_t^{\lambda_{i+1}}} \sum_{j=1}^{t} \delta_j \frac{S_j^L}{S_{j-1}^M} \\ &\leq C + \sum_{j=1}^{t} \delta_j \left(\frac{S_j^{L-\lambda_{i+1}}}{S_{j-1}^M}\right), \qquad \left(C = \sup_{t\geq 0} \frac{C_t}{S_t^{\lambda_{i+1}}}\right) \\ &\leq C + \sum_{j=1}^{t} \delta_j \left(\frac{S_j^{L-\lambda_{i+1}-1}}{S_{j-1}^{M-\lambda_{i+1}}}\right)\left(\frac{S_j}{S_{j-1}^{\lambda_{i+1}}}\right) \\ &\leq C + \sum_{j=1}^{t} \delta_j \left(\frac{S_j}{S_{j-1}^{\lambda_i}}\right)^{L-\lambda_{i+1}-1}\left(\frac{S_j}{S_{j-1}^{\lambda_{i+1}}}\right).\end{aligned}$$

Hence, the boundedness of $\{S_j/S_{j-1}^{\lambda_i}, j \geq 1\}$ implies the boundedness of $\{S_{t+1}/S_t^{\lambda_{i+1}}, t \geq 1\}$ by Lemma C.1. This completes the proof. □

Proof of Lemma 4.1: Without loss of generality, assume that $S_0 \geq 1$. We may also assume that $L \geq 1$ (since otherwise, Lemma C.1 can be applied to (46) to yield the desired result). By (69) we have

$$\frac{S_i^L}{S_{i-1}^M} \leq \left(\frac{S_i}{S_{i-1}}\right)^{L-1} \cdot S_i. \tag{71}$$

Hence, by (46)

$$\frac{S_{t+1}}{C_{t+1}} \leq 1 + \sum_{i=1}^{t} \delta_i \left(\frac{S_i}{S_{i-1}}\right)^{L-1} \cdot \frac{S_i}{C_i}, \qquad \forall i \geq 0.$$

From this and Lemma C.1, we know that if $\{S_i/S_{i-1}\}$ is bounded, then $S_{t+1}/C_{t+1} = O(1)$. Hence we only need to prove that $\{S_i/S_{i-1}\}$ is bounded. We consider two cases separately.

Case i) $1 < L \leq 2$: Note that (71) implies

$$\frac{S_i^{L-1}}{S_{i-1}^M} \leq \left(\frac{S_i}{S_{i-1}}\right)^{L-1} \leq \frac{S_i}{S_{i-1}}$$

and so by (46)

$$\frac{S_{t+1}}{S_t} \leq \frac{C_t}{S_t} + \sum_{i=1}^{t} \delta_i \frac{S_i^{L-1}}{S_{i-1}^M} \leq O(1) + \sum_{i=1}^{t} \delta_i \frac{S_i}{S_{i-1}}.$$

Hence, by Lemma C.1, $\{S_t/S_{t-1}\}$ is bounded.

Case ii) $L > 2$: Now, by (46) again we have

$$\frac{S_{t+1}}{S_t} \leq \frac{C_t}{S_t} + \sum_{i=1}^{t} \delta_i \left(\frac{S_i}{S_{i-1}^{\lambda}}\right)^{L-2}\left(\frac{S_i}{S_{i-1}}\right)$$
$$\lambda = \frac{M-1}{L-2}.$$

Hence, by Lemma C.1 we know that in order to prove boundedness of $\{S_{t+1}/S_t\}$, we only need to prove that $\{S_{t+1}/S_t^{\lambda}\}$ is bounded.

Now, consider the sequence $\{\lambda_i\}$ defined in Lemma C.2. Since $\lambda_0 = L-1$, by (69), (46), and Lemma C.1, it is easy to see that $\{S_{t+1}/S_t^{\lambda_0}\}$ is bounded. If $\lambda_0 \leq \lambda$, then obviously $\{S_{t+1}/S_t^{\lambda}\}$ is bounded. Otherwise, if $\lambda_0 > \lambda$, then by (68) it can easily be seen that $\lambda_1 \in [1, L-1)$, hence by Lemma C.3, we know that $\{S_{t+1}/S_t^{\lambda_1}\}$ is bounded. If $\lambda_1 \leq \lambda$, then the proof is finished. Otherwise, if $\lambda_1 > \lambda$, then similarly we can show that $\{S_{t+1}/S_t^{\lambda_2}\}$ is bounded $\cdots$. Continuing this argument, we know by Lemma C.2 that there must be a finite integer $i_0 \geq 0$ such that $\{S_{t+1}/S_t^{\lambda_{i_0}}\}$ is bounded and that $\lambda_{i_0} \leq \lambda$. Hence, $\{S_{t+1}/S_t^{\lambda}\}$ is bounded and the proof is completed.

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