In this case, neither of the roots of (31) lies on the imaginary axis.

For $\alpha_k > 0$ and $\alpha_m > 0$, because of the second expression in (32) at least one of two eigenvalues lies in the left half-plane. From the first expression in (31), the sum of the arguments of λ_{i_1} and λ_{i_2} has to be 2π. Thus other eigenvalues must lie in the left half-plane. Hence the system is stable. Thus an addition of dissipative force to the system in (29) does not destabilize the system.

For $\alpha_k = 0$ (static instability), one of the two eigenvalues will be at the origin.

For $\alpha_k < 0$ (with $\alpha_m > 0$), the sum of the arguments of λ_{i_1} and λ_{i_2} is π. Hence other eigenvalues must lie in the right-hand plane in order to satisfy the expressions in (32). Thus (27) stabilized by gyroscopic forces (29) for $\alpha_k < 0$ can be again destabilized by the addition of dissipative forces (31). For example, the top is stabilized by a gyroscopic moment as long as the spin is sufficiently large. Eventually the spin is decreased by friction (dissipative load) to make the top unstable.

V. Conclusions

In this paper, stability issues of matrix second-order dynamical systems are discussed. The necessary and sufficient conditions of asymptotic stability for time-invariant systems in matrix second-order form under various dynamic loadings (conservative/nonconservative) are derived and a physical interpretation is presented. The stability conditions in the sense of Lyapunov are also derived and analyzed. As the conditions are direct in terms of physical parameters of the system, the effect of different loadings on the system stability is lucid in the matrix second-order form approach. The conditions are shown to be useful in the designing controllers for matrix second-order systems.

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Fundamental Limitations of Discrete-Time Adaptive Nonlinear Control

Liang-Liang Xie and Lei Guo

***Abstract*—A particular polynomial is introduced in this paper which can be used to determine under what conditions a typical class of discrete-time nonlinear systems with uncertainties in both parameters and noises is not stabilizable by feedback, thus demonstrating the fundamental limitations of discrete-time adaptive nonlinear control. As a consequence, it is shown that for nonlinear systems with unknown parameters and noises, the systems may indeed be nonstabilizable, in general, whenever the usual linear growth condition is relaxed and the number of unknown parameters is large, even though the corresponding noise-free systems are globally stabilizable.**

***Index Terms*— Adaptive control, discrete-time, global stabilizability, nonlinear dynamics, stochastic systems.**

I. Introduction

Adaptive linear control and the related issues have been the main focus of adaptive control over the past several decades (see, e.g., [1]–[3], [6], [8], [9], and [11]). In recent years, attempts have been made toward a theory of adaptive nonlinear control. If the nonlinearity is only involved in the input part, or if the output part of a system is nonlinear but has a linear growth rate, then it is fairly well known that the existing adaptive control methods can still be applied as long as the unknown parameters enter the system linearly, whether the system is described in continuous-time or discrete-time (see, e.g., [13], [15], and [16]). However, the situation changes dramatically when one attempts to deal with systems with output nonlinearities having growth rates faster than linear. Neither of the existing methods are useful, nor do the similarities between adaptive control of continuous- and discrete-time systems remain. For a large class of continuous-time nonlinear systems, nonlinear-damping and/or back-stepping approaches can be successfully used in adaptive control design regardless of the growth rate of the nonlinearities (cf., e.g., [12]). This is so even in the case where external disturbances exist (c.f. [4] and [14]). For example, consider the following continuous-time stochastic control model:

$$dy_t = (\theta y_t^b + u_t)dt + dw_t, \qquad b > 0 \tag{1}$$

where θ is an unknown parameter and w_t is a standard Brownian motion, and y_t and u_t are the system output and input signals, respectively. Then it can be shown easily by using the Ito formula that (1) can be a.s. globally stabilized by the nonlinear damping control $u_t = -y_t - y_t|y_t^b|$ for *any* $b > 0$.

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However, the continuous-time approaches do not work in the discrete-time case when the nonlinear function has a growth rate faster than linear, as observed by several authors (see, e.g., [10] and [16]). Thus a question naturally arises: Can we find a stabilizing adaptive controller in this case?

A generally negative answer was recently given in [7], where a critical stabilizability phenomenon was found for a class of nonlinear control systems. To be precise, for the following typical control model:

$$y_{t+1} = \theta y_t^b + u_t + w_{t+1}, \qquad b > 0 \tag{2}$$

where θ is an unknown parameter and $\{w_t\}$ is a Gaussian white noise sequence, it has been shown in [7] that (2) is not a.s. globally stabilizable if and only if $b \geq 4$, or the following inequality has a solution:

$$z^2 - bz + b \leq 0, \qquad z \in (1, b). \tag{3}$$

The above result clearly demonstrates the limitations of adaptive control in the discrete-time case and shows that the discrete-time problems are much more complicated.

This paper is mainly concerned with discrete-time systems. We shall study nonlinear models with multi-unknown parameters, which are extensions of the scalar parameter case (2) as studied in [7]. Corresponding to (3), we shall introduce a generalized polynomial $P(z)$ in the multiparameter case, which will be used to determine when a nonlinear system is not stabilizable by feedback, thus demonstrating the fundamental limitations of adaptive nonlinear control in the discrete-time case. The bottom line of our main result is somewhat unexpected which shows that *for discrete-time nonlinear systems with random uncertainties in both parameters and disturbances, global adaptive stabilization is impossible, in general, without imposing the linear growth condition on the nonlinearities of the systems*.

II. Main Results

Consider the following discrete-time polynomial nonlinear regression model:

$$y_{t+1} = \theta_1 y_t^{b_1} + \theta_2 y_t^{b_2} + \cdots + \theta_n y_t^{b_n} + u_t + w_{t+1}, \qquad t \geq 0 \tag{4}$$

where y_t and u_t are the system output and input signals, respectively, $\theta_i (1 \leq i \leq n)$ are unknown parameters, and w_t is the noise signal. Assume the following.

A1) $b_i (1 \leq i \leq n)$ are nonnegative real numbers making (4) meaningful and satisfying $b_1 > b_2 > \cdots > b_n > 0$.

A2) $\{w_t\}$ is a Gaussian white noise sequence with distribution $N(0, 1)$.

A3) The unknown parameter vector $\theta \triangleq [\theta_1, \cdots, \theta_n]^\tau$ is independent of $\{w_t\}$ and has a Gaussian distribution $N(\overline{\theta}, I_n)$.

Our objective is to study the global stabilizability of (4) under the above conditions. First, we give a precise definition of stabilizability.

Definition 1: Let $\sigma\{y_i, 0 \leq i \leq t\}$ be the σ field generated by the observations $y_i, 0 \leq i \leq t$. System (4) is said to be a.s. globally stabilizable, if there exists a feedback control $u_t \in \mathcal{F}_t^y \triangleq \sigma\{y_i, 0 \leq i \leq t\}, t = 0, 1, \cdots$, such that for any initial condition $y_0 \in R^1$, $\limsup_{T\to\infty} 1/T \sum_{t=1}^T y_t^2 < \infty$, a.s.

Remark 1: We remark that the global stabilization of (4) is a trivial task in either the case where θ is known or the case where the noise is free (i.e., $w_t \equiv 0$). To be precise, if θ were known, we can put $u_t \equiv -(\theta_1 y_t^{b_1} + \theta_2 y_t^{b_2} + \cdots + \theta_n y_t^{b_n})$, which obviously globally stabilizes the system.

In the case where θ is unknown but the noise is free ($w_t \equiv 0$), we can obtain the true value of the parameter θ by solving n independent linear equations. For example, if in the first $(n+1)$ steps we choose $\{u_t, 0 \leq t \leq n\}$ to be independently identically distributed random variables with probability density function $p(x)$, then the true value of the parameter θ can be obtained easily by solving the following linear equation: $A \cdot \theta = [y_2 - u_1, y_3 - u_2, \cdots, y_{n+1} - u_n]^\tau$ where $A = (y_i^{b_j})_{n\times n}$ is a nonsingular matrix (cf. [5]). Hence, again, we can take the control as $u_t = -(\theta_1 y_t^{b_1} + \theta_2 y_t^{b_2} + \cdots + \theta_n y_t^{b_n})$ for $t > n$, which globally stabilizes the noise-free system. For more general parametric-strict-feedback models with no noise, a related but more complicated two-phase approach can also be applied to design a globally stabilizing adaptive controller regardless of the growth rate of the nonlinearities (cf., [17]).

Unfortunately, the main drawback of the above two-phase approach is that the resulting adaptive controller is not robust with respect to noise. In fact, the presence of noise will even change the stabilizability of discrete-time nonlinear systems dramatically if the growth rate of the nonlinearities is faster than linear, as will be shown by the following theorem together with its corollaries.

Theorem 1: Under Assumptions A1)–A3), system (4) is *not* a.s. globally stabilizable whenever the following inequality:

$$P(z) < 0, \qquad z \in (1, b_1) \tag{5}$$

has a solution, where $P(z)$ is a polynomial defined by

$$P(z) = z^{n+1} - b_1 z^n + (b_1 - b_2) z^{n-1} + \cdots + (b_{n-1} - b_n) z + b_n. \tag{6}$$

The proof is given in the next section.

Remark 2: Obviously, for $n = 1$, $P(z)$ coincides with the quadratic polynomial in (3). Note that a trivial necessary condition for (5) to have a solution is $b_1 > 1$, and when $b_1 \leq 1$ (4) is always a.s. globally stabilizable (see [15]).

To understand the implications of Theorem 1, we now give some detailed discussions on (5).

Corollary 1: If $b_i (1 \leq i \leq n)$ satisfies $b_1 > 1$ and $0 < b_i - b_{i+1} \leq \sqrt{b_1}/2(\sqrt{b_1} - 1)^2$, $1 \leq i \leq n-1$, then (5) has a solution whenever $n \geq 2 \log((\sqrt{b_1}+1)/(\sqrt{b_1}-1))/\log b_1$. Consequently, whenever $b_1 > 1$ and the number of unknown parameters n is suitably large, there always exist $0 < b_n < b_{n-1} < \cdots < b_1$ such that (4) is not a.s. globally stabilizable.

The proof of this corollary is given in the next section.

Remark 3: By Corollary 1 we know that the usual linear growth condition imposed on the nonlinear function $f(\cdot)$ of the general control model

$$y_{t+1} = \theta^\tau f(y_t, \cdots, y_{t-p}) + u_t + w_{t+1}, \qquad \theta \in R^n \tag{7}$$

cannot be essentially relaxed in general for global adaptive stabilization, unless additional conditions on the number n and the structure of $f(\cdot)$ are imposed.

Corollary 2: Let $b_1 > 2$, then for $n > 1 + 2\log(2/(b_1 - 2))/\log(b_1/2)$, (5) has a solution for any $\{b_i\}$ satisfying $1 \leq b_n < b_{n-1} < \cdots < b_2 < b_1$. On the other hand, if $b_1 \leq 2$, then for any n, there always exist $1 \leq b_n < b_{n-1} < \cdots < b_2 < b_1$ such that (5) has no solution.

Corollary 3: For any $n \geq 1$ and any $b_1 > b_2 > \cdots > b_n > 0$, we have the following.

1) A necessary condition for (5) to have a solution is $\sum_{i=1}^n b_i > 4$.

2) A sufficient condition for (5) to have a solution is either $b_1 > 4$, or $\sum_{i=1}^n b_i > (n+1)(1 + (1/n))^n$.

The proofs of Corollaries 2 and 3 are in [5], due to space limitations.

III. Proof of the Main Results

We first present the proof of Theorem 1, which is prefaced with two lemmas.

Lemma 1: Let $\{c_i, 1 \le i \le p\}$ and $\{d_i, 1 \le i \le q\}$ be two sequences of positive numbers satisfying $c_1 > c_2 > \cdots > c_p \ge 1$, $d_1 > d_2 > \cdots > d_q > 0$, $q \ge p$ and $c_i \ge c_{i+1}^{1+\delta}$, $1 \le i \le p-1$, for some $\delta > 0$. Also, let $\{i_1, \cdots, i_p\}$ and $\{j_1, \cdots, j_p\}$ be two sequences of integers arbitrarily taken from $\{1, 2, \cdots, q\}$, with $i_k \ne i_l$ and $j_k \ne j_l$, for $k \ne l$. If there exists an integer $m_0 (1 \le m_0 \le p)$ such that either $i_{m_0} \ne m_0$ or $j_{m_0} \ne m_0$, then

$$c_1^{d_{i_1}+d_{j_1}} c_2^{d_{i_2}+d_{j_2}} \cdots c_p^{d_{i_p}+d_{j_p}} \le \frac{c_1^{2d_1} c_2^{2d_2} \cdots c_p^{2d_p}}{c_p^{\min(1,\delta)\, \min_{1\le i\le p}(d_i - d_{i+1})}}$$

where by definition $d_{q+1} \triangleq 0$.

Proof: By the assumptions $d_1 > d_2 > \cdots > d_q > 0$ and $i_k \ne i_l$, $j_k \ne j_l$ for $k \ne l$, it is easy to see that

$$\sum_{k=1}^{m} 2d_k - \sum_{k=1}^{m} (d_{i_k} + d_{j_k}) \ge 0, \qquad \forall\, m \in [1, p]. \tag{8}$$

Moreover, without loss of generality assume that $m_0 = \inf\{m: i_m \ne m \text{ or } j_m \ne m, 1 \le m \le p\}$. Then

$$\sum_{k=1}^{m_0} 2d_k - \sum_{k=1}^{m_0} (d_{i_k} + d_{j_k}) = 2d_{m_0} - (d_{i_{m_0}} + d_{j_{m_0}}) \ge d_{m_0} - d_{m_0+1} > 0. \tag{9}$$

Now, by the assumption $c_i \ge c_{i+1}^{1+\delta}$, $c_p \ge 1$ and (8) and (9), we have

$$\begin{aligned}
&\frac{c_1^{2d_1} c_2^{2d_2} \cdots c_p^{2d_p}}{c_1^{d_{i_1}+d_{j_1}} c_2^{d_{i_2}+d_{j_2}} \cdots c_p^{d_{i_p}+d_{j_p}}} \\
&= c_1^{2d_1-(d_{i_1}+d_{j_1})} c_2^{2d_2-(d_{i_2}+d_{j_2})} \cdots c_p^{2d_p-(d_{i_p}+d_{j_p})} \\
&\ge c_2^{\delta(2d_1-d_{i_1}-d_{j_1})} c_2^{\sum_{k=1}^{2}(2d_k-d_{i_k}-d_{j_k})} \cdots c_p^{2d_p-d_{i_p}-d_{j_p}} \\
&\ge \prod_{m=1}^{p-1} c_{m+1}^{\delta \sum_{k=1}^{m}(2d_k-d_{i_k}-d_{j_k})} \cdot c_p^{\sum_{k=1}^{p}(2d_k-d_{i_k}-d_{j_k})} \\
&\ge \prod_{m=1}^{p-1} c_p^{\delta \sum_{k=1}^{m}(2d_k-d_{i_k}-d_{j_k})} \cdot c_p^{\sum_{k=1}^{p}(2d_k-d_{i_k}-d_{j_k})} \\
&\ge \prod_{m=1}^{p} c_p^{\min(1,\delta) \sum_{k=1}^{m}(2d_k-d_{i_k}-d_{j_k})} \\
&\ge c_p^{\min(1,\delta) \sum_{k=1}^{m_0}(2d_k-d_{i_k}-d_{j_k})} \\
&\ge c_p^{\min(1,\delta)(d_{m_0}-d_{m_0+1})} \\
&\ge c_p^{\min(1,\delta)\, \min_{1\le i\le p}(d_i-d_{i+1})}.
\end{aligned}$$

Hence Lemma 1 is true. □

The following lemma plays a key role in the proof of Theorem 1.

Lemma 2: Assume that for some $\delta > 0$ and $t \ge 1$, $|y_i| \ge |y_{i-1}|^{1+\delta}$, $i = 1, 2, \cdots, t$ and that the initial condition $|y_0| \ge 1$ is sufficiently large, then the determinant of the matrix P_{t+1}^{-1} satisfies

$$\tfrac{1}{2} y_t^{2b_1} y_{t-1}^{2b_2} \cdots y_{t-n+1}^{2b_n} \le |P_{t+1}^{-1}| \le \tfrac{3}{2} y_t^{2b_1} y_{t-1}^{2b_2} \cdots y_{t-n+1}^{2b_n} \tag{10}$$

where by definition $y_i \triangleq 1$ for $i < 0$, and P_{t+1} is defined recursively by

$$P_{t+1} = P_t - \frac{P_t \varphi_t \varphi_t^\tau P_t}{1 + \varphi_t^\tau P_t \varphi_t}, \quad P_0 = I \tag{11}$$

with

$$\varphi_t = [y_t^{b_1}, y_t^{b_2}, \cdots, y_t^{b_n}]^\tau, \qquad b_1 > b_2 > \cdots > b_n > 0. \tag{12}$$

Since the proof of this lemma is rather involved, we give it in the Appendix.

Proof of Theorem 1: We only need to prove that if (5) has a solution, then for any feedback control $u_t \in \mathcal{F}_t^y$, there always exists an initial condition y_0 and a set D_0 with positive probability such that the output signal y_t of the closed-loop control system tends to infinity at a rate faster than exponential on D_0.

Now consider the following state space equation $(t \ge 0)$:

$$\begin{cases} \theta_{t+1} = \theta_t, \quad \theta_0 = \theta, \\ y_{t+1} = \varphi_t^\tau \theta_t + u_t + w_{t+1} \end{cases} \tag{13}$$

where θ is the unknown parameter vector defined in Assumption A3) and φ_t is defined by (12).

By our Assumptions A2) and A3) and the fact that $u_t \in \mathcal{F}_t^y$, we know that (13) is a conditional Gaussian model, and hence the conditional expectation $\hat{\theta}_t = E[\theta | \mathcal{F}_t^y]$ can be generated by the Kalman filter and the conditional covariance matrix of the estimation error $(\theta - \hat{\theta}_t)$ can be generated by the Riccati equation (11) or

$$E[\tilde{\theta}_t \tilde{\theta}_t^\tau | \mathcal{F}_t^y] = P_t \tag{14}$$

where $\tilde{\theta}_t \triangleq \theta - \hat{\theta}_t$ (see, e.g., [3, Sec. 3.2]).

Next, by (13) we know that

$$y_{t+1} = \varphi_t^\tau \tilde{\theta}_t + (\varphi_t^\tau \hat{\theta}_t + u_t) + w_{t+1}. \tag{15}$$

Consequently, by the fact that $E[\tilde{\theta}_t | \mathcal{F}_t^y] = 0$ and $E[w_{t+1} | \mathcal{F}_t^y] = 0$ it follows from (14) and (15) that for any $u_t \in \mathcal{F}_t^y$

$$E[y_{t+1}^2 | \mathcal{F}_t^y] = \varphi_t^\tau P_t \varphi_t + (\varphi_t^\tau \hat{\theta}_t + u_t)^2 + 1 \ge \varphi_t^\tau P_t \varphi_t + 1. \tag{16}$$

Furthermore, by the matrix inversion formula it follows from (11) that $P_{t+1}^{-1} = P_t^{-1} + \varphi_t \varphi_t^\tau$. Then

$$\begin{aligned} |P_{t+1}^{-1}| &= |P_t^{-1} + \varphi_t \varphi_t^\tau| = |P_t^{-1}(I + P_t \varphi_t \varphi_t^\tau)| \\ &= |P_t^{-1}|(1 + \varphi_t^\tau P_t \varphi_t). \end{aligned}$$

Hence, it follows from this and (16) that

$$E[y_{t+1}^2 | \mathcal{F}_t^y] \ge \frac{|P_{t+1}^{-1}|}{|P_t^{-1}|}, \qquad \forall\, t \ge 0. \tag{17}$$

Let us define

$$D_0 = \bigcap_{t=0}^{\infty} \{\omega: E[y_{t+1}^2 | \mathcal{F}_t^y] \le (t+1)^{5/2} y_{t+1}^2\}.$$

Then by the conditional Gaussian property of the sequence $\{y_t\}$, a completely similar argument as that used in [7, Appendix B] shows that $\mathrm{Prob}(D_0) > 0$. Hence, by (17) we have

$$y_{t+1}^2 \ge \frac{1}{(t+1)^{5/2}} \frac{|P_{t+1}^{-1}|}{|P_t^{-1}|}, \qquad t \ge 0, \text{ on } D_0. \tag{18}$$

Now, let $z_0 \in (1, b_1)$ be a solution of (5). We proceed to prove that on D_0

$$|y_i| \ge |y_{i-1}|^{z_0}, \qquad i = 1, 2, \cdots. \tag{19}$$

We adopt the induction argument.

First, we consider the case where $i = 1$. Since $P_0^{-1} = I$, we have by (11)

$$|P_1^{-1}| = |P_0^{-1} + \varphi_0\varphi_0^\tau| = 1 + \|\varphi_0\|^2 > y_0^{2b_1}.$$

Therefore, by (18) we have for $|y_0| \geq 1$, $|y_1| \geq \sqrt{|P_1^{-1}|/|P_0^{-1}|} > |y_0|^{b_1} > |y_0|^{z_0}$, on D_0. Hence (19) is true for $i = 1$.

Now let us assume that for some $t \geq 1$, $|y_i| \geq |y_{i-1}|^{z_0}$, $\quad i = 1, 2, \cdots, t$, on D_0, then by Lemma 2, it follows that

$$\tfrac{1}{2} y_t^{2b_1} \cdots y_{t-n+1}^{2b_n} \leq |P_{t+1}^{-1}| \leq \tfrac{3}{2} y_t^{2b_1} \cdots y_{t-n+1}^{2b_n}$$

and

$$\tfrac{1}{2} y_{t-1}^{2b_1} \cdots y_{t-n}^{2b_n} \leq |P_t^{-1}| \leq \tfrac{3}{2} y_{t-1}^{2b_1} \cdots y_{t-n}^{2b_n}.$$

Consequently, by (18) we have on D_0

$$\begin{aligned} |y_{t+1}| &\geq \left\{ \frac{1}{(t+1)^{5/2}} \cdot \frac{|P_{t+1}^{-1}|}{|P_t^{-1}|} \right\}^{1/2} \\ &\geq \left\{ \frac{1}{3(t+1)^{5/2}} \cdot \frac{y_t^{2b_1} \cdots y_{t-n+1}^{2b_n}}{y_{t-1}^{2b_1} \cdots y_{t-n}^{2b_n}} \right\}^{1/2} \\ &= \frac{1}{\sqrt{3}(t+1)^{5/4}} \cdot \frac{|y_t|^{b_1} \cdots |y_{t-n+1}|^{b_n}}{|y_{t-1}|^{b_1} \cdots |y_{t-n}|^{b_n}} \\ &= \frac{1}{\sqrt{3}(t+1)^{5/4}} \cdot |y_t|^{b_1} |y_{t-1}|^{b_2 - b_1} \cdots \\ &\quad |y_{t-n+1}|^{b_n - b_{n-1}} |y_{t-n}|^{-b_n}. \end{aligned} \tag{20}$$

However, by the induction assumption we have $|y_{t-i}| \leq |y_t|^{z_0^{-i}}$, $\quad 1 \leq i \leq t$, and so by $b_{i+1} - b_i < 0 (1 \leq i \leq n)$

$$|y_{t-i}|^{b_{i+1} - b_i} \geq |y_t|^{(b_{i+1} - b_i) z_0^{-i}}, \qquad 1 \leq i \leq n. \tag{21}$$

Note that this inequality also holds for $i > t$, since $y_j \triangleq 1$ for $j < 0$ by definition.

Hence, it follows from (20) and (21) that on D_0

$$\begin{aligned} |y_{t+1}| &\geq \frac{1}{\sqrt{3}(t+1)^{5/4}} \\ &\quad \cdot |y_t|^{b_1 + (b_2 - b_1) z_0^{-1} + \cdots + (b_n - b_{n-1}) z_0^{-(n-1)} - b_n z_0^{-n}} \\ &\geq \frac{1}{\sqrt{3}(t+1)^{5/4}} \cdot |y_t|^{-z_0^{-n} P(z_0)} \cdot |y_t|^{z_0} \\ &\geq \frac{[|y_0|^{-z_0^{-n} P(z_0)}]^{z_0^t}}{\sqrt{3}(t+1)^{5/4}} \cdot |y_t|^{z_0} \geq |y_t|^{z_0} \end{aligned}$$

where the last inequality holds for sufficiently large $|y_0|$ because $-z_0^{-n} P(z_0) > 0$.

Hence, by induction, (19) is true. Thus for all large initial conditions $|y_0|$, the output process $|y_t|$ diverges to infinity at a rate faster than exponential on D_0. This completes the proof of Theorem 1. □

Proof of Corollary 1: Take $z_0 = \sqrt{b_1} \in (1, b_1)$. We need only verify that $P(z_0) < 0$.

By the assumption for n, we have $z_0^n = (\sqrt{b_1})^n \geq (\sqrt{b_1} + 1)/(\sqrt{b_1} - 1)$. Hence by (6) and the conditions on b_i, we have

$$\begin{aligned} P(z_0) &< z_0^n (z_0 - b_1) + \sqrt{\frac{b_1}{2}} \left(\sqrt{b_1} - 1\right)^2 \\ &\qquad \cdot (z_0^{n-1} + \cdots + 1) + b_1 \\ &= -z_0^n \sqrt{b_1} \left(\sqrt{b_1} - 1\right) + \sqrt{\frac{b_1}{2}} \left(\sqrt{b_1} - 1\right)^2 \frac{z_0^n - 1}{z_0 - 1} + b_1 \\ &= -z_0^n \sqrt{b_1} \left(\sqrt{b_1} - 1\right) + \frac{\sqrt{b_1}}{2} \left(\sqrt{b_1} - 1\right) z_0^n \\ &\qquad - \sqrt{\frac{b_1}{2}} \left(\sqrt{b_1} - 1\right) + b_1 \\ &= -\frac{\sqrt{b_1}}{2} \left(\sqrt{b_1} - 1\right) z_0^n + \frac{\sqrt{b_1}}{2} \left(\sqrt{b_1} + 1\right) \\ &\leq -\frac{\sqrt{b_1}}{2} \left[\left(\sqrt{b_1} - 1\right) \frac{\sqrt{b_1 + 1}}{\sqrt{b_1} - 1} - \left(\sqrt{b_1} + 1\right) \right] = 0. \end{aligned}$$

Hence, Corollary 1 holds. □

IV. Concluding Remarks

It is fairly well known that for nonlinear stochastic systems described by nonlinear regression models with linear unknown parameters, a globally stabilizing adaptive controller can be designed whenever the nonlinear function [say $f(x)$] involved has a linear growth rate, i.e., $|f(x)| = O(|x|)$, as $|x| \to \infty$. However, in contrast to the continuous-time case, essential difficulties emerge in the discrete-time case when the nonlinear function has a growth rate faster than linear. In fact, the nonlinear growth rate has been the crux in discrete-time adaptive nonlinear control for years. Naturally, one would ask the following questions: 1) Can we remove the usual linear growth condition in the discrete-time case? 2) How far can we go from linear growth to nonlinear growth for global stabilization?

A first step in this direction was recently made in [7], where it was shown that in the unknown scalar parameter case ($n = 1$), the nonlinear control system in question is globally stabilizable *if and only if* $|f(x)| = O(|x|^b)$ with $b < 4$. In the present paper, we have dealt with the general multiparameter case ($n \geq 1$) by considering the polynomial regression model described by (4). By introducing a new and more general polynomial (6), we have found a criterion about situations where (4) is not globally stabilizable (Theorem 1). Based on that, various explicit cases are discussed in Corollaries 1–3. Perhaps the most remarkable consequence of our main result is the following implication for general nonlinear regression models. *It is impossible in general to essentially relax the usual linear growth condition for global stabilization, unless additional conditions are imposed* (see Remark 3).

Appendix

Proof of Lemma 2: By the matrix inversion formula it follows from (11) that $P_{t+1}^{-1} = P_t^{-1} + \varphi_t \varphi_t^\tau$; hence, we have (21a) shown at the bottom of the next page.

Now, let us denote $\alpha_m(i) \triangleq [y_i^{b_1 + b_m}, y_i^{b_2 + b_m}, \cdots, y_i^{b_n + b_m}]^\tau$, $1 \leq m \leq n$, $0 \leq i \leq t$, and let $\alpha_m(-1) \triangleq e_m$, i.e., the mth column of the identity matrix I_m, then

$$|P_{t+1}^{-1}| = \det\left(\sum_{i=-1}^{t} \alpha_1(i), \sum_{i=-1}^{t} \alpha_2(i), \cdots, \sum_{i=-1}^{t} \alpha_n(i) \right).$$

By the elementary properties of determinants, we have

$$|P_{t+1}^{-1}| = \sum_{i_1, i_2, \cdots, i_n = -1}^{t} \det(\alpha_1(i_1), \alpha_2(i_2), \cdots, \alpha_n(i_n)). \tag{22}$$

It is clear that if in the group $(i_1, i_2, \cdots, i_n)$ there are at least two integers having the same value (but different from -1), then $\det(\alpha_1(i_1), \cdots, \alpha_n(i_n)) = 0$. So in the discussions below we will exclude this kind of zero-valued determinant.

We proceed to prove (10) by considering two cases separately.

Case I) $t < n-1$: In this case, in order that $\det(\alpha_1(i_1), \alpha_2(i_2), \cdots, \alpha_n(i_n)) \neq 0$, the number of (-1)'s in $(i_1, i_2, \cdots, i_n)$ must at least be $(n-1-t)$ and the other integers must be distinct. Then each term in the expansion of the nonzero determinant $\det(\alpha_1(i_1), \alpha_2(i_2), \cdots, \alpha_n(i_n))$ contains at most $[n-(n-1-t)] = (t+1)$ different factors, whose general form is

$$y_{i_1}^{b_{j_1}+b_{k_1}} y_{i_2}^{b_{j_2}+b_{k_2}} \cdots y_{i_p}^{b_{j_p}+b_{k_p}}, \qquad p \le t+1 \tag{23}$$

where $i_m \neq i_l, j_m \neq j_l$, and $k_m \neq k_l$ for $m \neq l$. Note that one such term is $y_t^{2b_1} y_{t-1}^{2b_2} \cdots y_0^{2b_{t+1}}$ (from the products of the main diagonal elements of the matrix $[\alpha_1(t), \cdots, \alpha_{t+1}(0), \alpha_{t+2}(-1), \cdots, \alpha_n(-1)]$), and it is different from other terms. Now, we proceed to prove that the absolute value of any other term is not greater than $1/[|y_0|^{\min(1,\delta)\cdot \min_{1\le i\le n-1}(b_i-b_{i+1})}]\, y_t^{2b_1} y_{t-1}^{2b_2} \cdots y_0^{2b_{t+1}}$.

We divide this proof into two subcases.

Subcase 1): If in the general form (23) $p = t+1$, then we can rewrite (23) as $y_t^{b_{j_1}+b_{k_1}} y_{t-1}^{b_{j_2}+b_{k_2}} \cdots y_0^{b_{j_{t+1}}+b_{k_{t+1}}}$. By our assumptions, it is easy to see that Lemma 1 is applicable, hence we have

$$\begin{aligned} &\left| y_t^{b_{j_1}+b_{k_1}} y_{t-1}^{b_{j_2}+b_{k_2}} \cdots y_0^{b_{j_{t+1}}+b_{k_{t+1}}} \right| \\ &\quad \le \frac{1}{|y_0|^{\min(1,\delta)\cdot \min_{1\le i\le n-1}(b_i-b_{i+1})}}\, y_t^{2b_1} y_{t-1}^{2b_2} \cdots y_0^{2b_{t+1}}. \end{aligned}$$

Subcase 2): If in the general form (23) $p < t+1$, then we can add some $(i_{p+1}, \cdots, i_{t+1})$ to $(i_1, \cdots, i_p)$ so that $(i_1, \cdots, i_p, i_{p+1}, \cdots, i_{t+1}) \in T(t, t-1, \cdots, 0)$. Here and hereafter, we use $T(t, t-1, \cdots, 0)$ to denote the class of all permutations of the integer sequence $(t, t-1, \cdots, 0)$. Then by $t+1 < n$, we can choose suitable $j_{p+1}, \cdots, j_{t+1}, k_{p+1}, \cdots, k_{t+1} \in \{1, 2, \cdots, n\}$ with $j_m \neq j_l$, $k_m \neq k_l$ for $m \neq l$, $1 \le m, l \le t+1$ such that the term

$$y_{i_1}^{b_{j_1}+b_{k_1}} \cdots y_{i_p}^{b_{j_p}+b_{k_p}} y_{i_{p+1}}^{b_{j_{p+1}}+b_{k_{p+1}}} \cdots y_{i_{t+1}}^{b_{j_{t+1}}+b_{k_{t+1}}}$$

is different from $y_t^{2b_1} y_{t-1}^{2b_2} \cdots y_0^{2b_{t+1}}$. Thus by the conclusion of Subcase 1), we have

$$\begin{aligned} &\left| y_{i_1}^{b_{j_1}+b_{k_1}} \cdots y_{i_p}^{b_{j_p}+b_{k_p}} \right| \\ &\quad \le \left| y_{i_1}^{b_{j_1}+b_{k_1}} \cdots y_{i_p}^{b_{j_p}+b_{k_p}} y_{i_{p+1}}^{b_{j_{p+1}}+b_{k_{p+1}}} \cdots y_{i_{t+1}}^{b_{j_{t+1}}+b_{k_{t+1}}} \right| \\ &\quad \le \frac{1}{|y_0|^{\min(1,\delta)\cdot \min_{1\le i\le n-1}(b_i-b_{i+1})}}\, y_t^{2b_1} \cdots y_0^{2b_{t+1}}. \end{aligned}$$

Hence, the desired inequality is true.

Now, rewrite (22) as $|P_{t+1}^{-1}| = R_t + y_t^{2b_1} y_{t-1}^{2b_2} \cdots y_0^{2b_{t+1}}$, where R_t denotes the summation of all the terms different from $y_t^{2b_1} y_{t-1}^{2b_2} \cdots y_0^{2b_{t+1}}$. It is obvious that R_t has at most $[(t+2)^n \cdot n! - 1] \le (n^n \cdot n! - 1)$ terms. Hence, by the results proved above, we obtain

$$|R_t| \le \frac{n^n \cdot n! - 1}{|y_0|^{\min(1,\delta)\min_{1\le i\le n-1}(b_i-b_{i+1})}} y_t^{2b_1} \cdots y_0^{2b_{t+1}}.$$

Therefore, by choosing the initial value $|y_0|$ large enough, we can make $|R_t|$ less than $(1/2) y_t^{2b_1} y_{t-1}^{2b_2} \cdots y_0^{2b_{t+1}}$. Consequently (10) follows.

Case II) $t \ge n-1$: First of all, any nonzero determinant $\det(\alpha_1(i_1), \alpha_2(i_2), \cdots, \alpha_n(i_n)), -1 \le i_1, \cdots, i_n \le t$ can be expanded as the summation of $n!$ terms whose general form is $y_{i_1}^{b_1+b_{j_1}} y_{i_2}^{b_2+b_{j_2}} \cdots y_{i_n}^{b_n+b_{j_n}}$, where $(j_1, \cdots, j_n) \in T(1, \cdots, n)$; $i_k \in \{-1, 0, \cdots, t\}$, $1 \le k \le n$, and as noted before, any two i_k's cannot have the same value different from (-1). Obviously one such term is $y_t^{2b_1} y_{t-1}^{2b_2} \cdots y_{t-n+1}^{2b_n}$ [one term in $\det(\alpha_1(t), \cdots, \alpha_n(t-n+1))$]. We now show that for any other terms $y_{i_1}^{b_1+b_{j_1}} y_{i_2}^{b_2+b_{j_2}} \cdots y_{i_n}^{b_n+b_{j_n}} (i_k \neq k$ or $j_k \neq k$, for some $1 \le k \le n)$, the following inequality holds:

$$\begin{aligned} &|y_{i_1}^{b_1+b_{j_1}} y_{i_2}^{b_2+b_{j_2}} \cdots y_{i_n}^{b_n+b_{j_n}}| \\ &\quad \le \frac{1}{|y_{t-n+1}|^{(\delta/(1+\delta))\ \min_{1\le i\le n}(b_i-b_{i+1})}}\, y_t^{2b_1} \cdots y_{t-n+1}^{2b_n}. \end{aligned} \tag{24}$$

We proceed to prove this by considering three subcases.

Subcase 1): If $(i_1, i_2, \cdots, i_n) \in T(t, t-1, \cdots, t-n+1)$, then we can directly apply Lemma 1 to get

$$\begin{aligned} &|y_{i_1}^{b_1+b_{j_1}} y_{i_2}^{b_2+b_{j_2}} \cdots y_{i_n}^{b_n+b_{j_n}}| \\ &\quad \le \frac{1}{|y_{t-n+1}|^{\min(1,\delta)\ \min_{1\le i\le n}(b_i-b_{i+1})}}\, y_t^{2b_1} \cdots y_{t-n+1}^{2b_n} \\ &\quad \le \frac{1}{|y_{t-n+1}|^{\delta/(1+\delta)\ \min_{1\le i\le n}(b_i-b_{i+1})}}\, y_t^{2b_1} \cdots y_{t-n+1}^{2b_n}. \end{aligned}$$

Subcase 2): If in the group $(i_1, i_2, \cdots, i_n)$ there are at least two integers less than $(t-n+1)$, then we may find $(i_1', i_2', \cdots, i_n') \in T(t, t-1, \cdots, t-n+1)$ such that $i_l \le i_l' (1 \le l \le n)$ and $y_{i_1'}^{b_1+b_{j_1}} y_{i_2'}^{b_2+b_{j_2}} \cdots y_{i_n'}^{b_n+b_{j_n}}$ is different from $y_t^{2b_1} y_{t-1}^{2b_2} \cdots y_{t-n+1}^{2b_n}$. Consequently, by the conclusion of Subcase 1), we have

$$\begin{aligned} &\left| y_{i_1}^{b_1+b_{j_1}} y_{i_2}^{b_2+b_{j_2}} \cdots y_{i_n}^{b_n+b_{j_n}} \right| \\ &\quad \le \left| y_{i_1'}^{b_1+b_{j_1}} y_{i_2'}^{b_2+b_{j_2}} \cdots y_{i_n'}^{b_n+b_{j_n}} \right| \\ &\quad \le \frac{1}{|y_{t-n+1}|^{(\delta/(1+\delta))\ \min_{1\le i\le n}(b_i-b_{i+1})}}\, y_t^{2b_1} \cdots y_{t-n+1}^{2b_n}. \end{aligned}$$

$$|P_{t+1}^{-1}| = \left| P_0^{-1} + \sum_{i=1}^{t} \varphi_i \varphi_i^\tau \right| = \begin{vmatrix} 1 + \sum_{i=0}^{t} y_i^{2b_1} & \sum_{i=0}^{t} y_i^{b_1+b_2} & \cdots & \sum_{i=0}^{t} y_i^{b_1+b_n} \\ \sum_{i=0}^{t} y_i^{b_1+b_2} & 1 + \sum_{i=0}^{t} y_i^{2b_2} & \cdots & \sum_{i=0}^{t} y_i^{b_2+b_n} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=0}^{t} y_i^{b_1+b_n} & \sum_{i=0}^{t} y_i^{b_2+b_n} & \cdots & 1 + \sum_{i=0}^{t} y_i^{2b_n} \end{vmatrix} \tag{21a}$$

Subcase 3): If in the group $(i_1, i_2, \cdots, i_n)$ there is just one integer less than $(t-n+1)$, i.e., $i_p \le t-n$, for some $1 \le p \le n$, then by $|y_{t-n}| \le |y_{t-n+1}|^{1/(1+\delta)}$ we have

$$\begin{aligned}
&|y_{i_1}^{b_1+b_{j_1}} \cdots y_{i_p}^{b_p+b_{j_p}} \cdots y_{i_n}^{b_n+b_{j_n}}| \\
&\quad \le \left|y_{i_1}^{b_1+b_{j_1}} \cdots y_{t-n}^{b_p+b_{j_p}} \cdots y_{i_n}^{b_n+b_{j_n}}\right| \\
&\quad \le \left|y_{i_1}^{b_1+b_{j_1}} \cdots y_{t-n+1}^{b_p+b_{j_p}} \cdots y_{i_n}^{b_n+b_{j_n}}\right| \Big/ \\
&\qquad |y_{t-n+1}|^{(\delta/(1+\delta))(b_p+b_{j_p})} \\
&\quad \le \left|y_t^{2b_1} y_{t-1}^{2b_2} \cdots y_{t-n+1}^{2b_n}\right| \Big/ |y_{t-n+1}|^{(\delta/(1+\delta))(b_p+b_{j_p})} \\
&\quad \le \left|y_t^{2b_1} y_{t-1}^{2b_2} \cdots y_{t-n+1}^{2b_n}\right| \Big/ |y_{t-n+1}|^{(\delta/(1+\delta))b_n} \\
&\quad \le \frac{1}{|y_{t-n+1}|^{(\delta/(1+\delta))\ \min_{1\le i\le n}(b_i-b_{i+1})}}\, y_t^{2b_1} y_t^{2b_2} \cdots y_{t-n+1}^{2b_n}.
\end{aligned}$$

Hence (24) is true.

Thus, similar to the arguments in Case I), we rewrite (22) as $|P_{t+1}^{-1}| = R_t + y_t^{2b_1} y_{t-1}^{2b_2} \cdots y_{t-n+1}^{2b_n}$, where R_t denotes the summation of $[(t+2)^n \cdot n! - 1]$ terms in the determinant expansions, which are different from $y_t^{2b_1} \cdots y_{t-n+1}^{2b_n}$. Then by (24) we know that

$$\begin{aligned}
|R_t| &\le \frac{(t+2)^n \cdot n! - 1}{|y_{t-n+1}|^{(\delta/(1+\delta))\ \min_{1\le i\le n}(b_i-b_{i+1})}}\, y_t^{2b_1} y_{t-1}^{2b_2} \cdots y_{t-n+1}^{2b_n} \\
&\le \frac{(t+2)^n \cdot n! - 1}{|y_0|^{(1+\delta)^{t-n+1}\cdot(\delta/(1+\delta))\cdot\min_{1\le i\le n}(b_i-b_{i+1})}} \\
&\quad \cdot y_t^{2b_1} y_{t-1}^{2b_2} \cdots y_{t-n+1}^{2b_n} \\
&\le \tfrac{1}{2} y_t^{2b_1} y_{t-1}^{2b_2} \cdots y_{t-n+1}^{2b_n}
\end{aligned}$$

where the last inequality holds for sufficiently large $|y_0|$. Hence, the proof of Lemma 2 is completed.

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Comments on the Computation of Interval Routh Approximants

Chyi Hwang and Shih-Feng Yang

***Abstract*—In recent papers [6], [7], the Routh approximation method was extended to derive reduced-order interval models for linear interval systems. In this paper, the authors show that: 1) interval Routh approximants to a high-order interval transfer function depend on the implementation of interval Routh expansion and inversion algorithms; 2) interval Routh expansion algorithms cannot guarantee the success in generating a full interval Routh array; 3) some interval Routh approximants may not be robustly stable even if the original interval system is robustly stable; and 4) an interval Routh approximant is in general not useful for robust controller design because its dynamic uncertainties (in terms of robust frequency responses) do not cover those of the original interval system.**

***Index Terms*— Interval systems, model reduction, Routh approximation.**

I. Introduction

In the last two decades, the Routh approximation method pioneered by Hutton and Friedland [1] and its variants [2]–[5] have been receiving much attention in the field of model reduction. The method is based on using the Routh stability array to derive reduced-order models for high-order linear systems. The main advantages of the Routh approximation method are that it has the ability to yield stable reduced-order models for stable original high-order systems, to produce a family of reduced-order models of different orders via a single set of algebraic computations, and to obtain reduced models that retain the first several time-moments and/or Markov parameters of the original systems.

Recently, the Routh approximation method has been extended to derive reduced-order interval models for high-order interval transfer functions [6], [7]. The extension is based on using interval arithmetic to perform Routh α–β or γ–δ canonical continued-fraction expansion and inversion. Surprisingly, it is observed from the literature [7]

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