On stability of Random Riccati equations

WANG Yuan (王 远) and GUO Lei (郭 雷)

(Institute of Systems Science, Chinese Academy of Sciences, Beijing 100080, China)

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Abstract Random Riccati equations (RRE) arise frequently in filtering, estimation and control, but their stability properties are rarely rigorously explored in the literature. First a suitable stochastic observability (or excitation) condition is introduced to guarantee both the L_r- and exponential stability of RRE. Then the stability of Kalman filter is analyzed with random coefficients, and the L_r boundedness of filtering errors is established.

Keywords: random Riccati equation, stochastic stability, Kalman filter, excitation.

Random Riccati equations (RRE) have several sources. First, for linear systems with random coefficients, the well-known Kalman filter and the LQG controller depend naturally on RRE. Second, even for linear systems with nonrandom coefficients, if the coefficients are unknown and adaptively estimated, then again the adaptive filter and the adaptive LQG controller depend explicitly on RRE. Third, consider nonlinear systems of the form

$$\begin{cases} x_{k+1} = f(x_k) + w_{k+1}, \\ y_k = g(x_k) + v_k, \end{cases}$$

where $f(\cdot)$ and $g(\cdot)$ are nonlinear functions, $\{w_k\}$ and $\{v_k\}$ are white noise processes, and $\{x_k\}$ is the state process to be estimated. The well-known and widely used extended Kalman filter (EKF)[1] is derived based on the following linearization:

$$\begin{cases} f(x_k) \approx f(\hat{x}_k) + \left.\dfrac{\partial f}{\partial x}\right|_{x=\hat{x}_k}(x_k - \hat{x}_k), \\ g(x_k) \approx g(\hat{x}_k) + \left.\dfrac{\partial g}{\partial x}\right|_{x=\hat{x}_k}(x_k - \hat{x}_k), \end{cases}$$

where $\hat{x}_k$ denotes the estimate of x_k at time k. Then, once again, the corresponding EKF depends on RRE since $\left.\dfrac{\partial f}{\partial x}\right|_{x=\hat{x}_k}$ and $\left.\dfrac{\partial g}{\partial x}\right|_{x=\hat{x}_k}$ are random matrices. Hence, it is necessary to study RRE in a variety of important situations.

Now, the general form of RRE is as follows:

$$P_{k+1} = F_kP_kF_k^{\mathrm{T}} - F_kP_kH_k^{\mathrm{T}}(H_kP_kH_k^{\mathrm{T}} + R_k)^{-1}H_kP_kF_k^{\mathrm{T}} + Q_k, \tag{1}$$

where F_k, H_k, Q_k and R_k are $d \times d$, $m \times d$, $d \times d$, and $m \times m$ random matrices respectively, and $Q_k \geqslant 0$, $R_k \geqslant 0$ are nonnegative definite matrices. A primary question concerning RRE is the stochastic stability (boundedness), i.e. under what kind of conditions on $\{F_k, H_k\}$, the process $\{P_k, k \geqslant 1\}$ is bounded in a stochastic sense. The standard results deal exclusively with deterministic matrices $\{F_k, H_k\}$, and the earlier results can be found in ref. [2]. For that study, the commonly used conditions are

$$\alpha_1 I \leqslant \sum_{i=k-N}^{k-1} \Phi(k, i+1) Q_i \Phi^{\mathrm{T}}(k, i+1) \leqslant \alpha_2 I, \ \forall k, \tag{2}$$

and

$$\beta_1 I \leqslant \sum_{i=k-N}^{k} \Phi^{\mathrm{T}}(i,k) H_i^{\mathrm{T}} R_i^{-1} H_i \Phi(i,k) \leqslant \beta_2 I, \ \forall k \tag{3}$$

for some deterministic positive constants $\alpha_1 \leqslant \alpha_2$, $\beta_1 \leqslant \beta_2$ and $N>0$, where $\Phi(k,i)$ is the transition matrix:

$$\Phi(k,i) = F_{k-1} \cdots F_i, \ \forall k \geqslant i+1; \ \Phi(i,i) = I, \ \Phi(i,k) = \Phi(k,i)^{-1}.$$

Unfortunately, as pointed out in ref. [3], Conditions (2) and (3) are mainly deterministic hypotheses and are unsuitable for models with random coefficients. In the standard linear regression framework where "$F_k = I$", Guo[3] introduced the following conditional richness condition:

$$E\left\{ \sum_{k=mh}^{(m+1)h-1} \frac{H_k^{\mathrm{T}} H_k}{1 + \| H_k \|^2} \middle| \mathscr{F}_{mh-1} \right\} \geqslant \delta I \quad \text{a.s. } \forall m \geqslant 0 \tag{4}$$

to ensure the L_r-boundedness of P_k, where $\delta > 0$ and $h > 0$ are two constants, and $\mathscr{F}_{mh-1} = \sigma\{H_k, k \leqslant mh-1\}$. This condition was further relaxed in ref. [4], allowing δ to be a random process. Later, the same method was used1) to investigate the case where F_k is a deterministic matrix F. However, it was required in footnote 1) that F satisfies the following property:

$$0 < \varepsilon_1 = \lambda_{\min}(FF^{\mathrm{T}}) \leqslant \lambda_{\max}(FF^{\mathrm{T}}) = \varepsilon_2 < 1.$$

Obviously, this property is stronger than the stability of F.

To the best of the authors' knowledge, there are only a few papers which treat RRE with general matrices $\{F_k, k \geqslant 1\}$ in a rigorous way (cf. refs. [5,6]). However, in all these works, stationarity of $\{F_k, H_k\}$ was required, and only weakly stochastically bounded (WSB) property was investigated.

In this paper, we are interested in nonstationary matrices $\{F_k, H_k\}$ and are concerned with the following two questions: (i) Under what conditions do we have the moment boundedness of RRE? (ii) Under what conditions do we have the exponential stability of RRE? Besides, we will analyse the stability of time-varying Kalman filter with random coefficients.

1 L_r-stability of RRE

Throughout the sequel, for simplicity of discussions, we assume that there are positive definite matrices $Q>0$ and $R>0$ such that $Q_k \geqslant Q$ and $R_k \geqslant R$.

Definition 1.1. A random matrix (or vector) sequence $\{A_k, k \geqslant 0\}$ defined on the basic probability space $(\Omega, \mathscr{F}, P)$ is called L_r-stable $(r>0)$ if $\sup_{k \geqslant 0} E \| A_k \|^r < \infty$, where the norm of a matrix X is defined as its maximum singular value, i.e. $\| X \| = \{\lambda_{\max}(XX)^{\mathrm{T}}\}^{1/2}$.

In the sequel, we will refer to $\| A_k \|_r$ defined by

$$\| A_k \|_r \triangleq \{E \| A_k \|^r\}^{1/r} \tag{5}$$

as the L_r-norm of A_k.

To analyse the RRE, we need the following assumptions:

A1. For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$P\{\lambda_{\min}(G(k+h,k)) > \delta \mid \mathscr{G}_{k-1}\} > 1 - \varepsilon, \ \forall k,$$

where $h>0$ is an integer, $G(k+h,k)$ is the observability Grammian

1) Wang, G. J., Discrete-time recursive algorithms analysis, *Ph. D. Dissertation*, Institute of Systems Science, Chinese Academy of Sciences, June, 1996.

$$G(k+h,k)=\sum_{i=k+1}^{k+h}\Phi^{\mathrm{T}}(i,k)H_i^{\mathrm{T}}H_i\Phi(i,k) \tag{6}$$

and $\mathcal{G}_k=\sigma\{F_i,\ H_i,\ i\leqslant k\}$ is the σ-algebra generated by $\{F_i,\ H_i,\ i\leqslant k\}$.

A2. For some $r\geqslant 1$, there exist positive constants M_1, M_2, M_3 and ε such that

(i) $\sup_k E\|H_k\|^{16r}<M_1<\infty$;

(ii) $\sup_{k\leqslant j\leqslant i\leqslant k+h} E\|\Phi(i,j)\|^{16r+\varepsilon}<M_2<\infty,\ \forall k$

and

$$\sup_k E[\|\Phi(k+h,k)\|^{8r}\mid\mathcal{G}_{k-1}]<M_3<\infty,$$

where $h>0$ is defined in Condition A1.

We remark that A1 is a form of stochastic observability condition of $\{F_k,\ H_k\}$, and A2 is automatically satisfied if $\{F_k,\ H_k\}$ is a bounded sequence as usually assumed in the literature.

Theorem 1.1. *Under Conditions A1 and A2, the random Riccati equation defined by (1) is L_r-stable.*

Proof. Denote $A_\delta(k)=\{\lambda_{\min}(G(k+h,k))>\delta\}$, $\delta>0$ and use $\{F_k,\ H_k,\ k\geqslant 0\}$ to construct an auxiliary time-varying linear system:

$$\begin{cases} x_{k+1}=F_kx_k+w_{k+1}, & (7)\\ y_k=H_kx_k+v_k, & (8)\end{cases}$$

where the initial condition x_0 has a Gaussian law with mean $\hat{x}_0$ and covariance matrix P_0, $\{w_k,\ v_k,\ k\geqslant 0\}$ is a sequence of independent Gaussian random vectors, independent of $\{F_k,\ H_k,\ k\geqslant 0\}$ with the following properties:

$$E[w_k]=0,\ E[w_kw_k^{\mathrm{T}}]=Q_k,\ E[v_k]=0,\ E[v_kv_k^{\mathrm{T}}]=R_k,\ E[w_kv_k^{\mathrm{T}}]=0.$$

Then, the minimum-variance linear estimate of x_k is determined by the following equations[6,7]:

$$\begin{aligned}\hat{x}_{k+1}&=F_k\hat{x}_k+F_kP_kH_k^{\mathrm{T}}(H_kP_kH_k^{\mathrm{T}}+R_k)^{-1}(y_k-H_k\hat{x}_k),\\ \hat{x}_k&=E(x_k\mid\mathcal{F}_{k-1}),\\ P_k&=E(\tilde{x}_k\tilde{x}_k^{\mathrm{T}}\mid\mathcal{F}_{k-1}),\\ \tilde{x}_k&=x_k-\hat{x}_k,\end{aligned} \tag{9}$$

where $\{P_k\}$ can be generated by (1) and $\mathcal{F}_k=\sigma(\mathcal{G}_\infty,\ y_0,\ y_1,\cdots,y_k\}$.

For $h>0$ defined in Condition A1, let us introduce another estimate of x_{k+h}, denoted by x_{k+h}^*, which is recursively defined by

$$x_{k+h}^* =\Phi(k+h,k)G^{-1}(k+h,k)\sum_{i=k+1}^{k+h}\Phi^{\mathrm{T}}(i,k)H_i^{\mathrm{T}}y_iI_{A_\delta(k)}+\Phi(k+h,k)x_k^*I_{A_\delta^c(k)}, \tag{10}$$

where the initial values $x_m^*\ (m=1,2,\cdots,h-1)$ are defined as

$$\begin{cases} x_0^*=\hat{x}_0, & (11)\\ x_{m+1}^*=F_mx_m^*, & (12)\end{cases}$$

with

$$E\|x_0-x_0^*\|^n=E\|x_0-\hat{x}_0\|^n<\infty,\ \forall n \tag{13}$$

because x_0 has a Gaussian distribution.

Now, from (7), (11) and (12), we have

$$x_{m+1} - x^*_{m+1} = \Phi(m+1,0)(x_0 - x_0^*) + \sum_{i=1}^{m+1}\Phi(m+1,i)w_i, \tag{14}$$

so by the C_r-inequality,

$$\| x_{m+1} - x^*_{m+1} \|^{4r} \leqslant (m+2)^{4r-1} \| \Phi(m+1,0) \|^{4r} \| x_0 - x_0^* \|^{4r} + (m+2)^{4r-1}\sum_{i=1}^{m+1} \| \Phi(m+1,\ i) \|^{4r} \| w_i \|^{4r}.$$

Thus, by the Schwarz inequality and Condition A2,

$$E \| x_{m+1} - x^*_{m+1} \|^{4r} \leqslant (m+2)^{4r-1}\sqrt{E \| \Phi(m+1,0) \|^{8r} E \| x_0 - x_0^* \|^{8r}} + (m+2)^{4r-1}\sum_{i=1}^{m+1}\sqrt{E \| \Phi(m+1,i) \|^{8r} E \| w_i \|^{8r}} < \infty, \quad \forall\, m = 0, \cdots, h-2. \tag{15}$$

Next, it follows from (7) that

$$x_i = \Phi(i,k)x_k + \sum_{j=k+1}^{i}\Phi(i,j)w_j, \quad \forall\, i \geqslant k, \tag{16}$$

so

$$x_{k+h} = \Phi(k+h,k)x_k + \sum_{j=k+1}^{k+h}\Phi(k+h,j)w_j. \tag{17}$$

Substituting (8) and (16) into (10) yields

$$\begin{aligned} x^*_{k+h} &= \Phi(k+h,k)G^{-1}(k+h,k)\sum_{i=k+1}^{k+h}\Phi^{\mathrm{T}}(i,k)H_i^{\mathrm{T}}[H_i x_i + v_i]I_{A_\delta(k)} + \Phi(k+h,k)x_k^* I_{A_\delta^c(k)} \\ &= \Phi(k+h,k)G^{-1}(k+h,k)\sum_{i=k+1}^{k+h}\Phi^{\mathrm{T}}(i,k)H_i^{\mathrm{T}}\left\{H_i\left[\Phi(i,k)x_k + \sum_{j=k+1}^{i}\Phi(i,j)w_j\right] + v_i\right\}I_{A_\delta(k)} + \Phi(k+h,\ k)x_k^* I_{A_\delta^c(k)} \\ &= \Phi(k+h,k)x_k I_{A_\delta(k)} + \Phi(k+h,k)G^{-1}(k+h,k)\sum_{i=k+1}^{k+h}\Phi^{\mathrm{T}}(i,k)H_i^{\mathrm{T}}\xi_i I_{A_\delta(k)} \\ &\quad + \Phi(k+h,k)x_k^* I_{A_\delta^c(k)}, \end{aligned} \tag{18}$$

where

$$\xi_i = H_i\sum_{j=k+1}^{i}\Phi(i,j)w_j + v_i. \tag{19}$$

From (17) and (18), it follows that

$$x_{k+h} - x^*_{k+h} = \Phi(k+h,k)(x_k - x_k^*)I_{A_\delta^c(k)} + \sum_{j=k+1}^{k+h}\Phi(k+h,j)w_j - \Phi(k+h,k)G^{-1}(k+h,k)\sum_{i=k+1}^{k+h}\Phi^{\mathrm{T}}(i,k)H_i^{\mathrm{T}}\xi_i I_{A_\delta(k)}. \tag{20}$$

By taking $k = ih$, $i = 1,2,\cdots$, from (20) we have,

$$\| x_{(i+1)h} - x^*_{(i+1)h} \|^{2r} \leqslant (2h+1)^{2r-1} \| \Phi((i+1)h, ih) \|^{2r} \| x_{ih} - x^*_{ih} \|^{2r} I_{A_\delta^c(ih)} + (2h+1)^{2r-1}\sum_{j=ih+1}^{(i+1)h} \| \Phi((i+1)h, j) \|^{2r} \| w_j \|^{2r}$$

$$+ (2h+1)^{2r-1} \| \Phi((i+1)h, ih) \|^{2r} \| G^{-1/2}((i+1)h, ih) \|^{2r}$$
$$\times \left\| G^{-1/2}((i+1)h, ih) \sum_{m=ih+1}^{(i+1)h} \Phi^{\mathrm{T}}(m, ih) H_m^{\mathrm{T}} \xi_m \right\|^{2r} I_{A_\delta(ih)}. \tag{21}$$

Let us denote $z_{i+1} \triangleq \| x_{(i+1)h} - x^*_{(i+1)h} \|^{2r}$ and $a_{i+1} \triangleq C_r \| \Phi((i+1)h, ih) \|^{2r} I_{A^c_\delta(ih)}$ with $C_r = (2h+1)^{2r-1}$. Also, denote the sum of the last two terms in (21) as Δ_{i+1}. Then we have

$$z_{i+1} \leqslant a_{i+1} z_i + \Delta_{i+1}. \tag{22}$$

Thus,

$$z_{i+1} \leqslant \left(\prod_{j=1}^{i+1} a_j \right) z_0 + \sum_{j=1}^{i+1} \left(\prod_{l=j+1}^{i+1} a_l \right) \Delta_j, \tag{23}$$

and

$$E(z_{i+1}) \leqslant \sqrt{E\left(\prod_{j=1}^{i+1} a_j \right)^2 E(z_0^2)} + \sum_{j=1}^{i+1} \sqrt{E\left(\prod_{l=j+1}^{i+1} a_l \right)^2 E(\Delta_j^2)}. \tag{24}$$

Let us define $\{\eta_i, i \geqslant j\}$ recursively by

$$\eta_{i+1} = a_{i+1}^2 \eta_i, \quad \eta_j = 1.$$

Then

$$\eta_{i+1} = \left(\prod_{l=j+1}^{i+1} a_l \right)^2, \tag{25}$$

and

$$E(\eta_{i+1} \mid \mathscr{G}_{ih-1}) = E(a_{i+1}^2 \eta_i \mid \mathscr{G}_{ih-1}) = \eta_i E(a_{i+1}^2 \mid \mathscr{G}_{ih-1}). \tag{26}$$

So,

$$E(\eta_{i+1}) = E[\eta_i E(a_{i+1}^2 \mid \mathscr{G}_{ih-1})]. \tag{27}$$

Now, by the definition of a_{i+1}, Condition A1 and the Schwarz inequality,

$$\begin{aligned} E(a_{i+1}^2 \mid \mathscr{G}_{ih-1}) &= C_r E[\| \Phi((i+1)h, ih) \|^{4r} I_{A^c_\delta(ih)} \mid \mathscr{G}_{ih-1}] \\ &\leqslant C_r \sqrt{E[\| \Phi((i+1)h, ih) \|^{8r} \mid \mathscr{G}_{ih-1}] E[I_{A^c_\delta(ih)} \mid \mathscr{G}_{ih-1}]} \\ &\leqslant C_r \sqrt{M_3} \sqrt{\varepsilon} \triangleq \rho < 1, \end{aligned} \tag{28}$$

where the last inequality can be guaranteed by taking ε small enough.

Hence, by (27),

$$E(\eta_{i+1}) \leqslant \rho E(\eta_i) \leqslant \cdots \leqslant \rho^{i-j+1} E(\eta_j) = \rho^{i-j+1}, \tag{29}$$

i.e.

$$E\left(\prod_{l=j+1}^{i+1} a_l \right)^2 \leqslant \rho^{i-j+1}. \tag{30}$$

In particular,

$$E\left(\prod_{j=1}^{i+1} a_j \right)^2 \leqslant 1. \tag{31}$$

Next, by the definition of Δ_{i+1}, we know that

$$\Delta_{i+1}^2 \leqslant (2h+1)^{2r} \sum_{j=ih+1}^{(i+1)h} \| \Phi((i+1)h, j) \|^{4r} \| w_j \|^{4r}$$
$$+ (2h+1)^{2r} \| \Phi((i+1)h, ih) \|^{4r} \| G^{-1/2}((i+1)h, ih) \|^{4r}$$

$$\times \left\| G^{-1/2}((i+1)h,\ ih) \sum_{m=ih+1}^{(i+1)h} \Phi^{\mathrm{T}}(m,ih) H_m^{\mathrm{T}} \xi_m \right\|^{4r} I_{A_\delta(ih)}$$

$$\stackrel{\Delta}{=} \alpha_{i+1} + \beta_{i+1}. \tag{32}$$

For the first term on the right hand side of (32),

$$E(\alpha_{i+1}) \leqslant (2h+1)^{2r} \sum_{j=ih+1}^{(i+1)h} \sqrt{E \| \Phi((i+1)h, j) \|^{8r} E \| w_j \|^{8r}} = O(1). \tag{33}$$

We now proceed to estimate β_{i+1}. Note that

$$\| G^{-1/2}((i+1)h,\ ih) \|^{4r} I_{A_\delta(ih)} \leqslant \frac{1}{\delta^{2r}} = O(1), \tag{34}$$

and

$$\left\| G^{-1/2}((i+1)h,\ ih) \sum_{m=ih+1}^{(i+1)h} \Phi^{\mathrm{T}}(m,ih) H_m^{\mathrm{T}} \xi_m \right\|^2 \leqslant \left\| \sum_{m=ih+1}^{(i+1)h} \xi_m \xi_m^{\mathrm{T}} \right\| \leqslant \sum_{m=ih+1}^{(i+1)h} \| \xi_m \|^2, \tag{35}$$

where we have used the fact that

$$W^{\mathrm{T}} M (M^{\mathrm{T}} M)^{-1} M^{\mathrm{T}} W \leqslant W^{\mathrm{T}} W$$

for any matrices M and W with $M^{\mathrm{T}} M$ invertible (cf. p. 6 of ref. [8]).

Hence, by the definition of β_{i+1} in (32),

$$\beta_{i+1} \leqslant O(1) \| \Phi((i+1)h,\ ih) \|^{4r} \sum_{m=ih+1}^{(i+1)/h} \| \xi_m \|^{4r}. \tag{36}$$

Furthermore, by (19) and Condition A2,

$$\begin{aligned} E \| \xi_m \|^{8r} &= E \left\| H_m \sum_{j=ih+1}^{m} \Phi(m,j) w_j + v_m \right\|^{8r} \\ &= \left\| H_m \sum_{j=ih+1}^{m} \Phi(m,j) w_j + v_m \right\|_{8r}^{8r} \\ &\leqslant \left(\left\| H_m \sum_{j=ih+1}^{m} \Phi(m,j) w_j \right\|_{8r} + \| v_m \|_{8r} \right)^{8r} \\ &\leqslant \left(\| H_m \|_{16r} \sum_{j=ih+1}^{m} \| \Phi(m,j) \|_{16r+\varepsilon} \| w_j \|_{\frac{16r(16r+\varepsilon)}{\varepsilon}} + \| v_m \|_{8r} \right)^{8r} \\ &= O(1). \end{aligned} \tag{37}$$

Therefore, by (36) and (37),

$$\begin{aligned} E(\beta_{i+1}) &\leqslant O(1) \sqrt{E \| \Phi((i+1)h, ih) \|^{8r}} \sqrt{E \left(\sum_{m=ih+1}^{(i+1)h} \| \xi_m \|^{4r} \right)^2} \\ &\leqslant O(1) \sqrt{E \left(\sum_{m=ih+1}^{(i+1)h} \| \xi_m \|^{8r} \right)} = O(1). \end{aligned} \tag{38}$$

Hence by (32), (33) and (38),

$$E(\Delta_{i+1}^2) = O(1). \tag{39}$$

Consequently, by (13), (31) and (39), it follows from (24) that

$$E(z_{i+1}) \leqslant O(1) + \sum_{j=1}^{i+1} \sqrt{\rho^{i-j+1} O(1)} = O(1). \tag{40}$$

On the other hand, by the optimality of the Kalman filter,

$$P_k \leqslant E[(x_k - x_k^*)(x_k - x_k^*)^{\mathrm{T}} \mid \mathscr{F}_{k-1}].$$

Hence,

$$\| P_k \| \leqslant \mathrm{tr}\, E[(x_k - x_k^*)(x_k - x_k^*)^{\mathrm{T}} \mid \mathscr{F}_{k-1}] = E[\| x_k - x_k^* \|^2 \mid \mathscr{F}_{k-1}],$$

and

$$\| P_k \|^r \leqslant \{E[\| x_k - x_k^* \|^2 \mid \mathscr{F}_{k-1}]\}^r \leqslant E[\| x_k - x_k^* \|^{2r} \mid \mathscr{F}_{k-1}]. \tag{41}$$

So,

$$E \| P_k \|^r \leqslant E \| x_k - x_k^* \|^{2r}. \tag{42}$$

Hence, by the definition of z_i and (40),

$$E \| P_{ih} \|^r \leqslant E(z_i) = O(1), \tag{43}$$

which means

$$\sup_i E \| P_{ih} \|^r < \infty. \tag{44}$$

In the same way, by taking $k = ih + 1, \cdots, (i+1)h - 1$ respectively in (20), it can be proved that

$$\sup_k E \| P_k \|^r < \infty. \tag{45}$$

Hence, Theorem 1.1 is true.

Now, we give an example to show that the key assumption A1 is necessary in some cases.

Example 1.1. Consider the one-dimensional case where $F_k = f > 1$, $h_k \overset{\Delta}{=} H_k$, $Q_k = q$ and $R_k = r$. Then by (1), $p_k \overset{\Delta}{=} P_k$ can be written as

$$p_{k+1} = f^2 \frac{r p_k}{h_k^2 p_k + r} + q. \tag{46}$$

Suppose that $\{h_k\}$ is an i.i.d. sequence. Then by Condition A1 we know that for any $\varepsilon > 0$ there exists $\delta > 0$, such that

$$P\left\{ \sum_{i=k+1}^{k+h} f^{2(i-k)} h_i^2 < \delta \right\} < \varepsilon.$$

Hence by the arbitrariness of ε, it is easy to conclude that Condition A1 is equivalent to

$$P\left\{ \sum_{i=k+1}^{k+h} f^{2(i-k)} h_i^2 = 0 \right\} = 0, \tag{47}$$

which in turn is equivalent to

$$P\{h_1 = 0\} = 0.$$

In fact, if $P\{h_1 = 0\} = 0$, then

$$P\left\{ \sum_{i=k+1}^{k+h} f^{2(i-k)} h_i^2 = 0 \right\} \leqslant P\{h_{k+1} = 0\} = P\{h_1 = 0\} = 0.$$

On the other hand, if $P\left\{ \sum_{i=k+1}^{k+h} f^{2(i-k)} h_i^2 = 0 \right\} = 0$, then

$$P\left\{ \bigcap_{i=k+1}^{k+h} (h_i = 0) \right\} = P\left\{ \sum_{i=k+1}^{k+h} f^{2(i-k)} h_i^2 = 0 \right\} = 0.$$

Since h_k is an i.i.d. sequence,

$$P\left\{ \bigcap_{i=k+1}^{k+h} (h_i = 0) \right\} = \prod_{i=k+1}^{k+h} P\{h_i = 0\} = P^h\{h_1 = 0\} = 0.$$

So, $P\{h_1 = 0\} = 0$, and the claimed equivalence is established.

Next, we show that if Condition A1 is not satisfied, i.e. $P(h_1 = 0) \neq 0$, we can find a sequence $\{h_k\}$ and a constant $f > 1$, such that

$$\sup_k Ep_k = \infty. \tag{48}$$

Let h_k have the following distribution,

$$P\{h_k = 0\} = a > 0,\ P\{h_k = 1\} = 1 - a. \tag{49}$$

They by (46)

$$\begin{aligned} E(p_{k+1} \mid \mathcal{G}_{k-1}) &= E\left(f^2 \frac{rp_k}{r + h_k^2 p_k} \Big| \mathcal{G}_{k-1}\right) + q \\ &= f^2 r \frac{p_k}{r} a + f^2 \frac{rp_k}{r + p_k}(1 - a) + q \\ &\geqslant af^2 p_k + q. \end{aligned} \tag{50}$$

Hence, if $f = \frac{2}{\sqrt{a}}$, then

$$E(p_{k+1}) \geqslant 4Ep_k + q \to \infty,$$

whenever $p_0 + q \neq 0$.

2 Exponential stability of RRE

Definition 2.1. A sequence of random matrices $\{A_k,\ k \geqslant 0\}$ is called L_r-exponentially stable if there exist constants $\lambda \in [0,1)$ and $M > 0$ such that

$$\left\| \prod_{j=i+1}^{k} A_j \right\|_r \leqslant M\lambda^{k-i},\ \forall k \geqslant i \geqslant 0. \tag{51}$$

The RRE (1) can be rewritten as the following random Lyapunov equations:

$$P_{k+1} = F_k(I - L_k H_k) P_k (I - L_k H_k)^{\mathrm{T}} F_k^{\mathrm{T}} + \bar{Q}_k, \tag{52}$$

where

$$L_k = P_k H_k^{\mathrm{T}} (H_k P_k H_k^{\mathrm{T}} + R_k)^{-1},\ \bar{Q}_k = F_k L_k R_k L_k^{\mathrm{T}} F_k^{\mathrm{T}} + Q_k. \tag{53}$$

The objective of this section is to show that the L_r-stability of $\{P_k\}$ implies L_r-exponential stability of $\{F_k(I - L_k H_k)\}$.

For this, let us introduce the following additional assumption:

B1. There exist constants N_1 and N_2 such that

$$\sup_{k \leqslant i \leqslant k+h} E[\| H_i \|^8 \mid \mathcal{G}_{k-1}] < N_1 < \infty,\ \forall k$$

and

$$\sup_{k \leqslant j \leqslant i \leqslant k+h} E[\| \Phi(i,j) \|^{8+\varepsilon} \mid \mathcal{G}_{k-1}] < N_2 < \infty,\ \forall k,$$

where $h > 0$ is defined in Condition A1.

Theorem 2.1. *Under Conditions A1, A2 and B1, the sequence* $\{F_k(I - L_k H_k),\ k \geqslant 0\}$ *is L_r-exponentially stable, where r is defined in Condition A2.*

We preface the proof of the theorem with several lemmas.

Lemma 2.1[3]. *Let* $\{a_k,\ \mathcal{D}_k\}$ *be an adapted process such that* $a_k \geqslant 1$, $Ea_0 < \infty$ *and*

$$E[a_k \mid \mathcal{D}_{k-1}] \leqslant \alpha a_{k-1} + \beta,\ \forall k \geqslant 1,$$

where $0 \leqslant \alpha < 1$, $0 < \beta < \infty$. *Then there exists* $\lambda \in [0,1)$ *such that* $\{1/a_k\} \in S^0(\lambda)$, *where* $S^0(\lambda)$ *is defined by*

$$S^0(\lambda) = \left\{ a : a_k \in [0,1],\ E\prod_{j=i+1}^{k}(1 - a_j) \leqslant M\lambda^{k-i},\right.$$

$$\forall k \geqslant i, \ \forall i \geqslant 0, \quad \text{for some } M > 0 \Big\}. \tag{54}$$

Lemma 2.2[4]. *Let $\{F_k\}$ be a sequence of $d \times d$ random matrices, and $\{\overline{Q}_k\}$ be a sequence of positive definite random matrices. Then for $\{P_k\}$ recursively defined by (52), we have, for all $n > m$,*

$$\left\| \prod_{k=m}^{n-1} F_k(I - L_k H_k) \right\|^2 \leqslant \prod_{k=m}^{n-1} \left(1 - \frac{1}{1 + \| \overline{Q}_k^{-1} P_{k+1} \|}\right) \| P_n \| \, \| P_m^{-1} \| . \tag{55}$$

Lemma 2.3. *Let $\{a_k : a_k \in [0,1], \ k \geqslant 0\}$ be a scalar sequence. If there exist $h > 0$, $c > 0$ and $\lambda \in [0,1)$, such that*

$$E \prod_{i=k}^{k+th-1} (1 - a_i) \leqslant c\lambda^{th}, \ \forall k \geqslant 0, \quad \forall t \geqslant 0,$$

then $\{a_k\} \in S^0(\lambda)$.

Proof. Let $n \geqslant m \geqslant 0$ and $n = m + th + s$ with $0 \leqslant s < h$. Then,

$$\begin{aligned} E\prod_{i=m}^{n-1}(1 - a_i) &= E \prod_{i=m}^{m+th-1} (1 - a_i) \prod_{j=m+th}^{n-1} (1 - a_i) \\ &\leqslant E \prod_{i=m}^{m+th-1} (1 - a_i) \leqslant c\lambda^{th} \\ &= \frac{c}{\lambda^s}(\lambda^{th}\lambda^s) \\ &\leqslant \frac{c}{\lambda^h}\lambda^{n-m} = M\lambda^{n-m}, \end{aligned}$$

where $M = \frac{c}{\lambda^h}$. Therefore, $\{a_k\} \in S^0(\lambda)$.

Lemma 2.4. *Consider the Lyapunov equation (52), and let Conditions A1 and B1 be satisfied. Then, the scalar sequence appearing on the RHS of (55) belongs to $S^0(\lambda)$, i.e.*

$$\left\{1 - \frac{1}{1 + \| \overline{Q}_k^{-1} P_{k+1} \|}, \ k \geqslant 0\right\} \in S^0(\lambda),$$

where $S^0(\lambda)$ is defined by (54), and $\lambda \in [0,1)$.

Proof. We adopt the notations introduced in the proof of Theorem 1.1. Let

$$\mathscr{D}_k = \mathscr{G}_{kh}, \ \mathscr{D}'_k = \sigma\{\mathscr{D}_k, \ x_0, \ w_0, \cdots, w_{kh+1}\},$$

$$z_k \triangleq \| x_{kh+1} - x^*_{kh+1} \|^2, \ a_k \triangleq 1 + d \, \| Q^{-1} \| \, E[z_k \mid \mathscr{D}_k].$$

We proceed to show that $\{a_k, \mathscr{D}_k\}$ satisfies the conditions of Lemma 2.1.

First, it is obvious that $a_k \geqslant 1$, and

$$\begin{aligned} E[a_0] &= 1 + d \, \| Q^{-1} \| \, E\{E[\, \| x_1 - x_1^* \|^2 \mid \mathscr{G}_0]\} \\ &= 1 + d \, \| Q^{-1} \| \, E[\, \| x_1 - x_1^* \|^2] < \infty. \end{aligned} \tag{56}$$

Next, by (34) and (35), it follows from (21) that

$$\begin{aligned} E[z_k \mid \mathscr{D}_{k-1}] \leqslant & C_1 E\left[\, \| \Phi(kh + 1, \ (k - 1)h + 1) \|^2 I_{A^c_\delta((k-1)h+1)} z_{k-1} \middle| \mathscr{D}_{k-1}\right] \\ & + C_1 \sum_{j=(k-1)h+2}^{kh+1} E\left[\, \| \Phi(kh + 1, j) \|^2 \, \| w_j \|^2 \middle| \mathscr{D}_{k-1}\right] \\ & + C_1 \frac{1}{\delta^2} E\left[\, \| \Phi(kh + 1, \ (k - 1)h + 1) \|^2 \sum_{m=(k-1)h+2}^{kh+1} \| \xi_m \|^2 \middle| \mathscr{D}_{k-1}\right] \end{aligned}$$

$$\stackrel{\Delta}{=} \alpha_{kh+1} + \beta_{kh+1} + \gamma_{kh+1}, \tag{57}$$

where $C_1 = 2h+1$.

For the first term on the RHS of (57),

$$\begin{aligned}\alpha_{kh+1} &= C_1 E\left\{E\left[\,\|\Phi(kh+1,(k-1)h+1)\|^2 I_{A_\delta^c((k-1)h+1)} z_{k-1}\,\middle|\,\mathscr{D}'_{k-1}\right]\,\middle|\,\mathscr{D}_{k-1}\right\} \\ &= C_1 E\left\{E\left[\,\|\Phi(kh+1,(k-1)h+1)\|^2 I_{A_\delta^c((k-1)h+1)}\,\middle|\,\mathscr{D}'_{k-1}\right] z_{k-1}\,\middle|\,\mathscr{D}_{k-1}\right\}. \end{aligned}\tag{58}$$

Note that $x_0, w_0, \cdots, w_{kh+1}$ are independent of $(F_i, H_i, i \geqslant 0)$. By the Schwarz inequality and Conditions A1, B1 and (58), we have

$$\begin{aligned}\alpha_{kh+1} &= C_1 E\left\{E\left[\,\|\Phi(kh+1,(k-1)h+1)\|^2 I_{A_\delta^c((k-1)h+1)}\,\middle|\,\mathscr{D}_{k-1}\right] z_{k-1}\,\middle|\,\mathscr{D}_{k-1}\right\} \\ &\leqslant C_1 E\left\{\sqrt{E[\,\|\Phi(kh+1,(k-1)h+1)\|^4\,|\,\mathscr{D}_{k-1}]E\left[I_{A_\delta^c((k-1)h+1)}\,\middle|\,\mathscr{D}_{k-1}\right]}\, z_{k-1}\,\middle|\,\mathscr{D}_{k-1}\right\} \\ &\leqslant C_1 \sqrt[8+\varepsilon]{N_2^2}\sqrt{\varepsilon} E[z_{k-1}\mid\mathscr{D}_{k-1}] = \rho_1 E[z_{k-1}\mid\mathscr{D}_{k-1}],\end{aligned}\tag{59}$$

where $\rho_1 \stackrel{\Delta}{=} C_1 \sqrt[8+\varepsilon]{N_2^2}\sqrt{\varepsilon}$ can be made less than one by taking ε small enough.

For the second term on the RHS of (57), by the Schwarz inequality and Condition B1, we have

$$\begin{aligned}\beta_{kh+1} &\leqslant C_1 \sum_{j=(k-1)h+2}^{kh+1} \sqrt{E[\,\|\Phi(kh+1,j)\|^4\mid\mathscr{D}_{k-1}]}\sqrt{E[\,\|w_j\|^4\mid\mathscr{D}_{k-1}]} \\ &\leqslant C_1 h \sqrt[8+\varepsilon]{N_2^2}\,O(1) = O(1).\end{aligned}\tag{60}$$

For the last term in (57), by the Schwarz inequality and Condition B1,

$$\begin{aligned}\gamma_{kh+1} &\leqslant C_1 \frac{1}{\delta^2}\sqrt[8+\varepsilon]{N_2^2}\sqrt{\varepsilon}\sqrt{E\left[\left(\sum_{m=(k-1)h+2}^{kh+1}\|\xi_m\|^2\right)^2\,\middle|\,\mathscr{D}_{k-1}\right]} \\ &\leqslant C_1 \frac{1}{\delta^2}\sqrt[8+\varepsilon]{N_2^2}\sqrt{\varepsilon}\sqrt{(h+1)\sum_{m=(k-1)h+2}^{kh+1} E\left[\,\|\xi_m\|^4\,\middle|\,\mathscr{D}_{k-1}\right]}.\end{aligned}\tag{61}$$

By the C_r-inequality, Hölder inequality and Condition B1, it follows from (19) that

$$\begin{aligned}E[\,\|\xi_m\|^4\mid\mathscr{D}_{k-1}] &= E\left[\left\|H_m \sum_{j=(k-1)h+2}^{m}\Phi(m,j)w_j + v_m\right\|^4\,\middle|\,\mathscr{D}_{k-1}\right] \\ &\leqslant 8E\left[\left\|H_m \sum_{j=(k-1)h+2}^{m}\Phi(m,j)w_j\right\|^4\,\middle|\,\mathscr{D}_{k-1}\right] + 8E[\,\|v_m\|^4\mid\mathscr{D}_{k-1}] \\ &\leqslant 8\sqrt{E[\,\|H_m\|^8\mid\mathscr{D}_{k-1}]}\sqrt{E\left[\left\|\sum_{j=(k-1)h+2}^{m}\Phi(m,j)w_j\right\|^8\,\middle|\,\mathscr{D}_{k-1}\right]} + O(1) \\ &\leqslant 8\sqrt{N_1}\sqrt{h^7\sum_{j=(k-1)h+2}^{m} N_2^{\frac{8}{8+\varepsilon}}O(1)} + O(1) \\ &= O(1).\end{aligned}\tag{62}$$

Hence, by (61) and (62),

$$\gamma_{kh+1} = O(1). \tag{63}$$

Finally, substituting (59), (60), and (63) into (57) yields

$$E[z_k\mid\mathscr{D}_{k-1}] \leqslant \rho_1 E[z_{k-1}\mid\mathscr{D}_{k-1}] + O(1).$$

Moreover,

$$E[a_k\mid\mathscr{D}_{k-1}] = 1 + d\,\|Q^{-1}\|\,E[z_k\mid\mathscr{D}_{k-1}]$$

$$\leqslant 1 + d \| Q^{-1} \| \{ \rho_1 E[z_{k-1} \mid \mathscr{D}_{k-1}] + O(1) \}$$
$$= \rho_1 a_{k-1} + O(1). \tag{64}$$

Thus, by Lemma 2.1, there exist some $M > 0$, $\lambda \in [0,1)$ such that

$$E \prod_{i=m}^{n} (1 - 1/a_i) \leqslant M\lambda^{n-m+1}, \quad \forall\, n \geqslant m \geqslant 0. \tag{65}$$

Furthermore,

$$\begin{aligned} a_k &= 1 + d \| Q^{-1} \| E[\| x_{kh+1} - x^*_{kh+1} \|^2 \mid \mathscr{G}_{kh}] \\ &\geqslant 1 + d \| Q^{-1} \| E[\| P_{kh+1} \| \mid \mathscr{G}_{kh}] \\ &= 1 + d \| Q^{-1} \| \, \| P_{kh+1} \|. \end{aligned} \tag{66}$$

Then, since $\| \overline{Q}_k^{-1} \| \leqslant \| Q^{-1} \|$, we have for $N \triangleq \left[\frac{k}{h} \right]$,

$$\begin{aligned} E \prod_{i=k}^{k+th-1} \left(1 - \frac{1}{1 + \| \overline{Q}_i^{-1} P_{i+1} \|} \right) &\leqslant E \prod_{i=k}^{k+th-1} \left(1 - \frac{1}{1 + d \| Q^{-1} \| \, \| P_{i+1} \|} \right) \\ &\leqslant E \prod_{i=N+1}^{N+t-1} \left(1 - \frac{1}{1 + d \| Q^{-1} \| \, \| P_{ih+1} \|} \right) \\ &\leqslant E \prod_{i=N+1}^{N+t-1} \left(1 - \frac{1}{1 + d \| Q^{-1} \| E[\| x_{ih+1} - x^*_{ih+1} \|^2 \mid \mathscr{G}_{ih}]} \right) \\ &= E \prod_{i=N+1}^{N+t-1} (1 - 1/a_i) \leqslant M\lambda^{t-1}. \end{aligned} \tag{67}$$

Let $C = M/\lambda$, $\lambda_1 = \lambda^{1/h}$. Then

$$E \prod_{i=k}^{k+th-1} \left(1 - \frac{1}{1 + \| \overline{Q}_i^{-1} P_{i+1} \|} \right) \leqslant C\lambda_1^{th}.$$

Thus, by Lemma 2.3, we know that

$$\left\{ 1 - \frac{1}{1 + \| \overline{Q}_k^{-1} P_{k+1} \|}, \ k \geqslant 0 \right\} \in S^0(\lambda_1),$$

and hence the proof of Lemma 2.4 is completed.

Proof of Theorem 2.1. By Lemma 2.2 and the fact that $P_m^{-1} \leqslant Q^{-1}$ we have

$$\begin{aligned} \left\| \prod_{k=m}^{n-1} F_k (I - L_k H_k) \right\|_r^r &\leqslant E \left\{ \left\{ \prod_{k=m}^{n-1} \left(1 - \frac{1}{1 + \| \overline{Q}_k^{-1} P_{k+1} \|} \right) \right\}^{r/2} \| P_n \|^{r/2} \| P_m^{-1} \|^{r/2} \right\} \\ &\leqslant \| Q^{-1} \|^{r/2} \sqrt{ E \left\{ \prod_{k=m}^{n-1} \left(1 - \frac{1}{1 + \| \overline{Q}_k^{-1} P_{k+1} \|} \right) \right\}^r } \sqrt{E \| P_n \|^r} \\ &\leqslant \| Q^{-1} \|^{r/2} \sqrt{ E \left\{ \prod_{k=m}^{n-1} \left(1 - \frac{1}{1 + \| \overline{Q}_k^{-1} P_{k+1} \|} \right) \right\} } \sqrt{E \| P_n \|^r}. \end{aligned}$$

Therefore, by Theorem 1.1 and Lemma 2.4, it is easy to see that Theorem 2.1 holds.

3 Applications to Kalman filter

Theorem 2.1 can be used to analyse the L_r-stability of Kalman filter with random coefficients.

Consider the time-varying linear system:

$$\begin{cases} x_{k+1} = F_k x_k + w_{k+1}, & (68) \\ y_k = H_k x_k + v_k, & (69) \end{cases}$$

where $\{w_k,\ v_k,\ k\geqslant 0\}$ is an independent noise process and satisfies

$$E[w_k]=0,\ E[w_k w_k^{\mathrm{T}}]=Q_k\geqslant Q>0;$$

and

$$E[v_k]=0,\ E[v_k v_k^{\mathrm{T}}]=R_k\geqslant aI>0;\ E[w_k v_k^{\mathrm{T}}]=0.$$

The initial condition x_0 is a random vector with mean $\hat{x}_0$ and covariance matrix P_0. Then the Kalman filtering equation of the above system is

$$\hat{x}_{k+1}=F_k\hat{x}_k+F_kP_kH_k^{\mathrm{T}}(H_kP_kH_k^{\mathrm{T}}+R_k)^{-1}(y_k-H_k\hat{x}_k),\tag{70}$$

where $\{P_k\}$ is defined by the RRE

$$P_{k+1}=F_kP_kF_k^{\mathrm{T}}-F_kP_kH_k^{\mathrm{T}}(H_kP_kH_k^{\mathrm{T}}+R_k)^{-1}H_kP_kF_k^{\mathrm{T}}+Q_k.\tag{71}$$

We assume that:

C1. For some $r\geqslant 1$, there exist positive constants c_1,c_2,c_3 and ε such that

(i) $\sup_k E\|H_k\|^{32r}<c_1<\infty$;

(ii) $\sup_{k\leqslant j\leqslant i\leqslant k+h} E\|\Phi(i,j)\|^{32r+\varepsilon}<c_2<\infty,\ \forall k$

and

$$\sup_k E[\|\Phi(k+h,k)\|^{16r}\mid \mathscr{G}_{k-1}]<c_3<\infty,$$

where $h>0$ is defined in Condition A1.

C2. $E\|x_0\|^{2r}<\infty,\ \sup_k E[\|w_{k+1}\|^{2r}+\|v_k\|^{4r}]<\infty$.

Note that here we do not assume that $\{F_k,\ H_k\}$ is independent of $\{w_k,\ v_k\}$, which allows us to include the adaptive case where F_k and H_k are measurable functions of the observations $\{y_0, y_1,\cdots,y_k\},\ \forall k$.

Theorem 3.1. *Under Conditions A1, B1, C1 and C2, the estimate error* $\tilde{x}_k=x_k-\hat{x}_k$ *produced by* (68) *and* (70) *satisfies*

$$\sup_k\|\tilde{x}_k\|_r<\infty.$$

Proof. By (68) and (70)

$$\tilde{x}_{k+1}=F_k(I-L_kH_k)\tilde{x}_k+w_{k+1}-F_kL_kv_k,\tag{72}$$

where L_k is defined by (53).

Denote $\Delta_{k+1}\triangleq w_{k+1}-F_kL_kv_k$. Then (72) gives

$$\tilde{x}_{k+1}=\prod_{i=0}^{k}(F_i(I-L_iH_i))\tilde{x}_0+\sum_{i=1}^{k+1}\prod_{j=i}^{k}(F_j(I-L_jH_j))\Delta_i.\tag{73}$$

By Theorem 2.1, Condition C2 and the Hölder inequality,

$$\begin{aligned}\|\tilde{x}_{k+1}\|_r&\leqslant\left\|\prod_{i=0}^{k}(F_i(I-L_iH_i))\right\|_{2r}\|\tilde{x}_0\|_{2r}+\sum_{i=1}^{k+1}\left\|\prod_{j=i}^{k}(F_j(I-L_jH_j))\right\|_{2r}\|\Delta_i\|_{2r}\\&\leqslant M\lambda^{k+1}O(1)+\sum_{i=1}^{k+1}M\lambda^{k-i+1}\|\Delta_i\|_{2r}\\&=O(1)+M\sum_{i=0}^{k}\lambda^i\|\Delta_{k-i+1}\|_{2r}.\end{aligned}$$

Note that

$$\begin{aligned}\|L_k\|^2&=\lambda_{\max}\{P_kH_k^{\mathrm{T}}(H_kP_kH_k^{\mathrm{T}}+R_k)^{-2}H_kP_k\}\\&\leqslant\mathrm{tr}\{P_kH_k^{\mathrm{T}}(H_kP_kH_k^{\mathrm{T}}+R_k)^{-2}H_kP_k\}\\&\leqslant\lambda_{\max}(P_k)\mathrm{tr}\{H_kP_kH_k^{\mathrm{T}}(H_kP_kH_k^{\mathrm{T}}+R_k)^{-2}\}\end{aligned}$$

$$\leqslant \| P_k \| \operatorname{tr}(H_k P_k H_k^{\mathrm{T}} + R_k)^{-1}$$
$$\leqslant \| P_k \| \operatorname{tr}(R_k^{-1}) \leqslant \| P_k \| m a^{-1}.$$

Hence, by Conditions A1, B1, C1 and C2 we get

$$\| \Delta_k \|_{2r} = O(1).$$

So

$$\| \tilde{x}_{k+1} \|_r = O(1) + O(1) \sum_{i=0}^{k} \lambda^i = O(1).$$

This completes the proof of Theorem 3.1.

4 Conclusions

Random Riccati equations play an important role in many problems of systems and control. In this paper, the L_r-stability of the Random Riccati equation is established under a very general excitation condition. The form of this excitation condition can be regarded as an extension of that introduced in ref. [3] for the case where $F = I$, and is shown to be necessary in a special case. Since in the present case F_k is random and time-varying, the analytic methods we have used here are different from that in references [3,4].

Comparing this paper with refs. [5,6], we do not need the stationary assumption on the random matrices $\{F_k, H_k\}$ and our L_r-stability results are much stronger than the weak stability (i.e. bounded in probability) established there. The results of this paper are expected to have more applications in adaptive systems where nonstationarity of signals is a key feature.

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