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Stabilization of stochastic systems with hidden Markovian jumps

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Abstract This paper considers the adaptive control of discrete-time hybrid stochastic systems with unknown randomly jumping parameters described by a finite-state hidden Markov chain. An intuitive yet longstanding conjecture in this area is that such hybrid systems can be adaptively stabilized whenever the rate of transition of the hidden Markov chain is small enough. This paper provides a rigorous positive answer to this conjecture by establishing the global stability of a gradient-algorithm-based adaptive linear-quadratic control.

Keywords: stochastic adaptive control, Markovian jump parameters, gradient algorithm, stability, linear quadratic control.

1 Introduction

1.1 System models

Fault-prone dynamic systems may experience abrupt changes in their structures and parameters, caused by such phenomena as component failures and changing subsystem interconnections. We can model such systems as operating in different "forms", each of which corresponds to some combination of these events. Thus, let us consider the following discrete-time hybrid systems with Markovian jumps:

$$x_{t+1} = A(r_t)x_t + B(r_t)u_t + F(r_t)w_{t+1},\ t \geq 0, \tag{1.1}$$

where $x_t \in \mathbb{R}^n$ and $u_t \in \mathbb{R}^m$ are the state and input of the system respectively, $w_t \in \mathbb{R}^n$ is the noise, $\{r_t,\ t=0,1,2,\cdots\}$ is an unknown (or hidden) homogeneous Markov chain taking values in $\mathscr{S} \triangleq \{1,2,\cdots,N\}$ with one-step transition probability matrix

$$P = \begin{bmatrix} p_{11} & \cdots & p_{1N} \\ \vdots & \ddots & \vdots \\ p_{N1} & \cdots & p_{NN} \end{bmatrix},$$

where

$$p_{ij} = P(r_{t+1} = j \mid r_t = i).$$

For $r_t = i$, we denote $A(r_t) = A_i$, $B(r_t) = B_i$ and $F(r_t) = F_i$, where the $A_i's$, $B_i's$ and $F_i's$ are, respectively, known matrices in $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times m}$ and $\mathbb{R}^{n\times n}$, such that $\|A_i - A_j\| + \|B_i - B_j\| + \|F_i - F_j\| \neq 0$ for $i \neq j$. Here and hereafter, $\|\cdot\|$ denotes the Frobenius norm of a matrix, defined by $\|A\| \triangleq \{Tr(AA^\tau)\}^{1/2}$ for any matrix A, where $Tr(\cdot)$ is the trace operator. Also, we will use $\|\cdot\|_2$ to denote the Euclidean norm of a vector or induced Euclidean norm of a matrix, which can be defined as $\|A\|_2 = \sqrt{\lambda_{\max}(A^\tau A)}$.

It may be worth noting that in the continuous-time case, the corresponding hybrid systems may be modelled by the following stochastic differential equation:

$$dx_t = [A(r'_t)x_t + B(r'_t)u_t]dt + F(r'_t)dw_t,\ t \geqslant 0, \tag{1.2}$$

$$\Phi_t = \Phi_0 + \Pi\int_0^t \Phi_s ds + m_t, \tag{1.3}$$

where $r'_t \in \mathcal{S}$ is a Markov process with N states, $\{w_t, \mathcal{F}_t\}$ is an $\mathbb{R}^n$-valued standard Wiener process on a probability space $(\Omega, P, \mathcal{F})$, $\Phi_t = [I_{(r'_t=1)}, \cdots, I_{(r'_t=N)}]^\tau$ is the indicator process for r'_t, and Π is the transition probability rate matrix. It should be noted that eq. (1.3) is a general representation of homogeneous finite state Markov chains[1].

The systems described by (1.1) or (1.2) and (1.3) are called hybrid systems, since they combine a part of the state that takes values continuously ($x \in \mathbb{R}^n$) and another part of the state that takes only discrete values ($r \in \mathcal{S}$).

Over the past three decades, such hybrid systems have attracted considerable research interest. The study of the continuous-time models can be traced back at least to the work of Krasovskii et al.[2], where the Markovian jump linear quadratic (JLQ) optimal control problem (corresponding to the case $F(r'_t) \equiv 0$ in (1.2)) was dealt with. The previous works can naturally be divided into two groups (nonadaptive and adaptive) according to the availability of the jump parameters.

1.2 Previous results on optimal (nonadaptive) control

For the simplest complete observation case, Sworder[3] and Wonham[4] solved the continuous-time JLQ problem with finite horizon, shortly after the work of Krasovskii et al.[2]. Sworder used a stochastic maximum principle while Wonham used the dynamic programming. Wonham also solved the infinite horizon version of the problem and found a set of sufficient conditions to guarantee the existence of a unique, steady-state control law. The discrete-time versions of the JLQ problems were solved by Blair et al.[5] and by Chizeck et al.[6] using dynamic programming. In refs. [7—9], necessary and sufficient conditions were derived for properly defined stochastic controllability and stabilizability. It was shown that the existence of a steady-state solution with finite cost-to-go in the JLQ problem can be attributed to stochastic stabilizability.

For the partial observation case where an output measurement of the states is available, Mariton[10] gave some necessary conditions for the optimality of the output feedback control law for continuous-time models. For the discrete-time jump linear quadratic Gaussian (JLQG) problem, Ji et al. designed an optimal controller based on a separation theorem by using the Kalman filter for state estimation[8].

1.3 Previous results on adaptive control

Adaptive control deals with the case where the Markovian parameters cannot be observed directly. For the continuous-time case, Wonham filtering is applicable to the estimation of the Markovian parameters. Caines et al.[11] used the Wonham filter and the dynamic programming approach to obtain the adaptive optimal control law with finite horizon for the continuous-time JLQG problem. In ref. [12], under the condition that both the jumping rate of the parameter process and the magnitude of the control Riccati equation together with its second derivatives are suitably small, Caines et al. gave a solution to the infinite time adaptive JLQG problem, showing that the closed-loop systems were stabilized in an average sense:

$$\limsup_{T\to\infty} \frac{1}{T} E \int_0^T (\| x_t \|^2 + \| u_t \|^2) dt < \infty .$$

In a related work, Dufour et al.[13] presented a different adaptive control law by introducing a set of algebraic conditions, which can be applied to a class of systems where the controllability condition used in ref. [12] fails.

In the more complicated case where both the states and the parameters are not directly observed, Sworder[14] studied the control problem by introducing an unconventional measurement architecture. However, since no explicit solution could easily be obtained, an approximation to the quadratic-optimal regulator problem has to be made. The solution is in a form quite similar to that obtained in the complete observation case, but the gain equation is much more complicated. In ref. [15], by introducing an exact transformation the adaptive JLQG control is embedded into an LQM (linear quadratic martingale) control with a completely observable stochastic control matrix. However, the martingales involved which drive the new linear systems are generally not Wiener processes and the optimal control policy is unlikely to be practically implementable. Therefore, suboptimal policies are considered as approximation solutions. In a recent work, Dufour et al.[16] considered adaptive control of hybrid systems with Markovian jump parameters, where two channels are included in the model for indirect measurement of the state and the jump parameter. Suboptimal estimations for the state, the Markovian parameter and approximate control law were investigated, and upper bounds for the estimation errors were given for justification of such an approach. However, only the finite horizon JLQG problem was studied in ref. [16], where the stability issue was not a concern.

1.4 The contribution of the present paper

From the above brief review, one can conclude that although considerable research has been conducted in the literature, the following basic problem still remains open: whether or not a stabilizing adaptive controller exists for the hybrid system (1.1) if, besides the controllability condition, one only knows that the transition rate of the hidden Markov chain is small enough. In this paper, we shall provide a positive answer to this problem. The adaptive controller is a certainty equivalent LQ control with parameter estimates given by a projected gradient algorithm[17]. It will be shown that whenever the transition rate of the Markovian parameters is sufficiently small and a certain controllability assumption is satisfied, the model (1.1) can be adaptively stabilized in the sense that

$$\sup_{t\geq 0} (E \| x_t \|^r + E \| u_t \|^r) < \infty ,$$

where $r \geq 1$ is a constant.

The remainder of this paper is organized as follows. In section 2, we describe the parameter estimation algorithm and the adaptive control law. The main stability results are given in section 3, and their proofs are given in section 4.

2 Adaptive controller design

Let us consider the following jump parameter linear model

$$x_{t+1} = A(r_t)x_t + B(r_t)u_t + w_{t+1}, \quad t \geq 0, \tag{2.1}$$

with a long run average quadratic index:

$$J(u) = \limsup_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} E(x_t^\tau Q x_t + u_t^\tau R u_t),$$

where Q and R are two positive definite matrices.

In this paper, we are interested in the case where both the Markov chain $\{r_t\}$ and its transition probabilities $\{p_{ij}\}$ are unknown *a priori*. Therefore, the controller has to be defined based on adaptive estimations of the jump parameters.

To describe the estimation algorithm, we introduce the following notations:

$$\theta_t \triangleq [A(r_t), B(r_t)]^\tau, \quad \varphi_t \triangleq [x_t^\tau, u_t^\tau]^\tau,$$

$$\mathscr{D} \triangleq \left\{\theta : \theta = \sum_{i=1}^{N} \lambda_i [A_i, B_i]^\tau, \ \sum_{i=1}^{N} \lambda_i = 1, \ \lambda_i \geqslant 0\right\}. \tag{2.2}$$

Note that $\mathscr{D}$ is the smallest convex set (a simplex) which contains the finite set $\{[A_i, B_i]^\tau, \ i = 1,2,\cdots,N\}$.

With these notations, the model (2.1) can be rewritten in a compact form:

$$x_{t+1} = \theta_t^\tau \varphi_t + w_{t+1}. \tag{2.3}$$

We estimate θ_t recursively by the following projected gradient algorithm[17]

$$\hat{\theta}_{t+1} = \Pi_{\mathscr{D}}\left\{\hat{\theta}_t + \frac{\varphi_t(x_{t+1}^\tau - \varphi_t^\tau \hat{\theta}_t)}{d + \|\varphi_t\|^2}\right\}, \tag{2.4}$$

where $\Pi_{\mathscr{D}}\{A\}$ denotes the projection on the convex set $\mathscr{D}$ of an $(m+n) \times n$ matrix A, i.e.

$$\Pi_{\mathscr{D}}\{A\} = \arg\min_{B \in \mathscr{D}} \|A - B\|, \tag{2.5}$$

and where d is a suitably large constant which will be specified later. The initial value $\hat{\theta}_0$ can be chosen as an arbitrary element in $\mathscr{D}$.

Now, the adaptive control law can be defined as

$$u_t = -(R + \hat{B}_t^\tau P_t \hat{B}_t)^{-1} \hat{B}_t^\tau P_t \hat{A}_t x_t \triangleq -K_t x_t, \tag{2.6}$$

where P_t is the solution to the following algebraic Riccati equation:

$$P_t = \hat{A}_t^\tau P_t \hat{A}_t - \hat{A}_t^\tau P_t \hat{B}_t (R + \hat{B}_t^\tau P_t \hat{B}_t)^{-1} \hat{B}_t^\tau P_t \hat{A}_t + Q,$$

$$\hat{\theta}_t \triangleq [\hat{A}_t, \ \hat{B}_t]^\tau. \tag{2.7}$$

Substituting (2.6) into (2.3) gives the closed-loop equation

$$x_{t+1} = F_t x_t + \tilde{\theta}_t^\tau \varphi_t + w_{t+1}, \tag{2.8}$$

where

$$F_t = \hat{A}_t - \hat{B}_t K_t, \ \tilde{\theta}_t = \theta_t - \hat{\theta}_t. \tag{2.9}$$

Note that by (2.9), the algebraic Riccati equation (2.7) can be rewritten as

$$P_t = F_t^\tau P_t F_t + K_t^\tau R K_t + Q. \tag{2.10}$$

We conclude this section by giving some interpretations about the adaptive control law (2.6). Given the existing results in the nonadaptive case where both the Markov chain and the transition probability are known, it is natural to design the certainty equivalence adaptive control via a set of coupled Riccati equations described as in refs. [5], [6] and [8], with the corresponding unknown parameters replaced by the estimated ones. However, in this framework, the estimation of the unknown transition probabilities $\{p_{ij}\}$ of the Markov chain presents a daunting challenge[18]. In order to avoid such a difficulty, it seems to be feasible to design the adaptive controller via the decoupled certainty equivalent Riccati equation (2.7), if the transition probabilities $p_{ij}, i \neq j$, are small enough. An additional advantage of this design is the simplicity of the adaptive controller structure.

3 The main result

To establish the global stability of the closed-loop system (2.8), we need the following assumptions:

A1) $\{w_t,\ t \geqslant 1\}$ is an independent noise sequence and there exist constants $\tau > 0$, $b \geqslant 0$, such that $\sup_{t \geqslant 1} E e^{\tau \| w_t \|^2} \leqslant e^b$.

A2) For any element $M \in \mathscr{D}$, the corresponding pair $[A, B]$ with $[A, B]^{\tau} \triangleq M$ is controllable, where $\mathscr{D}$ is defined in (2.2).

The main stability result is stated as follows, its proof is given in section 4.

Theorem 3.1. For the system (2.1) with both unknown Markovian jumps $\{r_t\}$ and unknown transition probability $\{p_{ij}\}$, let assumptions A1) and A2) hold, and the initial state x_0 satisfy $E \| x_0 \|^s < \infty$, $\forall s > 1$. Assume that d in the estimation algorithm (2.4) is suitably large. Then under the adaptive control law (2.6), there exists a real number $\Delta^* > 0$ such that whenever the the maximum transition rate $\Delta \triangleq \max_{i \neq j} \{p_{ij}\}$ satisfies $\Delta < \Delta^*$, the closed-loop adaptive system is L_r-stable in the following sense:

$$\sup_{t \geqslant 0} (E \| x_t \|^r + E \| u_t \|^r) < \infty,$$

where $r \geqslant 1$ is a constant.

Remark 3.1. For the more general model (1.1), if assumptions A1) and A2) are satisfied, then the stability result of Theorem 3.1 still holds with only a slight modification of the bounds on d and Δ. The independence restriction on the noise can also be relaxed. In fact, it can be replaced by the following general one:

$$E \exp\left\{ \sum_{t=i+1}^{j} \tau \| w_t \|^2 \right\} \leqslant e^{b(j-i)}, \forall j > i \geqslant 0,$$

for some constants $\tau > 0$ and $b > 0$, the stability analysis remains the same. Note that this condition includes a large class of correlated stochastic noises[19] as well as any bounded deterministic disturbances.

Remark 3.2. For systems with deterministic time-varying parameters $\{\theta_t^*\}$, the following assumption which depicts the time-varying mode of the parameters is standard[20,21]:

A) For any $t \geqslant 0$ and $T > 0$, the following inequality holds:

$$\sum_{i=t+1}^{t+T} \| \theta_{i+1}^* - \theta_i^* \| \leqslant \varepsilon T + C,$$

where $\varepsilon > 0$ is a sufficiently small number characterizing the rate of parameter variations, and $C > 0$ is a constant. From Proposition A in the appendix, it can be seen that for systems with jump-Markov parameters, this assumption cannot be satisfied in general, regardless of how small the rate of the transition is. Hence the existing deterministic treatments in the literature do not apply to the present case.

Remark 3.3. A rough, yet quantitative value for Δ^* in Theorem 3.1 may be found in the proof presented in the next section. It is not surprising that this value is quite small in general, since it reflects both the nature of adaptation and the key idea behind the controller design philosophy. However, we conjecture that under the conditions of Theorem 3.1, there exists a critical value Δ^*, such that adaptive stabilization of the jump parameter system is possible whenever the unknown transition probability satisfies $\Delta < \Delta^*$, and is impossible in general whenever $\Delta \geqslant \Delta^*$. Finding out such a critical value Δ^* would be important for further understanding the nature of adaptive control.

4 Stability analysis of the closed-loop systems

The proof of Theorem 3.1 is prefaced with several lemmas. We first present some properties on the Riccati equation (2.7).

Lemma 4.1. Under assumption A2), there exist constants $\beta > \alpha > 0$ and $L > 0$, such that for any $t \geqslant 0$,

$$\alpha I \leqslant P_t \leqslant \beta I, \tag{4.1}$$

$$\| P_{t+1} - P_t \|_2 \leqslant L \| \hat{\theta}_{t+1} - \hat{\theta}_t \|_2. \tag{4.2}$$

Proof. By assumption A2), the Riccati equation (2.7) has a unique positive definite solution $P_t \triangleq P(\hat{A}_t, \hat{B}_t)$ for any $t \geqslant 0$. Since this solution is analytic with respect to $[\hat{A}_t, \hat{B}_t]$ (see ref. [22]) and the convex set $\mathscr{D}$ is closed and bounded, it is obvious that there exist $\beta > \alpha > 0$ such that (4.1) holds. Similarly, (4.2) holds because the partial derivatives of P_t with respect to $[\hat{A}_t, \hat{B}_t]$ are bounded, which implies that the Lipschitz condition holds.

Let us now introduce the following notations:

$$\alpha_t \triangleq \frac{\| \varphi_t^\tau \tilde{\theta}_t \|^2}{d + \| \varphi_t \|^2}, \quad \| x \|_P \triangleq \sqrt{x^\tau P x}, \tag{4.3}$$

where $x \in \mathbb{R}^n$, and $P \in \mathbb{R}^{n \times n}$ is a positive definite matrix. It is easy to see that $\| \cdot \|_P$ is a norm, and that for any positive definite matrices P and U,

$$\begin{aligned} \| x \|_P &= \left\{ x^\tau U^{\frac{1}{2}} \left[I + U^{-\frac{1}{2}} (P - U) U^{-\frac{1}{2}} \right] U^{\frac{1}{2}} x \right\}^{1/2} \\ &\leqslant \left(1 + \sqrt{\| P - U \|_2 \cdot \| U^{-1} \|_2} \right) \| x \|_U, \quad x \in \mathbb{R}^n. \end{aligned} \tag{4.4}$$

In the following analyses of the closed-loop system (2.8), the above defined norm plays a key role.

Lemma 4.2. If assumption A2) is satisfied, then

$$\| x_{t+1} \|_{P_{t+1}} \leqslant \lambda_t \| x_t \|_{P_t} + \| w_{t+1} \|_{P_{t+1}} + M,$$

with

$$\begin{cases} \lambda_t = \lambda + \lambda \sqrt{\dfrac{L}{\alpha}} \cdot \sqrt[4]{\alpha_t} + c_1 \sqrt{\alpha_t} + \lambda \sqrt{\dfrac{L}{2\alpha}} \cdot \sqrt{\dfrac{\| w_{t+1} \|}{\sqrt[4]{d}}}, \\ \lambda = \sqrt{1 - \dfrac{\lambda_{\min}(Q)}{\beta}}, \\ c_1 = \sup\limits_{t \geqslant 0} (1 + \| K_t \|_2) \sqrt{\dfrac{\beta}{\alpha}}, \\ M = \sup\limits_{t \geqslant 0} \sqrt{\beta d \alpha_t}, \end{cases} \tag{4.5}$$

where α, β, L, K_t, P_t are defined by (4.1), (4.2), (2.6) and (2.7), respectively, and $\lambda_{\min}(Q)$ denotes the smallest eigenvalue of Q appearing in the cost functional.

Proof. Since $\| \cdot \|_P$ is a norm, by (2.8), we have

$$\| x_{t+1} \|_{P_{t+1}} \leqslant \| F_t x_t \|_{P_{t+1}} + \| w_{t+1} \|_{P_{t+1}} + \| \tilde{\theta}_t \varphi_t \|_{P_{t+1}}, \tag{4.6}$$

where P_{t+1} is the solution of the Riccati equation (2.7).

Now we proceed to analyze the right-hand-side of (4.6). By (4.4), we have

$$\| F_t x_t \|_{P_{t+1}} \leqslant \left(1 + \sqrt{\| P_{t+1} - P_t \|_2 \cdot \| P_t^{-1} \|_2} \right) \cdot \| F_t x_t \|_{P_t}. \tag{4.7}$$

Furthermore, by (2.10) we have

$$\| F_t x_t \|_{P_t}^2 = x_t^\tau (P_t - K_t^\tau R_t K_t - Q) x_t \leqslant x_t^\tau (P_t - Q) x_t \leqslant \left(1 - \frac{\lambda_{\min}(Q)}{\beta}\right) x_t^\tau P_t x_t, \tag{4.8}$$

where β is determined by (4.1). Define

$$\lambda = \sqrt{1 - \frac{\lambda_{\min}(Q)}{\beta}} \in (0,1).$$

Then, it follows from (4.8) that

$$\| F_t x_t \|_{P_t} \leqslant \lambda \| x_t \|_{P_t}. \tag{4.9}$$

On the other hand, by the property of projection in (2.4), we have

$$\begin{aligned} \| \hat{\theta}_{t+1} - \hat{\theta}_t \|_2 \leqslant \| \hat{\theta}_{t+1} - \hat{\theta}_t \| &\leqslant \frac{\| \varphi_t \|}{d + \| \varphi_t \|^2} \cdot (\| \varphi_t^\tau \tilde{\theta}_t \| + \| w_{t+1} \|) \\ &\leqslant \sqrt{\alpha_t} + \frac{\| w_{t+1} \|}{2\sqrt{d}}, \end{aligned} \tag{4.10}$$

where α_t is defined by (4.3).

Substituting (4.2), (4.9) and (4.10) into (4.7), we get

$$\| F_t x_t \|_{P_{t+1}} \leqslant \left[\lambda + \lambda \sqrt{\frac{L}{\alpha}} \cdot \left(\sqrt[4]{\alpha_t} + \sqrt{\frac{\| w_{t+1} \|}{2\sqrt{d}}} \right) \right] \| x_t \|_{P_t}, \tag{4.11}$$

where α and L are given by (4.1) and (4.2) respectively.

For the last term on the right-hand-side of (4.6), we have

$$\begin{aligned} \| \tilde{\theta}_t^\tau \varphi \|_{P_{t+1}} &\leqslant \sqrt{\alpha_t (d + \| \varphi_t \|^2) \| P_{t+1} \|} \\ &\leqslant \sqrt{\alpha_t d \| P_{t+1} \|} + \sqrt{\alpha_t \| P_{t+1} \|} \cdot \| \varphi_t \| \\ &= \sqrt{\alpha_t d \| P_{t+1} \|} + \sqrt{\alpha_t \| P_{t+1} \|} \cdot (\| x_t \| + \| K_t x_t \|) \\ &\leqslant \sqrt{\beta d \alpha_t} + \sqrt{\beta \alpha_t} \cdot (1 + \| K_t \|_2) \cdot \| x_t \| \\ &\leqslant \sqrt{\beta d \alpha_t} + \sqrt{\beta \alpha_t} \cdot (1 + \| K_t \|_2) \cdot \sqrt{\frac{x_t^\tau P_t x_t}{\alpha}} \\ &= \sqrt{\beta d \alpha_t} + (1 + \| K_t \|_2) \cdot \| x_t \|_{P_t} \cdot \sqrt{\frac{\beta \alpha_t}{\alpha}} \\ &\leqslant c_1 \sqrt{\alpha_t} \| x_t \|_{P_t} + M, \end{aligned} \tag{4.12}$$

where c_1 and M are defined by (4.5).

Substituting (4.11) and (4.12) into (4.6) yields Lemma 4.2.

Lemma 4.3. For α_t defined in (4.3), the following inequality holds,

$$\alpha_t \leqslant \mathrm{Tr}\{ \tilde{\theta}_t^\tau \tilde{\theta}_t - \tilde{\theta}_{t+1}^\tau \tilde{\theta}_{t+1} \} + \left(\frac{1}{d} + \frac{G}{2\sqrt{d}} \right) \| w_{t+1} \|^2 + \frac{G}{2\sqrt{d}} + 3nG^2 I_{(r_t \neq r_{t+1})},$$

where $\mathrm{Tr}\{\cdot\}$ denotes the trace of a matrix, $I_{(\cdot)}$ is the indicator function, n is the dimension of the state x_t in the system (2.1), r_t is the Markovian parameter, and G is a constant defined by

$$G \triangleq \max_{1 \leqslant i,j \leqslant N} \| [A_i, B_i] - [A_j, B_j] \|_2.$$

Proof. First of all, by Proposition B in the appendix, we know that

$$G \triangleq \max_{\theta', \theta'' \in \mathscr{D}} \| \theta' - \theta'' \|_2, \tag{4.13}$$

where $\mathscr{D}$ is defined in (2.2).

Now, suppose that T_1 and T_2 are two real matrices or vectors with the same dimension. By

$$(T_1 - T_2)^\tau (T_1 - T_2) = T_1^\tau T_1 + T_2^\tau T_2 - T_1^\tau T_2 - T_2^\tau T_1 \geqslant 0,$$

we have

$$T_1^\tau T_2 + T_2^\tau T_1 \leqslant T_1^\tau T_1 + T_2^\tau T_2. \tag{4.14}$$

Note that $\hat{\theta}_{t+1}, \theta_{t+1}$ and θ_t are all in $\mathscr{D}$, it follows from (4.13) and (4.14) that

$$\begin{aligned}
&[\hat{\theta}_{t+1} - \theta_{t+1}]^\tau [\hat{\theta}_{t+1} - \theta_{t+1}] \\
&= [\hat{\theta}_{t+1} - \theta_t + \theta_t - \theta_{t+1}]^\tau [\hat{\theta}_{t+1} - \theta_t + \theta_t - \theta_{t+1}] \\
&= [\hat{\theta}_{t+1} - \theta_t]^\tau [\hat{\theta}_{t+1} - \theta_t] + [\theta_t - \theta_{t+1}]^\tau [\theta_t - \theta_{t+1}] \\
&\quad + [\hat{\theta}_{t+1} - \theta_t]^\tau [\theta_t - \theta_{t+1}] + [\theta_t - \theta_{t+1}]^\tau [\hat{\theta}_{t+1} - \theta_t] \\
&\leqslant [\hat{\theta}_{t+1} - \theta_t]^\tau [\hat{\theta}_{t+1} - \theta_t] + 3G^2 I_{(r_t \neq r_{t+1})} \cdot I_{n\times n},
\end{aligned} \tag{4.15}$$

where $I_{n\times n}$ is the $n \times n$ identity matrix. Taking trace on both sides of (4.15) gives

$$\mathrm{Tr}\{\tilde{\theta}_{t+1}^\tau \tilde{\theta}_{t+1}\} \leqslant \mathrm{Tr}\{[\hat{\theta}_{t+1} - \theta_t]^\tau [\hat{\theta}_{t+1} - \theta_t]\} + 3nG^2 I_{(r_t \neq r_{t+1})}. \tag{4.16}$$

Denote

$$H_t = \theta_t - \hat{\theta}_t - \frac{\varphi_t [x_{t+1}^\tau - \varphi_t^\tau \hat{\theta}_t]}{d + \|\varphi_t\|^2}. \tag{4.17}$$

Then we have

$$H_t = \tilde{\theta}_t - \frac{\varphi_t \varphi_t^\tau \tilde{\theta}_t + \varphi_t w_{t+1}^\tau}{d + \|\varphi_t\|^2},$$

and so

$$\begin{aligned}
H_t^\tau H_t = {} & \tilde{\theta}_t^\tau \tilde{\theta}_t + \frac{\tilde{\theta}_t^\tau \varphi_t \varphi_t^\tau \varphi_t \varphi_t^\tau \tilde{\theta}_t}{(d + \|\varphi_t\|^2)^2} - \frac{2\tilde{\theta}_t^\tau \varphi_t \varphi_t^\tau \tilde{\theta}_t}{d + \|\varphi_t\|^2} + \frac{w_{t+1} \varphi_t^\tau \varphi_t w_{t+1}}{(d + \|\varphi_t\|^2)^2} \\
& - \frac{w_{t+1}\varphi_t^\tau}{d + \|\varphi_t\|^2} \cdot \left[I - \frac{\varphi_t \varphi_t^\tau}{d + \|\varphi_t\|^2} \right] \tilde{\theta}_t - \tilde{\theta}_t^\tau \left[I - \frac{\varphi_t \varphi_t^\tau}{d + \|\varphi_t\|^2} \right] \cdot \frac{\varphi\, w_{t+1}^\tau}{d + \|\varphi_t\|^2} \\
\leqslant {} & \tilde{\theta}_t^\tau \tilde{\theta}_t - \frac{\tilde{\theta}_t^\tau \varphi_t \varphi_t^\tau \tilde{\theta}_t}{d + \|\varphi_t\|^2} + \frac{w_{t+1} w_{t+1}^\tau}{d + \|\varphi_t\|^2} - \frac{d w_{t+1} \varphi_t^\tau \tilde{\theta}_t}{(d + \|\varphi_t\|^2)^2} - \frac{d \tilde{\theta}_t^\tau \varphi_t w_{t+1}^\tau}{(d + \|\varphi_t\|^2)^2}.
\end{aligned} \tag{4.18}$$

By the simple inequality $2\sqrt{d}\, \|\varphi_t\| \leqslant d + \|\varphi_t\|^2$ and the property (4.13), we have

$$\begin{aligned}
&\mathrm{Tr}\left\{ \frac{d w_{t+1} \varphi_t^\tau \tilde{\theta}_t}{(d + \|\varphi_t\|^2)^2} + \frac{d\, \tilde{\theta}_t^\tau \varphi_t w_{t+1}^\tau}{(d + \|\varphi_t\|^2)^2} \right\} \leqslant \frac{2d\, \|\varphi_t\| \cdot \|w_{t+1}\| \cdot G}{(d + \|\varphi_t\|^2)^2} \\
&\leqslant \frac{\sqrt{d}\, \|w_{t+1}\| \cdot G}{d + \|\varphi_t\|^2} \leqslant \frac{G}{\sqrt{d}} \|w_{t+1}\| \leqslant \frac{G}{2\sqrt{d}} (1 + \|w_{t+1}\|^2).
\end{aligned} \tag{4.19}$$

Taking trace on both sides of (4.18), using (4.3) and (4.19) we get

$$\mathrm{Tr}\{H_t^\tau H_t\} \leqslant \mathrm{Tr}\{\tilde{\theta}_t^\tau \tilde{\theta}_t\} - \alpha_t + \left(\frac{1}{d} + \frac{G}{2\sqrt{d}} \right) \|w_{t+1}\|^2 + \frac{G}{2\sqrt{d}}. \tag{4.20}$$

Now, since $\theta_t \in \mathscr{D}$, by Proposition C in the appendix, it follows from (2.4) and (4.17) that $\|\hat{\theta}_{t+1} - \theta_t\| \leqslant \|H_t\|$, which in conjunction with (4.20) yields

$$\begin{aligned}
&\mathrm{Tr}\{[\hat{\theta}_{t+1} - \theta_t]^\tau [\hat{\theta}_{t+1} - \theta_t]\} \leqslant \mathrm{Tr}\{H_t^\tau H_t\} \\
&\leqslant \mathrm{Tr}\{\tilde{\theta}_t^\tau \tilde{\theta}_t\} - \alpha_t + \left(\frac{1}{d} + \frac{G}{2\sqrt{d}} \right) \|w_{t+1}\|^2 + \frac{G}{2\sqrt{d}}.
\end{aligned} \tag{4.21}$$

Hence by (4.16) and (4.21), we have

$$\operatorname{Tr}\{\tilde{\theta}_{t+1}^{\tau}\tilde{\theta}_{t+1}\} \leqslant \operatorname{Tr}\{\tilde{\theta}_t^{\tau}\tilde{\theta}_t\} - \alpha_t + \left(\frac{1}{d} + \frac{G}{2\sqrt{d}}\right) \| w_{t+1} \|^2 + \frac{G}{2\sqrt{d}} + 3nG^2 I_{(r_t \neq r_{t+1})}, \tag{4.22}$$

and Lemma 4.3 follows easily from (4.22).

Lemma 4.4. Let $\{r_t, t \geqslant 0\}$ be the Markov chain in the model (2.1) with N states and with transition probability $\{p_{ij}\}$. Then for any $t > h > 0$,

$$E\exp\left\{ C \sum_{i=h+1}^{t} I_{(r_i \neq r_{i+1})} \right\} \leqslant \exp\{ e^C \Delta (N-1)(t-h) \},$$

where $\Delta = \max_{i \neq j} p_{ij}$, and C is an arbitrary constant.

Proof. Let $\mathscr{F}_t$ be the σ-algebra generated by $\{r_i, i \leqslant t\}$, i.e. $\mathscr{F}_t = \sigma(r_i, i \leqslant t)$. By the property of conditional expectation, it follows that

$$E\exp\left\{ C \sum_{i=h+1}^{t} I_{(r_i \neq r_{i+1})} \right\} = E\left\{ \exp\left(C \sum_{i=h+1}^{t-1} I_{(r_i \neq r_{i+1})} \right) \cdot E\left[e^{CI_{(r_t \neq r_{t+1})}} \,\middle|\, \mathscr{F}_t \right] \right\}. \tag{4.23}$$

By the Markov property[23], we have

$$\begin{aligned} E\exp\{ e^{CI_{(r_t \neq r_{t+1})}} \mid \mathscr{F}_t \} &= E\exp\{ e^{CI_{(r_t \neq r_{t+1})}} \mid \sigma(r_t) \} \\ &= E\{ I_{(r_t = r_{t+1})} + e^C I_{(r_t \neq r_{t+1})} \mid \sigma(r_t) \} \\ &\leqslant 1 + e^C E[I_{(r_t \neq r_{t+1})} \mid \sigma(r_t)], \end{aligned} \tag{4.24}$$

where $\sigma(r_t)$ denotes the σ-algebra generated by r_t. Now, we proceed to estimate the upper bound of $E[I_{(r_t \neq r_{t+1})} \mid \sigma(r_t)]$. First, note that

$$\begin{aligned} &E\{ I_{(r_t = i)} \cdot E[I_{(r_t \neq r_{t+1})} \mid \sigma(r_t)] \} = EI_{(r_t \neq r_{t+1})} \cdot I_{(r_t = i)} \\ &= P(r_t = i, r_{t+1} \neq i) = P(r_{t+1} \neq i \mid r_t = i) \cdot P(r_t = i) \\ &= \sum_{j \neq i} P(r_{t+1} = j \mid r_t = i) \cdot P(r_t = i) \leqslant (N-1)\Delta P(r_t = i). \end{aligned} \tag{4.25}$$

Since $E[I_{(r_t \neq r_{t+1})} \mid \sigma(r_t)]$ is a constant a.s. on $\{r_t = i\}$, by (4.25), it is easy to see that

$$E[I_{(r_t \neq r_{t+1})} \mid \sigma(r_t)] \leqslant (N-1)\Delta, \text{ a.s. on } \{r_t = i\},$$

which results in

$$E[I_{(r_t \neq r_{t+1})} \mid \sigma(r_t)] \leqslant (N-1)\Delta, \text{ a.s. on } \Omega. \tag{4.26}$$

By (4.23), (4.24) and (4.26), it is easy to conclude that

$$\begin{aligned} E\exp\left\{ C \sum_{i=h+1}^{t} I_{(r_i \neq r_{i+1})} \right\} &\leqslant E\exp\left\{ C \sum_{i=h+1}^{t-1} I_{(r_i \neq r_{i+1})} \right\} \cdot \{1 + e^C (N-1)\Delta\} \\ &\leqslant E\exp\left\{ C \sum_{i=h+1}^{t-1} I_{(r_i \neq r_{i+1})} \right\} \cdot \exp\{ e^C \Delta (N-1) \} \\ &\leqslant \cdots \\ &\leqslant \exp\{ e^C \Delta (N-1)(t-h) \}. \end{aligned}$$

Lemma 4.4 will play a key role in establishing the global stability. This lemma shows that the possible undesirable effects on stability resulting from parameter jumps can be dominated by the transition probability.

Proof of Theorem 3.1. First of all, for any $x \geqslant 0$, $\varepsilon > 0$, we have

$$x \leqslant \varepsilon + \frac{x^2}{4\varepsilon}. \tag{4.27}$$

By (4.27), it is easy to see that

$$x^2 \leqslant 2\varepsilon^2 + \frac{x^4}{8\varepsilon^2}. \tag{4.28}$$

Substituting (4.28) into the right-hand-side of (4.27), we have

$$x \leqslant \frac{3\varepsilon}{2} + \frac{x^4}{32\varepsilon^3}. \tag{4.29}$$

By taking x as $\sqrt{\alpha_t}$ and $\sqrt{\| w_{t+1} \|}$ respectively in (4.27), and x as $\sqrt[4]{\alpha_t}$ in (4.29), we can estimate the quantity λ_t defined in Lemma 4.2 as follows:

$$\begin{aligned}\lambda_t &\leqslant \lambda + \lambda\sqrt{\frac{L}{\alpha}} \cdot \left(\frac{3\varepsilon}{2} + \frac{\alpha_t}{32\varepsilon^3}\right) + c_1\left(\varepsilon + \frac{\alpha_t}{4\varepsilon}\right) + \sqrt{\frac{L}{2\alpha}} \cdot \frac{\lambda}{\sqrt[4]{d}} \cdot \left(1 + \frac{1}{4}\| w_{t+1} \|^2\right) \\ &= c_2 + c_3\alpha_t + c_4 \| w_{t+1} \|^2,\end{aligned} \tag{4.30}$$

where

$$\begin{cases} c_2 = \lambda + \left(\dfrac{3\lambda}{2}\sqrt{\dfrac{L}{\alpha}} + c_1\right)\varepsilon + \sqrt{\dfrac{L}{2\alpha}} \cdot \dfrac{\lambda}{\sqrt[4]{d}}, \\ c_3 = \dfrac{\lambda}{32\varepsilon^3}\sqrt{\dfrac{L}{\alpha}} + \dfrac{c_1}{4\varepsilon}, \\ c_4 = \dfrac{\lambda}{4}\sqrt{\dfrac{L}{2\alpha}} \cdot \dfrac{1}{\sqrt[4]{d}}. \end{cases} \tag{4.31}$$

It is obvious that

$$\sum_{t=i+1}^{j} \mathrm{Tr}[\tilde{\theta}_t^\tau \tilde{\theta}_t - \tilde{\theta}_{t+1}^\tau \tilde{\theta}_{t+1}] = \mathrm{Tr}[\tilde{\theta}_{i+1}^\tau \tilde{\theta}_{i+1} - \tilde{\theta}_{j+1}^\tau \tilde{\theta}_{j+1}] \leqslant \mathrm{Tr}(\tilde{\theta}_{i+1}^\tau \tilde{\theta}_{i+1}) \leqslant nG^2, \tag{4.32}$$

where n is the dimension of the state in model (2.1) and G is defined in Lemma 4.3. By Lemma 4.3 and (4.32), we have

$$\begin{aligned}\sum_{t=i+1}^{j} \alpha_t &\leqslant \sum_{t=i+1}^{j} \mathrm{Tr}[\tilde{\theta}_t^\tau \tilde{\theta}_t - \tilde{\theta}_{t+1}^\tau \tilde{\theta}_{t+1}] \\ &\quad + \sum_{t=i+1}^{j}\left[\left(\frac{1}{d} + \frac{G}{2\sqrt{d}}\right) \| w_{t+1} \|^2 + \frac{G}{2\sqrt{d}}\right] + \sum_{t=i+1}^{j} 3nG^2 I_{(r_t \neq r_{t+1})} \\ &\leqslant nG^2 + \sum_{t=i+1}^{j}\left(\frac{1}{d} + \frac{G}{2\sqrt{d}}\right) \| w_{t+1} \|^2 + \frac{G}{2\sqrt{d}}(j - i) + \sum_{t=i+1}^{j} 3nG^2 I_{(r_t \neq r_{t+1})}.\end{aligned} \tag{4.33}$$

For $x \geqslant 0$, it is easy to verify that $x \leqslant e^{x-1}$. By this, (4.30) and (4.33), we have for any $r \geqslant 1$, $p > 1$ and $j > i$,

$$\begin{aligned}\prod_{t=i+1}^{j} \lambda_t^{pr} &\leqslant \exp\left\{\sum_{t=i+1}^{j} pr(c_2 + c_3\alpha_t + c_4 \| w_{t+1} \|^2\right\} \cdot \exp\{-pr(j - i)\} \\ &\leqslant \exp\{prnc_3G^2\} \cdot \exp\left\{-pr\left(1 - c_2 - \frac{c_3G}{2\sqrt{d}}\right)(j - i)\right\} \\ &\quad \cdot \exp\left\{pr\left[c_4 + c_3\left(\frac{1}{d} + \frac{G}{2\sqrt{d}}\right)\right]\sum_{t=i+1}^{j} \| w_{t+1} \|^2\right\} \\ &\quad \cdot \exp\left\{3prnc_3G^2\sum_{t=i+1}^{j} I_{(r_t \neq r_{t+1})}\right\}.\end{aligned} \tag{4.34}$$

Let us now take Δ^* as

$$\Delta^* = \frac{r(1-\lambda)}{5(N-1)} e^{-3rnc_3 G^2}.$$

Then for any $\Delta \in (0, \Delta^*)$, there must exist a constant $p > 1$ such that

$$\Delta \leqslant \frac{p^2 r(1-\lambda)}{5(N-1)} e^{-3p^2 rnc_3 G^2}. \tag{4.35}$$

Now, let us denote $q \triangleq (1-p^{-1})^{-1}$ and $c_5 \triangleq c_4 + c_3\left(\frac{1}{d} + \frac{G}{2\sqrt{d}}\right)$, and take d large enough, such that $pqrc_5/\tau \leqslant 1$. This is possible since c_4 and hence c_5 is a decreasing function of d. By Jensen's inequality and assumption A1), it follows that

$$E^{1/q}\exp\{pqrc_5 \| w_{t+1} \|^2\} = E^{1/q}\exp\left\{\frac{pqrc_5}{\tau} \cdot \tau \| w_{t+1} \|^2\right\} \leqslant \exp\left\{prc_5 \cdot \frac{b}{\tau}\right\}. \tag{4.36}$$

Hence, by the independence of $\{w_t\}$, we have

$$E^{1/q}\exp\left\{pqrc_5 \sum_{t=i+1}^{j} \| w_{t+1} \|^2\right\} \leqslant \exp\left\{prc_5 \cdot \frac{b}{\tau} \cdot (j-i)\right\}. \tag{4.37}$$

To estimate the last term in (4.34), we have by Lemma 4.4

$$E^{1/p}\exp\left\{3p^2 rnc_3 G^2 \sum_{t=i+1}^{j} I_{(r_t \neq r_{t+1})}\right\} \leqslant \exp\left\{\frac{1}{p}\Delta(N-1)(j-i) e^{3p^2 rnc_3 G^2}\right\}. \tag{4.38}$$

Hence by (4.35), the Hölder inequality, (4.37) and (4.38), it follows that

$$E\prod_{t=i+1}^{j} \lambda_t^{pr} \leqslant \exp\{prnc_3 G^2\} \cdot \exp\left\{-pr\left(1 - c_2 - \frac{c_3 G}{2\sqrt{d}}\right)(j-i)\right\}$$
$$\cdot \exp\left\{pr(j-i)c_5 \cdot \frac{b}{\tau}\right\} \cdot \exp\left\{\frac{1}{p}\Delta(N-1)(j-i) e^{3p^2 rnc_3 G^2}\right\}. \tag{4.39}$$

To further analyse the RHS of (4.39), we now take ε appearing in (4.31) as

$$\varepsilon = \frac{1-\lambda}{5}\left(\frac{3\lambda}{2}\sqrt{\frac{L}{2\alpha}} + c_1\right)^{-1}.$$

then by (4.31),

$$c_2 = \lambda + \frac{1-\lambda}{5} + \sqrt{\frac{L}{2\alpha}} \cdot \frac{\lambda}{\sqrt[4]{d}}. \tag{4.40}$$

Next, we may choose d large enough to satisfy the following constraints:

$$\begin{cases} \sqrt{\frac{L}{2\alpha}} \cdot \frac{\lambda}{\sqrt[4]{d}} + \frac{c_3 G}{2\sqrt{d}} \leqslant \frac{1-\lambda}{5} \\ \frac{b}{\tau} c_5 \leqslant \frac{1-\lambda}{5} \\ qrc_5 \leqslant \tau. \end{cases} \tag{4.41}$$

Hence, substituting (4.35), (4.40) and (4.41) into (4.39), it is easy to verify that

$$E\prod_{t=i+1}^{j} \lambda_t^{pr} \leqslant \exp\{pnrc_3 G^2\} \cdot \exp\left\{-\frac{pr(1-\lambda)}{5}(j-i)\right\}. \tag{4.42}$$

Now we are in a position to prove the L_r-boundedness of x_t and u_t. By Lemma 4.2, we have

$$\| x_{t+1} \|_{P_{t+1}} \leqslant \xi_{t+1} + \left(\prod_{j=0}^{t} \lambda_j\right) \| x_0 \|_{P_0} + \sum_{i=1}^{t}\left(\prod_{j=i}^{t} \lambda_j\right)\xi_i, \tag{4.43}$$

where $\xi_i = \| w_i \|_{P_i} + M$.

Hence, by the Minkowski and the Hölder inequalities, it follows from (4.43) that

$$\{E \| x_{t+1} \|_{P_{t+1}}^r \}^{\frac{1}{r}}$$
$$\leqslant \{E\xi_{t+1}^r\}^{\frac{1}{r}} + \left\{E\prod_{j=0}^{t}\lambda_j^{pr}\right\}^{\frac{1}{pr}} \{E \| x_0 \|_{P_0}^{qr}\}^{\frac{1}{qr}} + \sum_{i=1}^{t}\left\{E\prod_{j=i}^{t}\lambda_j^{pr}\right\}^{\frac{1}{pr}} \{E\xi_i^{qr}\}^{\frac{1}{qr}}, \tag{4.44}$$

where p and q are positive numbers defined as above, which satisfy $p^{-1} + q^{-1} = 1$.

By (4.45) and the L_{qr}-boundedness of $\{ \| x_0 \|_{P_0}, \xi_i, i \geqslant 1\}$, it follows that

$$\sup_t E \| x_t \|_{P_t}^r < \infty, \tag{4.45}$$

which in conjunction with (2.6) and (4.1) yields the L_r-boundedness of $\{u_t\}$. Hence $\sup(E \| x_t \|^r + E \| u_t \|^r) < \infty$, and this completes the stability proof.

5 Conclusion

In this paper, the adaptive control of discrete-time hybrid systems with unknown Markovian jumps is considered. The projected gradient algorithm is used to estimate the unknown time-varying system parameters which are driven by the hidden Markov chain. A simple adaptive LQ controller is designed based on a decoupled certainty equivalent Riccati equation. Under a suitable controllability condition on the hybrid system, it is shown that the closed-loop adaptive system is globally stable whenever the transition probability of the hidden Markov chain is small enough, giving a complete solution to a basic unsolved problem in this area.

We have in this contribution concentrated on the stability issue of the adaptive LQ control of the jump Markov model (2.1), under the condition that the unknown transition probability of the hidden Markov chain is small enough. For further investigation, it would be of considerable interest to determine the largest possible value or critical value of the bound Δ^* for the transition probability, as explained in Remark 3.3. On the other hand, it would also be of interest to study the asymptotic optimality of the performance of the adaptive LQ control, as the rate of the Markovian transition diminishes, i.e. $\Delta^* \to 0$.

Appendix

Proposition A. Let $r_t \in \{1,2\}$ be a two-state Markov chain. If there exists $\delta > 0$ such that $p_{12} > \delta$ and $p_{21} > \delta$, then for any pair $C > 0$ and $\varepsilon > 0$, with $\varepsilon \leqslant \nu \triangleq \frac{1}{2} \| [A_1, B_1] - [A_2, B_2] \|$, there always exists T_0, such that for all t and $T \geqslant T_0$

$$P\left(\sum_{i=t+1}^{t+T} \| \theta_{i+1} - \theta_i \| > \varepsilon T + C\right) > 0.$$

Proof. Denote

$$S1 = \left\{r_{t+1} = 1,\ r_{t+2} = 2, \cdots, r_{t+T+1} = \frac{3+(-1)^{T+1}}{2}\right\},$$

$$S2 = \left\{r_{t+1} = 2,\ r_{t+2} = 1, \cdots, r_{t+T+1} = \frac{3+(-1)^{T}}{2}\right\}.$$

By the Markov property, we have

$$P(S1) = P\left(r_{t+T+1} = \frac{3+(-1)^{T+1}}{2} \,\middle|\, r_{t+T} = \frac{3+(-1)^{T}}{2}\right)$$

$$\begin{aligned}
&\cdot P\left(r_{t+T}=\frac{3+(-1)^{T}}{2},\cdots,r_{t+1}=1\right)\\
&\geqslant \delta P\left(r_{t+T}=\frac{3+(-1)^{T}}{2},\cdots,r_{t+1}=1\right)\\
&\geqslant \cdots\\
&\geqslant \delta^{T}P(r_{t+1}=1).
\end{aligned}\tag{A.1}$$

Similarly, we have

$$P(S2)\geqslant \delta^{T}P(r_{t+1}=2).\tag{A.2}$$

For $\varepsilon \leqslant \nu$, when $T>\frac{C}{\nu}$, it is obvious that

$$S1\cup S2\subseteq\left\{\sum_{i=t+1}^{t+T}\|\theta_{i+1}-\theta_i\|>\varepsilon T+C\right\}.\tag{A.3}$$

Hence, by (A.1)—(A.3), we have

$$P\left\{\sum_{i=t+1}^{t+T}\|\theta_{i+1}-\theta_i\|>\varepsilon T+C\right\}\geqslant \delta^{T}P(r_{t+1}=1)+\delta^{T}P(r_{t+1}=2)=\delta^{T}.$$

This completes the proof.

Proposition B. Let $\mathscr{D}$ be the set defined as in (2.2) and G be the number defined in Lemma 4.3. Then $G=\max_{\theta',\theta''\in\mathscr{D}}\|\theta'-\theta''\|_2$.

Proof. By definition (2.2), we need only to show that

$$\max_{\theta',\theta''\in\mathscr{D}}\|\theta'-\theta''\|_2\leqslant G.\tag{A.4}$$

For simplicity, denote $M_i=[A_i,B_i]^{\tau}$. We now prove (A.4) by induction for $N\geqslant 2$.

If $N=2$, then for any $\theta',\theta''\in\mathscr{D}$ with

$$\theta'=\lambda_1M_1+\lambda_2M_2,\ \lambda_1+\lambda_2=1,$$
$$\theta''=\mu_1M_1+\mu_2M_2,\ \mu_1+\mu_2=1,$$

where λ_i and $\mu_i(i=1,2)$ are nonnegative constants, it is easy to see that

$$\begin{aligned}
&\|\theta'-\theta''\|_2\\
&=\|(1-\lambda_2)M_1+\lambda_2M_2-(1-\mu_2)M_1-\mu_2M_2\|_2\\
&=\|(\mu_2-\lambda_2)M_1-(\mu_2-\lambda_2)M_2\|_2\\
&=|\mu_2-\lambda_2|\,\|M_1-M_2\|_2\leqslant\|M_1-M_2\|_2.
\end{aligned}$$

Hence, the proposition holds for $N=2$.

Next, let us assume that (A.4) holds for $N=k-1\geqslant 2$.

To prove (A.4) for $N=k$, we first show that for any matrices L, M and N with the same dimension, we have

$$\begin{aligned}
&\max_{\lambda_1,\mu_1\in[0,1]}\|(1-\lambda_1)L+\lambda_1M-(1-\mu_1)N-\mu_1M\|_2\\
&\leqslant\max\{\|L-M\|_2,\|L-N\|_2,\|N-M\|_2\}.
\end{aligned}\tag{A.5}$$

This can be easily verified by observing that

$$\begin{aligned}
&(1-\lambda_1)L+\lambda_1M-(1-\mu_1)N-\mu_1M\\
&=(1-\lambda_1)[(1-\mu_1)(L-N)+\mu_1(L-M)]+\lambda_1(1-\mu_1)(M-N).
\end{aligned}$$

Now, let $\theta',\theta''\in\mathscr{D}$ with

$$\theta'=\sum_{i=1}^{k}\lambda_iM_i,\ \theta''=\sum_{i=1}^{k}\mu_iM_i,$$

$$\sum \lambda_1 = 1, \ \sum \mu_i = 1, \ \lambda_i \geqslant 0, \ \mu_i \geqslant 0.$$

and set

$$L = \sum_{i=2}^{k} \frac{\lambda_i}{1 - \lambda_1} M_i, \quad N = \sum_{i=2}^{k} \frac{\mu_i}{1 - \mu_1} M_i,$$

by (A.5), we then have

$$\begin{aligned} \| \theta' - \theta'' \| &= \| (1 - \lambda_1) L + \lambda_1 M_1 - (1 - \mu_1) N - \mu_1 M_1 \|_2 \\ &\leqslant \max\{ \| L - M_1 \|_2, \| L - N \|_2, \| M_1 - N \|_2 \}. \end{aligned} \tag{A.6}$$

Now, by the induction assumption, we have

$$\| L - N \|_2 \leqslant \max_{2 \leqslant i,j \leqslant k} \| M_i - M_j \|_2. \tag{A.7}$$

Also, since $\sum_{i=2}^{k} \lambda_i / (1 - \lambda_1) = 1$,

$$\begin{aligned} \| L - M_1 \|_2 &= \left\| \sum_{i=2}^{k} \frac{\lambda_i}{1 - \lambda_1} (M_i - M_1) \right\|_2 \\ &\leqslant \max_{2 \leqslant i \leqslant k} \| M_i - M_1 \|_2 \cdot \sum_{i=2}^{k} \frac{\lambda_i}{1 - \lambda_1} \\ &= \max_{2 \leqslant i \leqslant k} \| M_i - M_1 \|_2. \end{aligned} \tag{A.8}$$

Similarly,

$$\| N - M_1 \|_2 \leqslant \max_{2 \leqslant i \leqslant k} \| M_i - M_1 \|_2. \tag{A.9}$$

Combining (A.6)—(A.9), we finally get

$$\| \theta' - \theta'' \|_2 \leqslant \max_{1 \leqslant i,j \leqslant k} \| M_i - M_j \|_2.$$

Hence, (A.4) holds for all $N \geqslant 2$, and the proof of Proposition B is completed.

Proposition C. Let $\Pi_{\mathscr{D}}\{\cdot\}$ be the projection operator defined by (2.5). Then for any matrix $A \in \mathbb{R}^{(n+m) \times n}$,

$$\| \Pi_{\mathscr{D}}(A) - B \| \leqslant \| A - B \|, \ \forall B \in \mathscr{D}.$$

Proof. By the definition of the projection in (2.5) we have

$$\| \Pi_{\mathscr{D}}(A) - A \|^2 \leqslant \| A - B \|^2, \ \forall B \in \mathscr{D}.$$

Now, rewriting $A - B$ as $[A - \Pi_{\mathscr{D}}(A)] + [\Pi_{\mathscr{D}}(A) - B]$ and expanding the square on the RHS, we get

$$2\mathrm{Tr}\{[A - \Pi_{\mathscr{D}}(A)]^{\tau}[\Pi_{\mathscr{D}}(A) - B]\} + \| \Pi_{\mathscr{D}}(A) - B \|^2 \geqslant 0, \ \forall B \in \mathscr{D}.$$

By the arbitrariness of $B \in \mathscr{D}$ and the convexity of $\mathscr{D}$, it is easy to show that

$$\mathrm{Tr}\{[A - \Pi_{\mathscr{D}}(A)]^{\tau}[\Pi_{\mathscr{D}}(A) - B]\} \geqslant 0, \ \forall B \in \mathscr{D}.$$

Hence,

$$\| A - B \|^2 = \| A - \Pi_{\mathscr{D}}(A) + \Pi_{\mathscr{D}}(A) - B \|^2 \geqslant \| \Pi_{\mathscr{D}}(A) - B \|^2,$$

which is the desired result.

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