with $\Phi_i(\cdot)$ a nonlinear function. Pre-assigning the values of Δ_i results in a nonlinear equation which is linear in the unknown parameters W_i. These parameters can be determined using a least-squares method [7]. The chosen values for Δ_i are equally spaced within the range of deflection of the asperity (the maximum value of z). The more elements used, the more accurate the approximation will be, but on the other hand the computational complexity is proportional to the number of elements used. The mimimum Δ_i-value is limited by the noise on the measurement results. If the Δ_i-value is smaller than the noise level, the model will create inner loops due to the noise and not to the change in deflection.

The advantage of the Maxwell slip implementation is the elimination of the stack overflow problem. Looking at the free response of a mass-spring system with limited friction from an initial state which does not correspond to an equilibrium, the position and the state variable z will have a lightly damped oscillating behavior, resulting in several velocity reversals without closing of inner loops, causing the addition of maximum and minimum values of $F_h(z)$ on the stacks $\min$ and $\max$. When the maximum lengths of the stacks are limited this can lead to the problem of stack overflow. The Maxwell slip method uses only a fixed number of memory places equal to the number of elements used in the implementation.

In the Maxwell slip implementation, the initial curve of the hysteresis behavior is implicitly taken into account in the equations. For the implementation described in [10], working on the initial curve and reentering the initial curve needs an extra implementation for those two cases.

V. Conclusion

This paper briefly discusses the integrated friction model structure, called the Leuven model, and proposes two improvements to this model. The first modification reformulates the nonlinear state equation in order to obtain always a continuous friction force. The second modification solves the problem of stack overflow, which may occur with the implementation method of the hysteresis force proposed in [10]. The General Maxwell slip model is a better way to implement the hysteresis force. Even with a limited number of elements, it is possible to approximate the hysteresis force accurately.

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A Limit to the Capability of Feedback

Yanxia Zhang and Lei Guo

***Abstract*—Feedback is ubiquitous and is a basic concept in the area of control, where it is used primarily for reducing internal or external uncertainties, or both. In this note, we will study the capability of feedback in dealing with both internal and external uncertainties for a class of pth order nonlinear autoregressive control systems. The size of the uncertainty is described by the Lipschitz constant (say L) of the uncertain nonlinear function in consideration. It is shown that if p and L satisfy a certain inequality, then there exists no globally stabilizing feedback for the corresponding class of uncertain systems, and thus finding a quantitative limit to the capability of the feedback mechanism in dealing with structural uncertainties.**

***Index Terms*—Feedback, nonlinear, stability, uncertainity.**

I. Introduction

Feedback is ubiquitous and is a basic concept in automatic control. Its primary objective is to reduce the effects of the plant uncertainty on the desired control performance. The uncertainty of a plant includes both internal (structure) uncertainty and external (disturbance) uncertainty and, in general, the former is harder to cope with than the latter. How to design efficient feedback laws to cope with various plant uncertainties has been a key issue in the development of automatic control [1]–[3]. However, from a philosophical point of view, there must be some limits to the capability of feedback in dealing with uncertainties. Finding such limits is of fundamental importance, since control scientists may not waste their time on constructing control laws for systems with uncertainties which are already beyond the capability of control, while control engineers may be more cautious (or confident) when applying their new control methods (robust, adaptive, intelligent, etc.) bravely to complex systems which practically contain many uncertainties.

Unfortunately, the question about the limits of feedback is a conundrum and on which only a few existing areas of control theory can shed some light. Robust control and adaptive control are two such areas where structural uncertainty of the plant is the main concern in the controller design.

Robust control usually requires that the true plant lies in a (small) ball centered at a known nominal model and often assumes that the controllers are selected from certain given classes of systems [4]. The need of a nominal model, with reliable model error bounds in robust

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The authors are with the Institute of Systems Science, Chinese Academy of Sciences, Beijing 100080, P.R. China (e-mail: Lguo@control.iss.ac.cn).

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control methods, motivated the extensive research activities in an area called control-oriented worst-case identification in the 1990s. During the same period, significant progress has also been made [8], [9] in linking the theories of identification, feedback, information and complexity following the framework and philosophy developed by Zames [3], [5]–[7].

Adaptive control is a nonlinear feedback technique which performs identification (or learning) and feedback control simultaneously in the same feedback loop, which is known to be a powerful tool in dealing with systems with large uncertainties. A well-developed theory is now available for the adaptive control of both continuous- and discrete-time linear systems since the end of the 1970s in [10]–[12]. Much progress has also been made for adaptive control of nonlinear continuous-time systems with linear unknown parameters [13]. However, essential difficulties emerge for adaptive control of discrete-time nonlinear systems when the nonlinearities have a nonlinear growth rate [14], [15].

The above analyzes show that, to study the limits of the feedback mechanism, we have to place ourselves in a framework that is somewhat beyond those of the classical robust control and adaptive control. First, the system structural uncertainty may be nonlinear and nonparametric and a useful or reliable ball containing the true plant and centered at a known nominal model, may not be available *a priori*; Second, we need to study the limits of the full feedback mechanism which includes all (nonlinear and time-varying) causal mappings, rather than confined to a fixed feedback law or a set of specific (e.g., linear) feedback laws. We shall also work with discrete-time control models, as they can reflect the limitations of actuator and sensor in a certain sense when implemented with digital computers. It is fairly well-known that in the present case, the high gain and nonlinear damping approaches which are so powerful in the continuous-time case are no longer effective now.

This note is a continuation of a series of studies on the limits of the feedback mechanism started in [14], where it was found and rigorously proved that for a typical class of nonlinear discrete-time systems with (even) scalar unknown parameters, the design of the globally stabilizing feedback is impossible when the growth rate of the nonlinearities is greater than $O(x^4)$. This result has been extended to a class of uncertain nonlinear systems with unknown vector parameters in [15]. For the more complicated nonparametric case, a natural way of constructing the adaptive control law is to use nonparametric estimation methods [18] which, however, can only be proven to be able to deal with open-loop stable nonlinear systems [19]. Recently, [16] made a significant step in this direction, investigated the capability and limits of feedback in controlling a class of first-order discrete-time dynamical control systems with nonparametric uncertainties. By introducing a suitable norm $\|\cdot\|$ (called the generalized Lipschitz norm) in the space of all nonlinear functions, the authors have given a complete characterization of the capability and limits of the feedback mechanism. To be precise, it was shown in [16] that the maximum uncertainty that can be dealt with by feedback is a ball with radius $3/2+\sqrt{2}$ in this normed function space. Analogously, for sampled-data control systems with uncertain nonparametric nonlinearities, it has been shown that if the sampling period is larger than a certain value, then globally stabilizing sampled-data feedback does not exist in general even if the nonlinearity has a linear growth rate [17].

The purpose of this note is to generalize some results in [16] to more general high order nonlinear control systems. We shall show that for a class of pth order nonlinear autoregressive control systems with the size of the uncertainty described by the Lipschitz constant (say L) of the uncertain nonlinear function in consideration, if p and L satisfy the following relationship:

$$L+\frac{1}{2} \geq \sqrt[1+p]{pL}\left(1+\frac{1}{p}\right), \quad pL>1$$

then there exists no globally stabilizing feedback law for the corresponding class of uncertain systems and thus finding a quantitative limit to the capability of the feedback mechanism in dealing with structural uncertainties.

The remainder of this note is organized as follows. In Section II, we will present the main results of the note. Two auxiliary lemmas are presented in Section III, which will be used in Section IV in the proofs of the main theorem.

II. MAIN RESULTS

Consider the following pth-order discrete-time nonlinear autoregressive control model:

$$y_{t+1}=g\left(y_t, y_{t-1}, \ldots, y_{t-p+1}\right)+u_t+w_{t+1}, \quad t \geq 0 \tag{1}$$

where $\{y_t\}$, $\{u_t\}$ and $\{w_t\}$ are the system output, input and noise sequences, respectively, $p \geq 1$ is an integer and the function $g(\cdot): \mathbb{R}^p \rightarrow \mathbb{R}^1$ is assumed to be completely unknown, but belongs to the following class of functions:

$$\mathcal{G}(L) \triangleq \left\{g(\cdot): \mathbb{R}^p \rightarrow \mathbb{R}^1 \,\middle|\, \left|g\left(X_1\right)-g\left(X_2\right)\right| \leq L\left\|X_1-X_2\right\|, \forall X_1, X_2 \in \mathbb{R}^p\right\} \tag{2}$$

where $\|X\| \triangleq \sum_{i=1}^p |x_i|$, $\forall X=(x_1, x_2, \ldots, x_p) \in \mathbb{R}^p$ and $L>0$ is a constant. Obviously, the larger the constant L, the larger the uncertainty of the class $\mathcal{G}(L)$ is. Hence, L may be regarded as a measure of the "size" of uncertainty for class $\mathcal{G}(L)$. It is well known that many "common" practical nonlinear phenomena, e.g., saturation and dead-zone etc., are included in the class of functions satisfying the Lipschitz condition.

In this note, we are primarily interested in the following question: What are the limits of feedback in dealing with uncertain systems (1) for any $g(\cdot) \in \mathcal{G}(L)$?

In order to give a rigorous answer to the above question, we need to give a precise definition of feedback first.

Definition: A sequence $\{u_t\}$ is called a feedback control law if at each step $t \geq 0$, u_t is a causal function of the observations $\{y_t\}$, i.e.,

$$u_t=h_t\left(y_i, i \leq t\right) \tag{3}$$

where $h_t(\cdot): \mathbb{R}^{t+p} \rightarrow \mathbb{R}^1$ can be an arbitrary(nonlinear and time-varying) mapping at each step t.

The main result of this note is stated in the following theorem, which provides a limit to the capability of feedback in terms of the Lipschitz constant L and the system order p.

Theorem 1: If $L>0$ and $p \geq 1$ satisfy

$$L+\frac{1}{2} \geq \sqrt[1+p]{pL}\left(1+\frac{1}{p}\right), \quad pL>1 \tag{4}$$

then there exists an unbounded domain $D \subset \mathbb{R}^p$ such that for any initial values $(y_0, y_{-1}, \ldots, y_{-p+1}) \in D$ and any feedback control law $\{h_t(\cdot), t \geq 0\}$ in (3), there always exists some $g \in \mathcal{G}(L)$ such that the corresponding closed-loop system (1) with (3) is unstable, i.e., $\sup_{t \geq 0}|y_t|=\infty$.

Remark 1: Obviously, the "negative" result established in the above theorem holds also true for any model classes, as long as the model class (1) is included as a subclass. We remark that for the case where $p=1$, the inequality (4) becomes $L \geq 3/2+\sqrt{2}$, which has been shown to be a critical case for feedback stabilization [16], i.e., it is also a necessary condition for nonstabilizability. However, in the case where $p>1$, whether or not the condition (4) is necessary for nonstabilizability is still an open question.

III. Two Auxiliary Lemmas

In this section, we present two auxiliary lemmas which will be needed in the proofs of the main theorem stated in the last section.

Lemma 1: If $L > 0$, then the inequality (4) is the necessary and sufficient condition for the following equation to have a real root in (1, $+\infty$):

$$x^{p+1} - \left(L + \frac{1}{2}\right) x^p + L = 0. \quad (5)$$

Proof: ***Sufficiency:***
By (4), we have

$$L + \frac{1}{2} \geq \sqrt[p+1]{pL}\left(1 + \frac{1}{p}\right) > 1 + \frac{1}{p} \quad (6)$$

and

$$\left(L + \frac{1}{2}\right)^{p+1} \geq \frac{L(p+1)^{p+1}}{p^p}. \quad (7)$$

Now, denote

$$b(x) \triangleq x^{p+1} - \left(L + \frac{1}{2}\right) x^p + L \quad (8)$$

then

$$\begin{aligned} b'(x) =& (p+1)x^p - P\left(L + \frac{1}{2}\right) x^{p-1} \\ =& (p+1)x^{p-1}\left(x - \left(L + \frac{1}{2}\right)\frac{p}{p+1}\right). \end{aligned} \quad (9)$$

Let $x_p \triangleq (L + 1/2)p/p + 1$, then from (6) and (7) we have

$$x_p > 1 \quad \text{and} \quad b(x_p) \leq 0. \quad (10)$$

Then from the fact that $b(1) = 1/2 > 0$ and that $b(x)$ is continuous, we know that there must exist $x_0 \in (1, x_p]$ which satisfies $b(x_0) = 0$. Hence, the sufficiency of Lemma 1 is true.

Necessity: Suppose that (5) has a real root in $(1, +\infty)$, then $\min_{x\geq 1} b(x) \leq 0$. By (9), We know that $b(x)$ must reach the minimum in $(1, +\infty)$ at the point x_p. Hence, (10) holds. Substituting the value of x_p into (10), we have (4). □

Lemma 2: Under the assumption of (4), if a sequence $\{x_k\}$ satisfies

$$x_{-p+1} = x_{-p+2} = \cdots = x_0 = 0, \; x_1 > 0 \quad (11)$$

and

$$x_{k+1} \geq \left(L + \frac{1}{2}\right) x_k - Lx_{k-p}, \; k \geq 1 \quad (12)$$

then

$$x_{k+1} - x_k > 0, \; \forall k \geq 0 \; and \; \lim_{k\to\infty} x_k = \infty \quad (13)$$

Moreover,

$$\left(L - \frac{1}{2}\right) x_k - Lx_{k-p} > 0, \; \forall k \geq 1. \quad (14)$$

Proof: We can rewrite (12) as

$$x_{k+1} - \beta_0 x_k \geq \beta_1 (x_k - \beta_0 x_{k-1}) + \cdots + \beta_p (x_{k-p+1} - \beta_0 x_{k-p}) \quad (15)$$

where $\beta_0, \beta_1, \ldots, \beta_p$ satisfy the following equations:

$$\begin{cases} \beta_0 + \beta_1 = L + \frac{1}{2} \\ \beta_2 = \beta_1\beta_0 \\ \vdots \\ \beta_p = \beta_{p-1}\beta_0 \\ \beta_p\beta_0 = L \end{cases}. \quad (16)$$

Substitute the first p equations into the last one, we then have

$$\beta_0^{p+1} - \left(L + \frac{1}{2}\right)\beta_0^p + L = 0. \quad (17)$$

By Lemma 1, we know that the above equation has a real root in (1, $+\infty$). Let β_0 be the smallest root of (17) in (1, $+\infty$), then by (16), we have

$$\beta_0 > 1 \quad \text{and} \quad \beta_i > 0, \quad i = 1, 2, \ldots, p. \quad (18)$$

Denote $z_k \triangleq x_k - \beta_0 x_{k-1}$, $\forall k \geq -p+2$. By (11), we have

$$\begin{aligned} z_{-p+2} =& z_{-p+3} = \cdots = z_0 = 0, \; z_1 = x_1 > 0 & (19) \\ z_{k+1} \geq& \beta_1 z_k + \beta_2 z_{k-1} + \cdots + \beta_p z_{k-p+1}, \; \forall k \geq 1. & (20) \end{aligned}$$

By (18)–(20), it is easy to see that $z_k > 0, \quad \forall k \geq 1$. Hence

$$x_{k+1} > \beta_0 x_k > \beta_0^2 x_{k-1} > \cdots > \beta_0^k x_1 > 0. \quad (21)$$

So, by $\beta_0 > 1$ and $x_1 > 0$, we have

$$x_{k+1} - x_k > 0, \; \forall k \geq 0 \quad \text{and} \quad \lim_{k\to\infty} x_k = \infty.$$

Hence, (13) holds. Moreover, from (12) and (15), it follows that:

$$\begin{aligned} \left(L - \frac{1}{2}\right) x_k - Lx_{k-p} =& (\beta_0 - 1)x_k \\ &+ \beta_1 z_k + \beta_2 z_{k-1} + \cdots + \beta_p z_{k-p+1}. \end{aligned} \quad (22)$$

From this, (14) holds. □

IV. The Proof of the Main Theorem

Proof of Theorem 1: We first introduce some notations. Note that $y_0, y_{-1}, \ldots, y_{-p+1}$ are initial values and y_t is the output at step t. Define

$$\bar{b}_t \triangleq \max_{-p+1\leq i\leq t} y_i, \quad \underline{b}_t \triangleq \min_{-p+1\leq i\leq t} y_i \quad (23)$$

and

$$\begin{aligned} B_t \triangleq& \left[\underline{b}_t, \bar{b}_t\right], \quad t \geq -p+1 \quad \text{and} \quad \triangle B_{-p+1} \triangleq B_{-p+1}, \\ \triangle B_t \triangleq& B_t - B_{t-1}, \quad t \geq -p+2, \end{aligned} \quad (24)$$

and

$$\begin{aligned} |B_t| \triangleq& \bar{b}_t - \underline{b}_t \quad t \geq -p+1 \quad \text{and} \quad |\triangle B_{-p+1}| \triangleq 0, \\ |\triangle B_t| \triangleq& |B_t| - |B_{t-1}|, \quad t \geq -p+2. \end{aligned} \quad (25)$$

By (23)

$$\bar{b}_t \geq \bar{b}_{t-1}, \quad \underline{b}_t \leq \underline{b}_{t-1} \quad \text{and} \quad \left(\bar{b}_t - \bar{b}_{t-1}\right)\left(\underline{b}_t - \underline{b}_{t-1}\right) = 0$$

we know that the interval sequence $\{B_t, \; t \geq -p+1\}$ is nondecreasing and that $\triangle B_t$ is also an interval (can be a null set $\emptyset$). Note that

$$B_t = \bigcup_{i=-p+1}^{t} \triangle B_i, \quad \text{and} \quad \triangle B_i \bigcap \triangle B_j = \emptyset, \quad i \neq j. \quad (26)$$

For any point $a \in \mathbb{R}^1$ and any set $B \subset \mathbb{R}^1$, define a distance function $d(\cdot, \cdot)$ as

$$d(a, B) \triangleq \inf_{b\in B} |a - b| \quad (27)$$

and if $B = \{b\}$, we rewrite $d(a, B)$ as $d(a, b) \triangleq |a - b|$.

Then, it is clear that $|\triangle B_t| = d(y_t, B_{t-1}), t \geq -p+2$.

According to the above notations, we now introduce the domain D for the value of the initial conditions, shown in (28) at the bottom of the page. It is easy to see that D is an unbounded domain of $\mathbb{R}^p$. Now, we will introduce a class of functions on $\mathbb{R}^1$. For a given L, define

$$\mathcal{F}(L) \triangleq \left\{ f : \mathbb{R}^1 \to \mathbb{R}^1 \,\middle|\, |f(x_1) - f(x_2)| \le L|x_1 - x_2|, \forall x_1, x_2 \in \mathbb{R}^1 \right\}. \tag{29}$$

It is obvious that if $f_i \in \mathcal{F}(L), i = 1, 2, \ldots, p$ and

$$g(x_1, x_2, \ldots, x_p) = f_1(x_1) + f_2(x_2) + \cdots + f_p(x_p)$$

then $g \in \mathcal{G}(L)$, where $\mathcal{G}(L)$ is defined by (2).

In the following, we will construct $f_i \in \mathcal{F}(L), i = 1, 2, \ldots, p$, such that for any initial value $(y_0, y_{-1}, \ldots, y_{-p+1}) \in D$ and any feedback law $u_t = h_t(y_i, i \le t)$, the following closed-loop system:

$$y_{t+1} = f_1(y_t) + f_2(y_{t-1}) + \cdots + f_p(y_{t-p+1}) + u_t + w_{t+1}, \; t \ge 0$$

is unstable.

We divide our analysis into three steps.

Step 1) For any $(y_0, y_{-1}, \ldots, y_{-p+1}) \in D$, we can choose $a_{ij} \in \mathbb{R}$ to define the values of $f_i(y_j), i = 1, 2, \ldots, p-1, j = -i, -i-1, \ldots, -p+1$, under the condition of $f_i \in \mathcal{F}(L)$, i.e.,

$$f_i(y_j) = a_{ij}, i = 1, 2, \ldots, p-1, \quad j = -i, \ldots, -p+1; \quad f_p(y_{-p+1}) = a_{p,-p+1}. \tag{30}$$

Also, define

$$\mathcal{F}_{-1} \triangleq \left\{ (f_1, \ldots, f_p) \,\middle|\, f_i \in \mathcal{F}(L), i = 1, 2, \ldots, p, \text{and satisfies}(30) \right\}. \tag{31}$$

Obviously, $\mathcal{F}_{-1} \ne \emptyset$, where $\emptyset$ denote the null set.

In the following, we will determine:

$$\{y_1, f_1(y_0), \ldots, f_{p-1}(y_{-p+2})\}$$
$$\{y_2, f_1(y_1), \ldots, f_p(y_{-p+2})\}, \ldots,$$
$$\{y_n, f_1(y_{n-1}), \ldots, f_p(y_{n-p})\} \ldots$$

successively.

By (26), we know that $|B_t| = \sum_{i=-p+1}^{t} |\triangle B_i|$. Since $(y_0, y_{-1}, \ldots, y_{-p+1}) \in D$, by the definition of D and the above equality, we have

$$|\triangle B_{-p+i}| \ge \left(L - \frac{1}{2}\right)|B_{-p+i-1}|, \quad i = 3, 4, \ldots, p. \tag{32}$$

From (4), we know that $L > 1/2$, so $|\triangle B_{-p+i}| > 0$, i.e., $y_{-p+i} \notin B_{-p+i-1}, i = 2, 3, \ldots, p$. Define

$$b_{-p+i-1} \triangleq \begin{cases} \bar{b}_{-p+i-1}, & \text{if } y_{-p+i} > \bar{b}_{-p+i-1} \\ \underline{b}_{-p+i-1}, & \text{if } y_{-p+i} < \underline{b}_{-p+i-1} \end{cases} \tag{33}$$

where $i = 2, 3, \ldots, p$. By the definition of $\mathcal{F}_{-1}$, we know that $f_i(b_{-i}), i = 1, 2, \ldots, p-1$ are constants for all $(f_1, f_2, \ldots, f_p) \in \mathcal{F}_{-1}$.

Since $f_i \in \mathcal{F}(L)$, $f_i(y_{-i+1})$ can be any value in the interval

$$[f_i(b_{-i}) - L|\triangle B_{-i+1}|, f_i(b_{-i}) + L|\triangle B_{-i+1}|], \quad i = 1, 2, \ldots, p-1.$$

Define (34) and (35) as shown at the bottom of the page. Then, for any $(f_1', f_2', \ldots, f_p') \in \mathcal{F}_0'$, $(f_1'', f_2'', \ldots, f_p'') \in \mathcal{F}_0''$ and any $u_0 = h_0(y_0, y_{-1}, \ldots, y_{-p+1})$, $w_1 \in \mathbb{R}$, we have

$$\begin{aligned} &\Big| f_1'(y_0) + f_2'(y_{-1}) + \cdots + f_p'(y_{-p+1}) + u_0 + w_1 \\ &\quad - \left(f_1''(y_0) + f_2''(y_{-1}) + \cdots + f_p''(y_{-p+1}) + u_0 + w_1\right)\Big| \\ &= 2L(|\triangle B_0| + |\triangle B_{-1}| + \cdots + |\triangle B_{-p+2}|) \\ &= 2L|B_0|. \end{aligned} \tag{36}$$

From this, it is obvious that

$$\begin{aligned} \max\Big\{ & d\left(f_1'(y_0) + \cdots + f_p'(y_{p+1}) + u_0 + w_1, \frac{\underline{b}_0 + \bar{b}_0}{2}\right), \\ & d\left(f_1''(y_0) + \cdots + f_p''(y_{p+1}) + u_0 + w_1, \frac{\underline{b}_0 + \bar{b}_0}{2}\right)\Big\} \\ \ge & L|B_0|. \end{aligned} \tag{37}$$

$$D \triangleq \left\{ (y_0, y_{-1}, \ldots, y_{-p+1}) \in \mathbb{R}^p \,\middle|\, \begin{array}{l} |\triangle B_{-p+2}| > 0 \\ |\triangle B_{-p+3}| \ge L|\triangle B_{-p+2}| - \frac{1}{2}|B_{-p+2}| \\ \vdots \\ |\triangle B_0| \ge L|\triangle B_{-1}| + \cdots + L|\triangle B_{-p+2}| - \frac{1}{2}|B_{-1}| \end{array} \right\}. \tag{28}$$

$$\mathcal{F}_0' \triangleq \left\{ (f_1, f_2, \ldots, f_p) \in \mathcal{F}_{-1} \,\middle|\, \begin{array}{l} f_1(y_0) = f_1(b_{-1}) + L|\triangle B_0| \\ f_2(y_{-1}) = f_2(b_{-2}) + L|\triangle B_{-1}| \\ \vdots \\ f_{p-1}(y_{-p+2}) = f_{p-1}(b_{-p+1}) + L|\triangle B_{-p+2}| \end{array} \right\} \ne \emptyset \tag{34}$$

and

$$\mathcal{F}_0'' \triangleq \left\{ (f_1, f_2, \ldots, f_p) \in \mathcal{F}_{-1} \,\middle|\, \begin{array}{l} f_1(y_0) = f_1(b_{-1}) - L|\triangle B_0| \\ f_2(y_{-1}) = f_2(b_{-2}) - L|\triangle B_{-1}| \\ \vdots \\ f_{p-1}(y_{-p+2}) = f_{p-1}(b_{-p+1}) - L|\triangle B_{-p+2}| \end{array} \right\} \ne \emptyset. \tag{35}$$

Thus

$$\max\left\{d\left(f_1'(y_0)+\cdots+f_p'(y_{p+1})+u_0+w_1,B_0\right),\ d\left(f_1''(y_0)+\cdots+f_p''(y_{p+1})+u_0+w_1,B_0\right)\right\} \geq \left(L-\frac{1}{2}\right)|B_0|. \tag{38}$$

Define (39) as shown at the bottom of the page. Obviously, $\mathcal{F}_0 \neq \emptyset$ and for any $(f_1, f_2, \ldots, f_p) \in \mathcal{F}_0$, $y_1, f_1(y_0), \ldots, f_p(y_{-p+1})$ are all uniquely determined values. Moreover

$$\begin{aligned}|B_1| &= d(y_1, B_0)\\ &= d(f_1(y_0)+\cdots+f_p(y_{-p+1})+u_0+w_1,B_0)\\ &\geq \left(L-\frac{1}{2}\right)|B_0|>0.\end{aligned} \tag{40}$$

Hence, $y_1 \notin B_0$. Define

$$b_0 \triangleq \begin{cases}\bar{b}_0, & \text{if } y_1 > \bar{b}_0\\ \underline{b}_0, & \text{if } y_1 < \underline{b}_0\end{cases}. \tag{41}$$

By now we can see $f_1(b_0), f_2(b_{-1}), \ldots, f_p(b_{-p+1})$ are all constants for all $(f_1, f_2, \ldots, f_p) \in \mathcal{F}_0$.

Step 2) Suppose that $\{y_j, f_1(y_{j-1}), \ldots, f_p(y_{j-p}),\ j \leq k,\ k \geq 1\}$ has been determined, then $f_i(b_{k-i})$, $i = 1, 2, \ldots, p$ are all constants for any $(f_1, f_2, \ldots, f_p) \in \mathcal{F}_{k-1}$. Now, we will determine $\{y_{k+1}, f_1(y_k), f_2(y_{k-1}), \ldots, f_p(y_{k-p+1})\}$. Define (42) and (43) as shown at the bottom of the page.

Then, similar to Step 1), for any $(f_1', f_2', \ldots, f_p') \in \mathcal{F}_k'$, $(f_1'', f_2'', \ldots, f_p'') \in \mathcal{F}_k''$, we can obtain

$$\begin{aligned}\max&\left\{d\left(f_1'(y_k)+\cdots+f_p'(y_{k-p+1})+u_k+w_{k+1},B_k\right),\ d\left(f_1''(y_k)+\cdots+f_p''(y_{k-p+1})+u_k+w_{k+1},B_k\right)\right\}\\ &\geq L\left(|\triangle B_k|+|\triangle B_{k-1}|+\cdots+|\triangle B_{k-p+1}|\right)-\frac{1}{2}|B_k|\\ &=\left(L-\frac{1}{2}\right)|B_k|-L|B_{k-p}|.\end{aligned} \tag{44}$$

Define (45) as shown at the bottom of the page. Then, for any $(f_1, f_2, \ldots, f_p) \in \mathcal{F}_k$, the values $y_{k+1}, f_1(y_k), f_2(y_{k-1}), \ldots, f_p(y_{k-p+1})$ have all been determined. Moreover, $|\triangle B_{k+1}|$ is constant and

$$\begin{aligned}|\triangle B_{k+1}| &= d(f_1(y_k)+\cdots+f_p(y_{k-p+1})+u_k+w_{k+1},\ B_k)\\ &\geq \left(L-\frac{1}{2}\right)|B_k|-L|B_{k-p}|>0\end{aligned} \tag{46}$$

where the last inequality follows from Lemma 2. Hence, $y_{k+1} \notin B_k$, and we can define

$$b_k \triangleq \begin{cases}\bar{b}_k, & \text{if } y_{k+1} > \bar{b}_k\\ \underline{b}_k, & \text{if } y_{k+1} < \underline{b}_k\end{cases}. \tag{47}$$

Step 3) Finally, we prove that $\overline{\lim}_{t\to\infty}|y_t| = \infty$. Since $(y_0, y_{-1}, \ldots, y_{-p+1}) \in D$, by the definition of D, we have

$$\begin{cases}|B_{-p+2}| = |\triangle B_{-p+2}| > 0\\ |B_{-p+3}| \geq \left(L+\frac{1}{2}\right)|B_{-p+2}|\\ \vdots\\ |B_0| \geq \left(L+\frac{1}{2}\right)|B_{-1}|\end{cases}. \tag{48}$$

From Steps 1) and 2), we have

$$\begin{cases}|B_1| \geq \left(L+\frac{1}{2}\right)|B_0|\\ \vdots\\ |B_{k+1}| \geq \left(L+\frac{1}{2}\right)|B_k|-L|B_{k-p}|, \quad k\geq 1\end{cases}. \tag{49}$$

For convenience of analysis, we denote $B_{-2p+2} = B_{-2p+3} = \cdots = B_{-p} = B_{-p+1} = 0$, so that (48) and (49) can be rewritten as

$$|B_{k+1}| \geq \left(L+\frac{1}{2}\right)|B_k|-L|B_{k-p}|, \quad k\geq -p+2. \tag{50}$$

Then, by Lemma 2, we have

$$|B_{k+1}| > |B_k|, \quad k \geq -p+1 \quad \text{and} \quad \lim_{k\to\infty}|B_k| = \infty. \tag{51}$$

$$\mathcal{F}_0 \triangleq \begin{cases}\mathcal{F}_0', & \text{if } d\left(f_1'(y_0)+\cdots+f_p'(y_{-p+1})+u_0+w_1,B_0\right) \geq \left(L-\frac{1}{2}\right)|B_0|\\ \mathcal{F}_0'', & \text{otherwise.}\end{cases} \tag{39}$$

$$\mathcal{F}_k' \triangleq \left\{(f_1,f_2,\ldots,f_p)\in\mathcal{F}_{k-1}\ \middle|\ \begin{array}{l}f_1(y_k)=f_1(b_{k-1})+L|\triangle B_k|\\ f_2(y_{k-1})=f_2(b_{k-2})+L|\triangle B_{k-1}|\\ \vdots\\ f_p(y_{k-p+1})=f_p(b_{k-p})+L|\triangle B_{k-p+1}|\end{array}\right\}\neq\emptyset \tag{42}$$

and

$$\mathcal{F}_k'' \triangleq \left\{(f_1,f_2,\ldots,f_p)\in\mathcal{F}_{k-1}\ \middle|\ \begin{array}{l}f_1(y_k)=f_1(b_{k-1})-L|\triangle B_k|\\ f_2(y_{k-1})=f_2(b_{k-2})-L|\triangle B_{k-1}|\\ \vdots\\ f_p(y_{k-p+1})=f_p(b_{k-p})-L|\triangle B_{k-p+1}|\end{array}\right\}\neq\emptyset. \tag{43}$$

$$\mathcal{F}_k \triangleq \begin{cases}\mathcal{F}_k', & \text{if } d\left(f_1'(y_k)+\cdots+f_p'(y_{k-p+1})+u_k+w_{k+1},B_k\right) \geq \left(L-\frac{1}{2}\right)|B_k|-L|B_{k-p}|\\ \mathcal{F}_k'', & \text{otherwise}\end{cases}. \tag{45}$$

$$f_i^\infty(x) \triangleq \begin{cases} \text{linear interpolation of } \left(y_k, f_i^{k+i-1}(y_k)\right), & x < \overline{b}_\infty \\ \overline{x}_i^\infty, & x \geq \overline{b}_\infty \end{cases}. \quad (57)$$

$$f_i^\infty(x) \triangleq \begin{cases} \underline{x}_i^\infty, & x \leq \underline{b}_\infty \\ \text{linear interpolation of } \left(y_k, f_i^{k+i-1}(y_k)\right), & x > \underline{b}_\infty \end{cases}. \quad (58)$$

So, by the definition of $|B_k|$, we have $\overline{\lim}_{t\to\infty}|y_t| = \infty$.

Define

$$\mathcal{F}_\infty \triangleq \Big\{(f_1, f_2, \ldots, f_p) \in \mathcal{F}_{-1} \;\Big|\; f_i(y_{k-i+1}) = f_i^k(y_{k-i+1}),\ \left(f_1^k, f_2^k, \ldots, f_p^k\right) \in \mathcal{F}_k, k \geq 0\Big\} \quad (52)$$

which is well defined since $y_{k+1} \notin B_k$, $\forall k \geq -p+1$ and $(f_1^k(y_k), f_2^k(y_{k-1}), \ldots, f_p^k(y_{k-p+1}))$ is independent of the particular choice of $(f_1^k, f_2^k, \ldots, f_p^k) \in \mathcal{F}_k$.

Denote

$$\overline{b}_\infty \triangleq \lim_{k\to\infty} \overline{b}_k \quad \underline{b}_\infty \triangleq \lim_{k\to\infty} \underline{b}_k. \quad (53)$$

Then, from (51), we know there are three possible cases.

Case 1) $\underline{b}_\infty = -\infty$, $\overline{b}_\infty < \infty$. First, if there exists $R > 0$ such that $y_R = \overline{b}_\infty$, then denote $\overline{x}_i^\infty \triangleq f_i^{R+i-1}(y_R)$, the value of which is given in $\mathcal{F}_{R+i-1}$ or $\mathcal{F}_{-1}$.

Otherwise, there must exist a subsequence $\{y_{j_k}\}$ of $\{y_j\}$ such that

$$\left|\triangle B_{j_{k+1}}\right| = y_{j_{k+1}} - y_{j_k}, \quad \text{and} \quad \lim_{k\to\infty} y_{j_k} = \overline{b}_\infty. \quad (54)$$

Hence, $\{y_{j_k}\}$ is a Cauchy sequence. $\forall m > n$, we have

$$\left|f_i^{j_m+i-1}(y_{j_m}) - f_i^{j_n+i-1}(y_{j_n})\right| = \left|f_i^{j_m+i-1}(y_{j_m}) - f_i^{j_m+i-1}(y_{j_n})\right| \leq L\left|y_{j_m} - y_{j_n}\right| \quad (55)$$

where the value of $f_i^{j_k+i-1}(y_{j_k})$ is given in $\mathcal{F}_{j_k+i-1}$. Hence, the sequence $\left\{f_i^{j_k+i-1}(y_{j_k})\right\}$ is also a Cauchy sequence. Denote

$$\overline{x}_i^\infty \triangleq \lim_{k\to\infty} f_i^{j_k+i-1}(y_{j_k}). \quad (56)$$

Then, for $i = 1, 2, \ldots, p$, define (57) as shown at the top of the page.

Case 2) $\underline{b}_\infty > -\infty$, $\overline{b}_\infty = \infty$. Similar to Case 1), we can define $\underline{x}_\infty^i$ and (58) shown at the top of the page.

Case 3) $\underline{b}_\infty = -\infty$ $\overline{b}_\infty = \infty$. Define

$$f_i^\infty(x) \triangleq \text{linear interpolation of } \left(y_k, f_i^{k+i-1}(y_k)\right), \quad \underline{b}_\infty < x < \overline{b}_\infty. \quad (59)$$

Obviously, $\left(f_1^\infty, f_2^\infty, \ldots, f_p^\infty\right) \in \mathcal{F}_\infty$ for any case. Hence, $\mathcal{F}_\infty \neq \emptyset$. $\forall \left(f_1^\infty, f_2^\infty, \ldots, f_p^\infty\right) \in \mathcal{F}_\infty$ define

$$g(x_1, x_2, \ldots, x_p) \triangleq f_1^\infty(x_1) + f_2^\infty(x_2) + \cdots + f_p^\infty(x_p) \quad (60)$$

then $g \in \mathcal{G}(L)$. So, there actually exists some $g \in \mathcal{G}(L)$ such that the following closed-loop system:

$$y_{t+1} = g(y_t, y_{t-1}, \ldots, y_{t-p+1}) + u_t + w_{t+1} \quad (61)$$

is unstable. Hence, the proof of the theorem is completed.

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