beginning of this section gives the partition of $x = \mathrm{col}(x^1, x^2)$ with $x^1 = \mathrm{col}(x_1, x_2, x_4)$ and $x^2 = x_3$ and the following mappings:

$$x^1 = \sigma(x^2, v) = \begin{bmatrix} v_1 \\ v_2 \\ -_3 v_2 - v_1 \end{bmatrix}$$

$$K(x, v) = \begin{bmatrix} v_2 \\ -\frac{(1+x_3+x_4)v_2 - x_3 v_1}{1+v_2\sin(v_1 v_2)} \end{bmatrix}$$

as well as the zero dynamics of (3.3)

$$\begin{bmatrix} \dot{x}_3 \\ \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} x_3 + (-x_3 v_2 - v_1) - \sin(v_1 v_2)\frac{x_3\left(v_2 - v_2^2 - v_1\right) + v_2 - v_1 v_2}{1+\sin(v_1 v_2) v_2} \\ v_2 \\ -v_1 \end{bmatrix}.$$

As a result, $\mathbf{x}_3(v)$ can be obtained by solving the following center manifold equation:

$$\frac{\partial \mathbf{x}_3(v)}{\partial v} a(v) = \mathbf{x}_3(v) + (-\mathbf{x}_3(v) v_2 - v_1) - \sin(v_1 v_2)\frac{\mathbf{x}_3(v)\left(v_2 - v_2^2 - v_1\right) + v_2 - v_1 v_2}{1 + \sin(v_1 v_2) v_2}.$$

Therefore, the solution of the regulator equations is given by

$$\mathbf{x}(v) = \begin{bmatrix} v_1 \\ v_2 \\ \mathbf{x}_3(v) \\ -\mathbf{x}_3(v) v_2 - v_1 \end{bmatrix}$$

$$\mathbf{u}(v) = \begin{bmatrix} v_2 \\ -\frac{\mathbf{x}_3(v)\left(v_2 - v_2^2 - v_1\right) + v_2 - v_1 v_2}{1+\sin(v_1 v_2) v_2} \end{bmatrix}.$$

IV. Conclusion

For a general class of MIMO nonlinear systems, we have shown that the pertinent regulator equations are solvable if the composite system has a well defined vector relative degree at the origin, and the equilibrium of the zero dynamics of the given plant with $v = 0$ can be made hyperbolic. Our approach only involves straightforward algebraic manipulations, and reduces the solution of the regulator equations into a set of well defined algebraic equations and a type of center manifold equation.

The approach and results can be generalized to more general nonlinear systems as described by (1.1). In fact, we can still define the relative degree for this class of general nonlinear systems [9]. The only complexity is that the equation $E_a(x, v) + D_a(x, v)u = 0$ has to be replaced by an equation nonlinear in u. Therefore, the function $K(x, v, u^2)$ has to be defined through the Implicit Function Theorem.

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On Quadratic Lyapunov Functions

Daizhan Cheng, Lei Guo, and Jie Huang

***Abstract*—A topological structure, as a subset of $[0, 2\pi)^L \times \mathbb{R}_+^{n-1}$, is proposed for the set of quadratic Lyapunov functions (QLFs) of a given stable linear system. A necessary and sufficient condition for the existence of a common QLF of a finite set of stable matrices is obtained as the positivity of a certain integral. The structure and the conditions are considerably simplified for planar systems. It is also proved that a set of block upper triangular matrices share a common QLF, iff each set of diagonal blocks share a common QLF.**

***Index Terms*—Common quadratic Lyapunov function (QLF), stabilization, switched system.**

I. Introduction

In recent years, the problem of stability and stabilization of switched systems has attracted a considerable amount of attention [12]. The stability of a switched system can be assured by a common Lyapunov function of the different models for arbitrary switching. Particularly, when the switching models are linear, the problem of common quadratic Lyapunov functions (QLFs) [8], [11] arises. The problem for diagonal quadratic Lyapunov functions was solved in [3] and [9]. When two stable matrices are commutative it was proved in [14] that they share a common QLF. Some special classes of matrices sharing a common QLF were investigated in [7], [13]. Certain Lie algebra structure and matrix inequalities were used to solve the problem [5], [15]. The problem of constructing Lyapunov functions has been discussed in [4] for some particular forms of P. Some recent results showed that if the Lie algebra generated by the set of matrices is solvable, then the common QLF exists [11]. In [2], the set of matrices, which share a given common QLF was investigated. The first necessary and sufficient condition was given for planar systems [16]. Numerical solution of common quadratic Lyapunov function has been discussed in [4], [10].

In this note, we shall give a topological description for the set of common QLFs of a finite set of stable matrices. Based on it, the existence of common QLFs depends on whether or not an integral is positive.

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II. Topological Structure of QLFs

Definition 2.1: A matrix A is said to have a QLF if there exists a positive–definite matrix $P > 0$, such that

$$PA + A^T P < 0. \tag{2.1}$$

P is briefly called a QLF of A. If in addition P is diagonal, A is said to have a diagonal QLF.

The following easily provable lemma is the starting point of our new approach.

Lemma 2.2: Assume a set of matrices $\{A_\lambda \mid \lambda \in \Lambda\}$ are stable, i.e., $Re\sigma(A_\lambda) < 0$, (where $\sigma(A)$ is the set of eigenvalues of A) and there exists a common QLF, then there exists an orthogonal matrix $T \in SO(n, R)$ such that $\{\tilde{A}_\lambda = T^T A_\lambda T | \lambda \in \Lambda\}$ have a common diagonal QLF.

According to Lemma 2.2, instead of searching a common QLF we can search a common diagonal QLF under a common orthogonal transformation on $\{A_\lambda\}$.

Let Φ_n be the set of $n \times n$ positive–definite matrices, and $\Xi_n \subset \Phi_n$ be its diagonal subset. Then

$$\Phi_n = \{TDT^T \mid T \in SO(n, R),\ D \in \Xi_n\}.$$

Now, since in searching a QLF, $P \sim kP$, $k > 0$ (where "$\sim$" stands for equivalence), we need only to consider the quotient set

$$\Psi_n = \Xi_n/\mathbb{R}_+ = \{(1, x_1, \ldots, x_{n-1}) | x_1 > 0, \ldots, x_{n-1} > 0\}.$$

Giving Ψ_n the conventional topology of R_+^{n-1}, the set of QLF has the structure as

$$\Phi_n/\mathbb{R}_+ := \{TDT^T \mid T \in SO(n, \mathbb{R}),\ D \in \Psi_n\} \tag{2.2}$$

and then we define a mapping $\pi : SO(n, \mathbb{R}) \times \mathbb{R}_+^{n-1} \to \Phi_n/\mathbb{R}_+$ as

$$(T, x) \mapsto T\operatorname{diag}(1, x_1, \ldots, x_{n-1})T^T$$

which is obviously a surjective mapping. So we can search the common QLF over $SO(n, \mathbb{R}) \times \mathbb{R}_+^{n-1}$.

For later discussion, it is not convenient to use the conventional topology of $SO(n, \mathbb{R})$ for searching P. We turn to its Lie algebra.

Let $s^{ij} \in o(n, \mathbb{R})$ be an element in the orthogonal algebra $o(n, \mathbb{R})$, defined as

$$s^{ij}(i, j) = -1 \quad s^{ij}(j, i) = 1 \quad s^{ij}(p, q) = 0 \qquad (p, q) \neq (i, j) \text{ and } (p, q) \neq (j, i).$$

Note that the connected Lie group generated by s^{ij}, denoted by S^{ij}, is a one-dimensional subgroup of $SO(n, \mathbb{R})$, and

$$S^{ij} \cong SO(2, \mathbb{R})$$

where $\cong$ stands for group isomorphism. It is easy to prove the following.

Lemma 2.3: There exist $L = n(n-1)/2$ one dimensional subgroups $S^{ij} < SO(n, \mathbb{R}), i < j$, such that

$$(S^{12}S^{13} \ldots S^{1n})(S^{23}S^{24} \ldots S^{2n}) \ldots \left(S^{(n-1)n}\right) = SO(n, \mathbb{R}). \tag{2.3}$$

Lemma 2.3 provides a surjective mapping $\Pi: [0, 2\pi)^L \to SO(n, R)$ as

$$\Pi(t_1, \ldots, t_L) = \exp(s^{12}t_1), \ldots, \exp(s^{1n}t_n), \ldots, \exp\left(s^{(n-1)n}t_L\right). \tag{2.4}$$

Throughout the rest of this note, we will not distinguish Ψ_n with $\Psi_n/\mathbb{R}_+$, unless elsewhere stated.

Based on Lemmas 2.2 and 2.3 one sees that we can give the topological structure of $[0, 2\pi)^L \times \mathbb{R}_+^{n-1}$ to the set of positive definite matrices (under equivalence). So, in the sequel, we will search the common QLF over this set.

III. Necessary and Sufficient Condition

Definition 3.1: Let $x = (x_1, \ldots, x_n)$ be a fixed coordinate frame in $\mathbb{R}^n$. A set of limits (L, U), with lower limits L and upper limits U are described as$L = \{L_1, L_2(x_1), \ldots, L_n(x_1, \ldots, x_{n-1})\}$, $U = \{U_1, U_2(x_1), \ldots, U_n(x_1, \ldots, x_{n-1})\}$. From (L, U), a nonnegative set of limits $(\tilde{L}, \tilde{U})$ is deduced as $\tilde{L}_i = \max\{L_i, 0\}$, $\tilde{U}_i = \max\{U_i, 0\}$, $i = 1, \ldots, n$.

For a given (L, U), the positive cub $C(L, U) \subset \mathbb{R}^n$ is defined as $x = (x_1, \ldots, x_n) \in C(L, U)$, iff

$$\tilde{L}_1 < x_1 < \tilde{U}_1 \quad \tilde{L}_2 < x_2 < \tilde{U}_2(x_1), \ldots \quad \tilde{L}_n(x_1, \ldots, x_{n-1}) < x_n < \tilde{U}_n(x_1, \ldots, x_{n-1}). \tag{3.1}$$

In fact, $C(L, U)$ is a domain of n-dimensional integration. It is the intersection of the domain bounded by (L_i, U_i), $i = 1, \ldots, n$ with the first quadrant.

We need the following lemma, which can be proved by a straightforward computation.

Lemma 3.2: Assume a matrix A has a diagonal QLF, then its diagonal elements are all negative, i.e., $a_{ii} < 0, i = 1, \ldots, n$.

Now, we are ready to give a complete characterization of the set of QLF for a given stable matrix. Let A be a given $n \times n$ stable matrix. According to Lemma 2.2, P is a QLF of A, iff ξ is a diagonal QLF of $T^T AT$ for some $T = \Pi(t)$. For a fixed t, we denote

$$A(t) := T(t)^T AT(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \ldots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \ldots & a_{2n}(t) \\ & & \ldots & \\ a_{n1}(t) & a_{n2}(t) & \ldots & a_{nn}(t) \end{pmatrix}$$

and $\xi = \operatorname{diag}(1, x_1, \ldots, x_{n-1})$ with $x_i > 0$. Then, it is required that (3.2), as shown at the bottom of the page, holds true. To make the $Q(t)$ negative–definite, it is enough to show that the determinants of the principal minors of odd orders are negative and those of even orders are

$$Q(t) := \xi A(t) + A^T(t)\xi = \begin{pmatrix} 2a_{11}(t) & a_{12}(t) + a_{21}(t)x_1 & \ldots & a_{1n}(t) + a_{n1}(t)x_{n-1} \\ a_{12}(t) + a_{21}(t)x_1 & 2a_{22}(t)x_1 & \ldots & a_{2n}(t) + a_{n2}(t)x_{n-1} \\ & & \ldots & \\ a_{1n}(t) + a_{n1}(t)x_{n-1} & a_{2n}(t) + a_{n2}(t)x_{n-1} & \ldots & 2a_{nn}(t)x_{n-1} \end{pmatrix} < 0. \tag{3.2}$$

positive. We show that these requirements lead to the required boundary functions.

According to Lemma 3.2, $a_{ii}(t)$, $i = 1, \ldots, n$ should be negative. Denote the kth principal minor by D_k. When $k = 2$, we have

$$D_2(t, x_1) = \begin{pmatrix} 2a_{11}(t) & a_{12}(t) + a_{21}(t)x_1 \\ a_{12}(t) + a_{21}(t)x_1 & 2a_{22}(t)x_1 \end{pmatrix}.$$

Let $\det(D2) > 0$, we get a quadratic inequality about x_1. It is easy to see that the solution of the inequality can be expressed as

$$L_1(t) < x_1 < U_1(t).$$

Denote $\Delta = (2a_{11}(t)a_{22}(t) - a_{12}(t)a_{21}(t))^2 - (a_{12}(t)a_{21}(t))^2$, the boundary can be calculated as

$$\begin{cases} \begin{cases} U_1(t) = +\infty \\ L_1(t) = \frac{a_{12}^2(t)}{4a_{11}(t)a_{22}(t)}, \quad a_{21}(t) = 0 \end{cases} \\ \begin{cases} U_1(t) = 0 \\ L_1(t) = 0 \quad a_{21}(t) \neq 0, \qquad \Delta \geq 0 \end{cases} \\ \begin{cases} U_1(t) = \frac{(2a_{11}(t)a_{22}(t) - a_{12}(t)a_{21}(t)) + \sqrt{\Delta}}{(a_{21}(t))^2} \\ L_1(t) = \frac{(2a_{11}(t)a_{22}(t) - a_{12}(t)a_{21}(t)) - \sqrt{\Delta}}{(a_{21}(t))^2}, \quad a_{21}(t) \neq 0, \qquad \Delta > 0. \end{cases} \end{cases} \tag{3.3}$$

Using mathematical induction, we assume there exists, for $s \leq k-1, U_s(t, x_1, \ldots, x_{s-1})$ and $L_s(t, x_1, \ldots, x_{s-1})$, such that $\det(D_{s+1}) > 0$ for odd s or $\det(D_{s+1}) < 0$ for eve s, iff

$$L_s(t, x_1, \ldots, x_{s-1}) < x_s < U_s(t, x_1, \ldots, x_{s-1}).$$

If there is some $\alpha \leq k - 1$, such that

$$\{x_a |\, L_a(t, x_1, \ldots, x_{a-1}) < x_a < U_a(t, x_1, \ldots, x_{a-1})\}$$

is an empty set, then we set $U_s = 0$ and $V_s = 0$,$s \geq \alpha$. Otherwise, we consider $\det(D_{k+1})$ and denote

$$\begin{aligned} E_k &= (a_{1,k+1}(t), a_{2,k+1}(t), \ldots, a_{k,k+1}(t))^T \\ F_k &= (a_{k+1,1}(t), a_{k+1,2}(t), \ldots, a_{k+1,k}(t))^T. \end{aligned}$$

Then

$$D_{k+1}(t, x_1, \ldots, x_k) = \begin{pmatrix} D_k(t, x_1, \ldots, x_{k-1}) & E_k + F_k x_k \\ E_k^T + F_k^T x_k & 2a_{k+1,k+1}(t)x_k \end{pmatrix}.$$

A computation with some skill shows that

$$\begin{aligned} \det(D_{k+1}(t, x_1, \ldots, x_k)) = {} & \det\begin{pmatrix} D_k & E_k \\ E_k^T & 0 \end{pmatrix} \\ & + 2\det\begin{pmatrix} D_k & F_k \\ E_k^T & a_{k+1,k+1}(t) \end{pmatrix} x_k \\ & + \det\begin{pmatrix} D_k & F_k \\ F_k^T & 0 \end{pmatrix} x_k^2. \end{aligned} \tag{3.4}$$

Denote (3.4) simply by

$$\begin{aligned} \det(D_{k+1}) = {} & a_k(t, x_1, \ldots, x_{k-1})x_k^2 + 2b_k(t, x_1, \ldots, x_{k-1})x_k \\ & + c_k(t, x_1, \ldots, x_{k-1}). \end{aligned} \tag{3.5}$$

We also denote $\Delta_k = b_k^2 - a_k c_k$. To get the coefficients of (3.4) we use the following simple fact. Let A be an $n \times n$ nonsingular matrix and b,$c \in R^n$. Then

$$\det\begin{pmatrix} A & b \\ c^T & e \end{pmatrix} = -\det(A)(c^T A^{-1} b) + e \det(A).$$

Applying it to (3.4), a straightforward computation shows that

$$\begin{cases} a_k(t, x_1, \ldots, x_{k-1}) = -\det(D_k)\left(F_k^T D_k^{-1} F_k\right) \\ b_k(t, x_1, \ldots, x_{k-1}) = -\det(D_k) E_k^T D_k^{-1} F_k \\ \qquad\qquad + \det(D_k) a_{k+1,k+1}(t) \\ c_k(t, x_1, \ldots, x_{k-1}) = -\det(D_k) E_k^T D_k^{-1} E_k. \end{cases} \tag{3.6}$$

Assume k is odd. Then $\det(D_k) < 0$ and since D_k is negative definite, then $a_k \leq 0$ and $a_k = 0$ iff $F_k = 0$. Note that $c_k = -\det(D_k) E_k^T D_k^{-1} E_k \leq 0$, and when $F_k = 0$, $b_k = \det(D_k) a_{k+1,k+1}(t) > 0$. Since we require $\det(D_{k+1}) > 0$, then the boundary functions can be expressed as

$$\begin{cases} \begin{cases} U_k(t, x_1, \ldots, x_{k-1}) = +\infty \\ L_k(t, x_1, \ldots, x_{k-1}) = \frac{-c_k}{2b_k}, \quad F_k = 0 \end{cases} \\ \begin{cases} U_k(t, x_1, \ldots, x_{k-1}) = 0 \\ L_k(t, x_1, \ldots, x_{k-1}) = 0, \quad F_k \neq 0, \ \Delta_k \leq 0 \end{cases} \\ \begin{cases} U_k(t, x_1, \ldots, x_{k-1}) = \frac{-b_k - \sqrt{\Delta_k}}{a_k} \\ L_k(t, x_1, \ldots, x_{k-1}) = \frac{-b_k + \sqrt{\Delta_k}}{a_k}, \quad F_k \neq 0, \qquad \Delta_k > 0. \end{cases} \end{cases} \tag{3.7}$$

Assume k is even. Then $\det(D_k) > 0$ and since D_k is negative definite, then $a_k \geq 0$ and $a_k = 0$ iff $F_k = 0$. Note that now $c_k \geq 0$, and when $F_k = 0$, $b_k < 0$. Now, we require $\det(D_{k+1}) < 0$. Taking this into consideration, a formula can be obtained, which is very similar to (3.7), but with an opposite sign before the square roots. Then a general formula, which covers (3.3), (3.7), and the case of even k, can be expressed in a unified form as

$$\begin{cases} \begin{cases} U_k(t, x_1, \ldots, x_{k-1}) = +\infty \\ L_k(t, x_1, \ldots, x_{k-1}) = \frac{-c_k}{2b_k}, \quad F_k = 0 \end{cases} \\ \begin{cases} U_k(t, x_1, \ldots, x_{k-1}) = 0 \\ L_k(t, x_1, \ldots, x_{k-1}) = 0, \quad F_k \neq 0, \ \Delta_k \leq 0 \end{cases} \\ \begin{cases} U_k(t, x_1, \ldots, x_{k-1}) = \frac{-b_k + (-1)^k \sqrt{\Delta_k}}{a_k} \\ L_k(t, x_1, \ldots, x_{k-1}) = \frac{-b_k - (-1)^k \sqrt{\Delta_k}}{a_k}, \qquad F_k \neq 0, \ \Delta_k > 0, \end{cases} \\ \qquad k = 1, 2, \ldots, n-1. \end{cases} \tag{3.8}$$

To assure $x_i > 0$ we need only to show that (L_k, U_k) is a positive interval. We need only to prove it for the case $\Delta_k > 0$. Observe that for either odd k or even k we always have: 1). $-b_k = a_k > 0$ and 2). $a_k c_k \geq 0$. It follows that $(L_k, U_k) \subset \mathbb{R}_+$.

Summarizing the aforementioned argument, we have the following.

Theorem 3.3: Let A be a given $n \times n$ stable matrix. Then, P is a QLF of A iff there exists a set $t = (t_1, \ldots, t_L) \in [0, 2\pi)^L$, a positive cube $C_t(L, U) \subset \Phi_n$ such that

$$P = T\xi T^T \tag{3.9}$$

where $\xi = \mathrm{diag}(1, x_1, \ldots, x_{n-1})$, with $(x_1, \ldots, x_{n-1}) \in C_t(L, U)$, $T = \Pi(t)$, which is defined by (2.4). The bounds ($L(t, x)$, $U(t, x)$) are determined by (3.8).

Now, for a set of matrices $\{A_1,\ldots,A_N\}$, say the boundary functions are obtained as $U_k^i(t,x_1,\ldots,x_{k-1})$ and $L_k^i(t,x_1,\ldots,x_{k-1})$, for $i=1,\ldots,N$, respectively, then we can define

$$\begin{cases} U_k(t,x_1,\ldots,x_{k-1}) = \min_{1\le i\le N} U_k^i(t,x_i,\ldots,x_{k-1}) \\ L_k(t,x_1,\ldots,x_{k-1}) = \max_{1\le i\le N} L_k^i(t,x_i,\ldots,x_{k-1}). \end{cases} \quad (3.10)$$

Summarizing the aforementioned argument, one sees easily that a set of matrices $\{A_1,\ldots,A_N\}$ have a common QLF, iff there exists a $t\in[0,2\pi)^L$ such that

$$U_k(t,x_1,\ldots,x_{k-1}) > x_k > L_k(t,x_1,\ldots,x_{k-1}),\ k=1,\ldots,n-1 \quad (3.11)$$

have a solution $x=(x_1,\ldots,x_{n-1})$. As a consequence, we have the following.

Theorem 3.4: A set of stable matrices $\{A_1,\ldots,A_N\}$ have a common QLF, iff

$$\int_0^{2\pi} dt_1 \ldots \int_0^{2\pi} dt_L \int_{L_1(t)}^{V_1(t)} dx_1 \int_{L_2(t,x_1)}^{V_2(t,x_1)} dx_2 \ldots \int_{L_{n-2}(t,x_1,\ldots,x_{n-3})}^{V_{n-2}(t,x_1,\ldots,x_{n-3})} dx_{n-2} \int_{L_{n-1}(t,x_1,\ldots,x_{n-2})}^{V_{n-1}(t,x_1,\ldots,x_{n-2})} dx_{n-1} > 0 \quad (3.12)$$

where

$$V_k(t,x_1,\ldots,x_{k-1}) = \max\{U_k(t,x_1,\ldots,x_{k-1}), L_k(t,x_1,\ldots,x_{k-1})\}, \qquad k=1,\ldots,n-1.$$

It is well known that [11], all the leading principal minors of a Hermitian matrix are real. A Hermitian matrix H is positive definite if and only if all its leading principal minors are positive. Using this fact, we can prove that Theorem 3.3 remains true for the set of complex matrices. It will be used in Section V.

Corollary 3.5: Theorem 3.3 remains true when the set of matrices are complex matrices.

IV. On Planar Systems

For the planar case, the orthogonal transformations $T\in SO(2,\mathbb{R})$ can be expressed as

$$T_t = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}, \qquad 0\le t<2\pi. \quad (4.1)$$

Consider a stable matrix $A=\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. According to Lemma 2.2, we first consider when

$$A(t) = T_t^T A T_t, \qquad 0\le t<\pi$$

has negative diagonal elements. Set

$$a=\frac{\alpha+\delta}{2},\ b=\frac{\alpha-\delta}{2},\ c=\frac{\beta+\gamma}{2},\ d=\frac{\beta-\gamma}{2},\ r=\sqrt{b^2+c^2}. \quad (4.2)$$

When $r\ne 0$, we define $\mu\in[0,2\pi)$ by

$$\cos(\mu)=\frac{c}{\tau} \quad \sin(\mu)=\frac{b}{\tau}, \qquad 0\le\mu<2\pi.$$

Then, we have the following. (Since space is limited, we refer to [6] for some missed less important proofs hereafter).

Proposition 4.1: Given a stable matrix $A=\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. When $r\ge -a$, the diagonal elements of

$$A(t) = T_t^T A T_t, \qquad 0\le t<\pi$$

are negative, iff t satisfies

$$r\,|\sin(2t+\mu)| < -a. \quad (4.3)$$

Equivalently

$$\frac{k\pi-\sin^{-1}\left(\frac{|a|}{r}\right)-\mu}{2} < t < \frac{k\pi+\sin^{-1}\left(\frac{|a|}{r}\right)-\mu}{2}, \qquad k\in Z. \quad (4.4)$$

When $r<-a$, the diagonal elements are always negative.

Remark: From the structure of $A(t)$, one sees easily that we do not need to consider whole $0\le t<2\pi$. It is enough to consider the problem only for $0\le t<\pi$.

Denote by

$$\Theta = \{t \mid 0\le t<\pi, \qquad r\,|\sin(2t+\mu)|<-a\}.$$

Note that Proposition 4.1 provides the set of t, which assures that the rotated matrix, $A(t)$, has negative diagonal elements. Later on, we will prove that it is exactly the set for the corresponding rotated matrix to have diagonal QLFs.

Now, we can start to search the diagonal QLFs $P(x)=\mathrm{diag}(1,x)$, where $x>0$. For notational ease, let $RS=r\sin(2t+\mu)$ and $RC=r\cos(2t+\mu)$. Then

$$P(x)A(t)+A^T(t)P(x) = \begin{pmatrix} 2(a+RS) & (-d+RC)x+(d+RC) \\ (-d+RC)x+(d+RC) & 2(a-RS)x \end{pmatrix}. \quad (4.5)$$

Define $D(t,x)=\det[P(x)A(t)+A^T(t)P(x)]$. Now, finding a QLF P is equivalent to finding $t\in\Theta$ and $x>0$ such that $D(t,x)>0$. It is equivalent to

$$-D(t,x)=E(t)x^2+2F(t)x+G(t)<0 \quad (4.6)$$

where $E(t)=(RC-d)^2$, $F(t)=(RC)^2-d^2+2(RS)^2-2a^2$, $G(t)=(RC+d)^2$.

Since $t\in\Theta$, (4.5) is negative definite, iff there exists $x>0$ such that (4.6) is satisfied.

Observing (4.6), it is obvious that $F(t)<0$ is a necessary condition for the existence of $x>0$. Fortunately, for $t\in\Theta$ this condition can be satisfied automatically.

Lemma 4.2: When t satisfies (4.3), then $F(t)=(RC)^2-d^2+2(RS)^2-2a^2<0$.

Then, we can prove the following.

Theorem 4.3: For each $t\in\Theta$, there exists an open nonempty interval $I_t=(L(t),U(t))\subset(0,+\infty)$ such that $P=\mathrm{diag}(1,x)$ is a diagonal QLF of A_t, iff, $x \in I_t$. Here, $L(t)$ and $U(t)$ are determined by

$$\begin{cases} L(t)=\begin{cases}1+2\left(\frac{a}{d}\right)^2-\left(\frac{2|a|}{|d|}\right)\sqrt{\left(\frac{a}{d}\right)^2 1+1}, & r=0\\ \frac{(RC+d)^2}{-2F}, & r>0, RC=d\\ \frac{-F-\sqrt{F^2((RC)^2-d^2)^2}}{(RC-d)^2}, & r>0, RC\neq d\end{cases}\\ U(t)=\begin{cases}1+2\left(\frac{a}{d}\right)^2+\left(\frac{2|a|}{|d|}\right)\sqrt{\left(\frac{a}{d}\right)^2+1}, & r=0\\ +\infty, & r>0, RC=d\\ \frac{-F+\sqrt{F^2-((RC)^2-d^2)^2}}{(RC-d)^2}, & r>0, RC\neq d \quad t\in\Theta.\end{cases}\end{cases} \tag{4.7}$$

We can also prove the following.

Proposition 4.4: If (t, x) is a feasible QLF with $t < \pi/2$, then $(t+\pi/2, 1/x)$ is also a feasible QLF. Conversely, if (t, x) is a feasible QLF with $t \geq \pi/2$, then $(t-\pi/2, 1/x)$ is also a feasible QLF.

This proposition tells us that to search the common QLFs we have only to search over $[0, \pi/2)$. Using a, b, c, d, and r as in above, the set Θ can be precisely described as the follows.

Assume $r < |a|$, then $\Theta = [0, \pi/2)$. Otherwise, we first calculate μ as

$$\mu = \begin{cases} \sin^{-1}\left(\frac{|b|}{r}\right), & b\geq 0, c\geq 0\\ \pi-\sin^{-1}\left(\frac{|b|}{r}\right), & b\geq 0, c<0\\ \pi+\sin^{-1}\left(\frac{|b|}{r}\right), & b<0, c<0\\ 2\pi-\sin^{-1}\left(\frac{|b|}{r}\right), & b<0, c\geq 0.\end{cases} \tag{4.8}$$

Then, we can get the feasible region of the rotations, which assure that the rotated $A(t)$ has diagonal QLFs. Set

$$\theta_1=\sin^{-1}\left(\frac{|a|}{r}\right),\ \theta_2=\pi-\theta_1,\ \theta_3=\pi+\theta_1,\ \theta_4=2\pi-\theta_1.$$

Then

$$\Theta=\begin{cases} \left\{\left[0, \frac{\theta_1-\mu}{2}\right)\right\}\cup\left\{\left(\frac{\theta_2-\mu}{2}, \frac{\pi}{2}\right)\right\}, & \mu<\theta_1\\ \left\{\left(\frac{\theta_2-\mu}{2}, \frac{\theta_3-\mu}{2}\right)\right\}, & \theta_1\leq\mu<\theta_2\\ \left\{\left[0, \frac{\theta_3-\mu}{2}\right)\right\}\cup\left\{\left(\frac{\theta_4-\mu}{2}, \frac{\pi}{2}\right)\right\}, & \theta_2\mu<\theta_3\\ \left\{\left(\frac{\theta_4-\mu}{2}, \frac{2\pi+\theta_1-\mu}{2}\right)\right\}, & \theta_3\leq\mu<\theta_4\\ \left\{\left[0, \frac{2\pi+\theta_1-\mu}{2}\right)\right\}\cup\left\{\left(\frac{2\pi+\theta_2-\mu}{2}, \frac{\pi}{2}\right)\right\}, & \theta_4\leq\mu<2\pi.\end{cases} \tag{4.9}$$

Using (4.9), we can construct the feasible set of t for each matrix, as Θ_k. Moreover, over each Θ_k the boundary functions $L_k(t)$ and $U_k(t)$ as in (4.7) are defined. Then, we construct the common feasible set as

$$\Theta=\cap_{k=1}^N\Theta_k\subset\left[0, \frac{\pi}{2}\right).$$

We know that it consists of only finite intervals. Then, define

$$L(t)=\max_{1\leq k\leq N}L_k(t)\quad U(t)=\min_{1\leq k\leq N}U_k(t)$$
$$V(t)=\max\{U(t), L(t)\}, \qquad t\in\Theta.$$

Summarizing the previous argument, we have the following.

Theorem 4.5: The set of stable 2×2 matrices $\{A_1, A_2, \ldots, A_N\}$ share a common QLF, iff

$$\int_{t\in\Theta}(V(t)-L(t))\,dt>0. \tag{4.10}$$

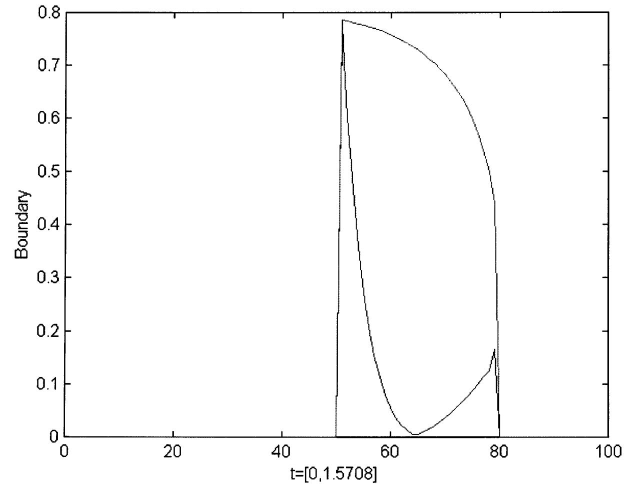

Fig. 1. Set of common QLF for A, B, and C.

Example 4.6: Consider three matrices

$$A=\begin{pmatrix}-4 & -1\\ 2 & 0.1\end{pmatrix}\quad B=\begin{pmatrix}-2 & 3\\ 1 & -2\end{pmatrix}\quad C=\begin{pmatrix}-5 & 2\\ 3 & -2\end{pmatrix}. \tag{4.11}$$

We skip the tedious elementary computation and give the domains of $A(t)$, $B(t)$, and $C(t)$, respectively, as

$$\Theta_A=(0.0764, 1.2552)\quad\Theta_B=[0, \pi/2)\quad\Theta_C=[0, \pi/2).$$

So

$$\Theta=\Theta_A\cap\Theta_B\cap\Theta_C=(0.0764, 1.2552).$$

It is easy to calculate the integration as

$$\int_{t\in\Theta}(V(t)-L(t))\,dt=0.2524>0.$$

So A, B, and C share a common QLF. Fig. 1 shows $V(t)$ (above curve), $L(t)$ (below curve), and the set of common QLFs.

From Fig. 1, it is easy to find out a common QLF. Say, $(t, x) = (0.3\pi, 0.5)$ is obviously in the feasible region. Hence, we can choose

$$\begin{aligned} P&=\begin{pmatrix}\cos(0.3\pi) & -\sin(0.3\pi)\\ \sin(0.3\pi & \cos(0.3\pi)\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0.5\end{pmatrix}\\ &\times\begin{pmatrix}\cos(0.3\pi) & \sin(0.3\pi)\\ -\sin(0.3\pi & \cos(0.3\pi)\end{pmatrix}\\ &=\begin{pmatrix}0.6727 & 0.2378\\ 0.2378 & 0.8273\end{pmatrix}.\end{aligned}$$

□

V. Systems of Same Block Upper Triangular Form

In this section, we show that when a set of matrices have the same block upper-triangular form, the complexity of the searching common QLF will be reduced tremendously. The main result is the following.

Theorem 5.1: Assume a finite set of block triangular (complex) matrices with same diagonal block structure as

$$A^i=\begin{pmatrix}A_{11}^i & A_{12}^i & \ldots & A_{1n}^i\\ 0 & A_{22}^i & \ldots & A_{2n}^i\\ & & \ldots & \\ 0 & 0 & \ldots & A_{nn}\end{pmatrix}, \qquad i=1,\ldots,N \tag{5.1}$$

where the same kth diagonal blocks A_{kk}^i have same dimensions for all i. Then, A_i share a common QLF, iff for every k the diagonal blocks $\{A_{kk}^i \mid i = 1, \ldots, N\}$ share a common QLF.

Proof: Without loss of generality, we have only to prove it for $n = 2$. Then, by mathematical induction we can prove it for any n for both necessity and sufficiency.

(Sufficiency) Denote by

$$A_i = \begin{pmatrix} X_i & Y_i \\ 0 & Z_i \end{pmatrix}, \qquad i = 1, \ldots, N. \tag{5.2}$$

Assume $dim(X_i) = p$ and $dim(Z_i) = q$. Let P and Q be the common QLF of $\{X_i\}$ and $\{Z_i\}$, respectively. Since $U_i := -(PX_i + X_i^T P) > 0$, $i = 1, \ldots, N$. There exists a positive real number $\delta > 0$, such that all the eigenvalues of U_i are greater then δ. Similarly, let $V_i := -(QX_i + X_i^T Q) > 0, i = 1, \ldots, N$, and all the eigenvalues of V_i are greater then some positive $\epsilon > 0$.

We claim that for large enough $\mu > 0$, $W = \operatorname{diag}(P, \mu Q)$ is a common QLF of A_i. Calculate

$$H_i := WA_i + A_i^T W = \begin{pmatrix} -U_i & PY_i \\ Y_i^* P & -\mu V_i \end{pmatrix}.$$

To show $H_i < 0$, choose $\xi \in \mathbb{C}^p$ and $\eta \in \mathbb{C}^q$. Then

$$\begin{aligned}(\xi^*, \eta^*) H_i \begin{pmatrix} \xi \\ \eta \end{pmatrix} &= -\, \xi^* U_i \xi + \xi^* P Y_i \eta + \eta^* Y_i^* P \xi - \mu \eta^* V_i \eta \\ &\le -\, \xi^* U_i \xi + \delta \|\xi\|^2 + \frac{1}{\delta} \eta^* \left(Y_i^* P^2 Y_i\right) \eta \\ &\quad - \mu \eta^* V_i \eta \\ &= \xi^* (-U_i + \delta I_p) \xi + \eta^* \\ &\quad \times \left[-\mu V_i + \frac{1}{\delta} \left(Y_i^* P^2 Y_i\right) \right] \eta. \end{aligned} \tag{5.3}$$

Choosing

$$\mu > \frac{1}{\delta \epsilon} \max_{1 \le i \le N} \left\| Y_i^* P^2 Y_i \right\|$$

then it is obvious that (5.3) is less than or equal to zero, and it equals zero, iff, $\xi = 0$ and $\eta = 0$.

(Necessity) Assume $A_i, i = 1, \ldots, t$ share a common QLF, P. According to the structure of A_i, we split P as $\begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$. Then

$$PA_i + A_i^* P = \begin{pmatrix} P_{11} X_i + X_i^* P_{11} & \times \\ \times & \times \end{pmatrix} < 0$$

where X stands for some uncertain elements. Then, $P_{11} X_i + X_i^T P_{11} < 0$, which means P_{11} is the common QLF of X_i.

Let $H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = P^{-1} > 0$. Then

$$H(PA_i + A_i^* P)H < 0$$

which leads to

$$A_i H + H A_i^* = \begin{pmatrix} \times & \times \\ \times & H_{22} Z_i^* + Z_i H_{22} \end{pmatrix} < 0.$$

It is easy to see than

$$H_{22}^{-1} Z_i + Z_i^* H_{22}^{-1} < 0.$$

That is, H_{22}^{-1} is a common QLF of Z_i. □

So, if $A_i, i = 1, \ldots, N$ can be converted into a same block upper triangular form, verifying the existence of common QLF becomes much easier.

VI. Conclusion

In this note, we considered the common QLF of a set of matrices. $[0, 2\pi)^L \times \mathbb{R}_+^{n-1}$, was proposed as the topological space of the set of QLFs for a set of stable matrices. Based on this structure, a necessary and sufficient condition for the existence of a common QLF was presented. The condition is described by the positivity of an integral. In fact, it provides a region because the integrand is 1. Comparing with other numerical methods, this condition provides a precise description for the set of all QLF.

As for planar systems the structure of the set of common QLFs and the necessary and sufficient conditions become very simple. We may compare it with [16]. The result in [16] says that the existence of a common QLF iff every three-tuple of systems have a common QLF. So if the number of matrices is 100, by [16] about 160 700 three-tuples have to be verified. But according to ours, only two curves need to be considered.

For a set of same structure of block upper triangular matrices, it was proved that they share a common QLF iff each set of same position diagonal blocks share a common QLF.

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