On global asymptotic controllability of planar affine nonlinear systems

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Abstract In this paper, we present a necessary and sufficient condition for globally asymptotic controllability of the general planar affine nonlinear systems with single-input. This result is obtained by introducing a new method in the analysis, which is based on the use of some basic results in planar topology and in the geometric theory of ordinary differential equations.

Keywords: affine nonlinear system, globally asymptotic controllability, stabilization, vector field, Jordan curve theorem, control curve.

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1 Introduction

The stabilization of the nonlinear systems has been studied extensively over the past three decades, and considerable progress has been made in either analysis or synthesis by introducing some powerful methods, including the well-known differential geometric method and backstepping method. However, the differential geometric method can only give local stabilization results of the nonlinear control systems in general, see refs. [1–3] for example. For the study of global stabilization, ref. [4] gave some necessary conditions by the methods of the homotopy and topological degree. As for sufficient conditions, to the authors' knowledge, there has been no general methods or results up to now. For example, the popular backstepping design method and its variants can only provide global stabilization results for a class of the nonlinear systems with the special structure essentially (e.g. triangular structure[2,5−7]). Even for the global stabilization of the seemingly simple planar affine nonlinear systems with single input, the problem is highly nontrivial, which has been investigated by many authors from various aspects[8−11], but most of the results are still on local stabilization, except the case where the system has a special triangular structure[12].

In this paper, we consider the planar affine nonlinear control systems and investigate the global asymptotic controllability, a concept which is closely related to but some-what weaker than the global stabilization. The asymptotic controllability has been studied previously[3,11,13], but most of the results are of local nature. In this paper, we present a necessary and sufficient condition for global asymptotic controllability of the planar affine nonlinear control systems. Our new approach used to establish the main result is based on the Jordan curve-like theorem, Poincare-Bendixson theorem, Whitney's smooth extension theorem, and some other basic facts in the geometric theory of ordinary differential equations, as well as on some basic results established recently in ref. [14].

The rest of this paper is organized as follows: Section 2 presents the main results together with some basic concepts, and several illustrative examples are given in section 3, and the proof of the main theorem is given in section 4. Finally, section 5 concludes the paper.

2 Main results

Consider the following planar affine nonlinear control systems

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) + g_1(x_1, x_2)u, \\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)u, \end{aligned} \tag{2.1}$$

where $f_i(x_1, x_2), g_i(x_1, x_2), i = 1, 2$ are smooth functions of the state $\boldsymbol{x} = (x_1, x_2)^T$ in $\mathbb{R}^2$, and u is the system input function taking values on $\mathbb{R}$. Denote $\boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}),\ f_2(\boldsymbol{x}))^T$, $\boldsymbol{g}(\boldsymbol{x}) = (g_1(\boldsymbol{x}), g_2(\boldsymbol{x}))^T$, and assume that $\boldsymbol{f}(0) = 0$, $\boldsymbol{g}(\boldsymbol{x}) \neq 0, \forall\ \boldsymbol{x} \in \mathbb{R}^2$. As usual, a smooth function means a function with continuous partial derivatives.

Definition 2.1. The system (2.1) is said to be **locally asymptotic controllable** at the origin, if there exist two neighborhood U_1 and U_2 of the origin, such that for any initial point $\boldsymbol{x}(0) = \boldsymbol{x}^0 \in U_1$, there exists a smooth control function of state $u_{x^0}(\boldsymbol{x})$ which keeps the trajectory $\boldsymbol{x}(t), t \geqslant 0$ in U_2 and drives the state converging to zero, i.e., $\boldsymbol{x}(t) \to 0$ as $t \to +\infty$. If $U_1 = U_2 = \mathbb{R}^2$, then the system (2.1) is said to be **globally asymptotic controllable**1).

The local stabilization issue has been well studied in the literature[1,2,11], for example, if $\left(\frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial \boldsymbol{x}}|_{x=0}, \boldsymbol{g}(0)\right)$ is controllable, then a locally stabilizing controller can be easily constructed. Hence, the locally asymptotic controllability of some systems is easy to be validated.

The purpose of this paper, however, is to study the more difficult global asymptotic controllability problems. To this end, we need to introduce several new concepts.

The well-known Jordan curve theorem in topology[15] says that a simple closed curve $\mathcal{C}$ separates the plane into two disjoint components, of which $\mathcal{C}$ is the common boundary. The proof of the Jordan curve theorem in ref. [15] actually gives also the following assertion: the curve which is homeomorphic to the straight-line with its two ends extending to infinite separates the plane into two

1) In most literature, the control function needs not to depend explicitly on the state, e.g. refs.[3,11], and in some literature the state trajectory does not need to be kept in U_2, e.g. ref. [3].

disjoint components[1)] . We call this result the Jordan curve-like theorem (see Fig. 1)

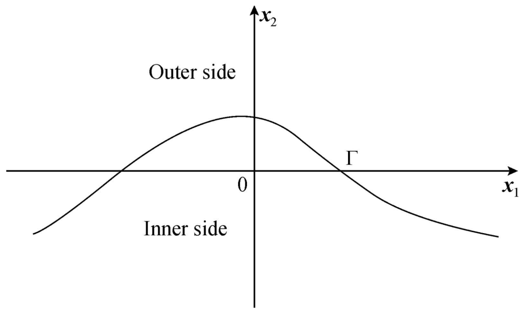

Fig. 1.

We are now in a position to give the following definitions.

Definition 2.2. The **inner side** of a curve which does not pass through the origin and is homeomorphic to the straight-line with its two ends extending to infinite, is defined as one of the two above-mentioned components which contain the origin. The other component is accordingly called the **outer side** (see Fig. 1).

Definition 2.3. A smooth curve $\Gamma : \gamma(s) \in \mathbb{R}^2, s \in \mathbb{R}$ which satisfies the conditions in Definition 2.2 is called a *P*-**curve** of system (2.1), if there exists $s_1 \in \mathbb{R}$ such that

$$L(s_1) < 0$$

holds for some function $u(x)$, where $L(s) = \langle \boldsymbol{f}(\gamma(s)) + \boldsymbol{g}(\gamma(s))u(\gamma(s)), \boldsymbol{p}(s)\rangle$, and $\boldsymbol{p}(s)$ is a non-zero normal vector of $\gamma(s)$ which points to the outer side of Γ.

Definition 2.4[14]. A **control curve** of system (2.1) is defined to be a solution $(x_1(t),\, x_2(t))$ of the following differential equation on the plane which does not pass through the origin:

$$\begin{aligned} \dot{x}_1 &= g_1(x_1, x_2), \\ \dot{x}_2 &= g_2(x_1, x_2), \end{aligned} \tag{2.2}$$

where $g_i(\boldsymbol{x}), i = 1, 2$ are the same as those in (2.1).

Lemma 2.1[14]. Any control curve of the system (2.1) is homeomorphic to the straight-line with its two ends extending to infinite.

Proposition 2.1. A control curve $\Gamma : \gamma(s)$ of system (2.1) is a P-curve if and only if there exists s_1, such that $\langle \boldsymbol{f}(\gamma(s_1)), \boldsymbol{p}(s_1)\rangle < 0$, where $\boldsymbol{p}(s)$ is non-zero normal vector of $\gamma(s)$, which points to the outer side of Γ.

This proposition is obvious, because $\boldsymbol{p}(s)$ is perpendicular to $\boldsymbol{g}(\gamma(s))$ for any s. Moreover, for any given control curve $\gamma(s)$ of system (2.1), $\boldsymbol{p}(s)$ can be taken

1) The two ends of a curve $\Gamma(t), t \in \mathbb{R}$ extending to infinite means that $\|\Gamma(t)\| \to +\infty$, when $t \to +\infty$ and $-\infty$.

as either $(-g_2, g_1)^T$ or $(g_2, -g_1)^T$. Consequently, $L(s)$ can be represented as

$$L(s) = \pm\{g_1(\gamma(s))f_2(\gamma(s)) - g_2(\gamma(s))f_1(\gamma(s))\}.$$

From this, we can immediately get the following easily verifiable condition for a P-curve.

Proposition 2.2. If the following function

$$g_1(\gamma(s))f_2(\gamma(s)) - g_2(\gamma(s))f_1(\gamma(s))$$

changes its sign over a control curve $\gamma(s)$, then it is a P-curve.

The main result of this paper is stated below, which characterizes the additional condition needed from local asymptotic controllability to global asymptotic controllability.

Theorem 2.1. Let the system (2.1) be locally asymptotic controllable at the origin. Then the necessary and sufficient condition for global asymptotic controllability of the control system is that any control curve $\Gamma : \gamma(t)$ of the system is a P-curve.

Remark 2.1. From the proof of Theorem 2.1 to be given in Section 4, one can see that if there exists a control curve for the system (2.1) which is not a P-curve, then the system cannot be globally stabilized by any measurable control function u_t (not necessarily required to be smooth).

Let us now give an intuitive interpretation of the theorem whose rigorous proof will be given in section 4.

According to Lemma 2.1 and the Jordan curve-like theorem, any control curve separates the plane into two disjoint components, and by the uniqueness of solutions, the plane $\mathbb{R}^2$ is partitioned by all the control curves corresponding to all initial conditions in $\mathbb{R}^2$. These results in a foliation which can be viewed as Fig. 2 on the whole, where $\Gamma_i, i = 1, 2, \cdots$ are some control curves of (2.1), the point $\boldsymbol{x}^0$ is assumed to lie on the curve Γ_1, and another curve Γ_3 passes through the origin.

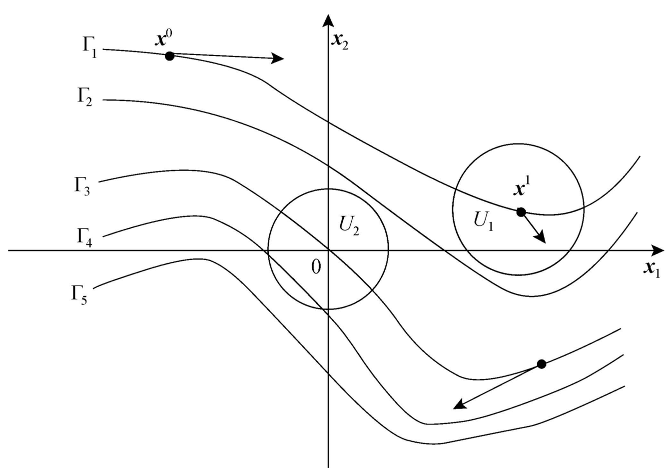

Fig. 2.

If Γ_1 is a P-curve, then there must exist a point $\boldsymbol{x}^1 \in \Gamma_1$ and a $u(\boldsymbol{x}^1)$, such that the positive semi-trajectory of the control systems with the initial point $\boldsymbol{x}^1$ will go into the inner side of the curve Γ_1. Hence, there exists a neighborhood U_1 of the point $\boldsymbol{x}^1$, such that for any point in U_1 together with the corresponding control $u(\boldsymbol{x})$, the positive semi-trajectory with initial point in U_1 will go into the inner side. On the other hand, according to the local asymptotic controllability, there exists a neighborhood U_2 of the origin such that for each $\xi \in U_2$ there is a control $u_\xi(\boldsymbol{x})$ steering the system (2.1) from $\boldsymbol{x} = \xi$ to 0 as the time approaches to infinity.

Hence, let the control function $u(\boldsymbol{x})$ be sufficiently large on a tubular neighborhood of Γ_1 which contains both $\boldsymbol{x}^0$ and $\boldsymbol{x}^1$ with the direction of the vector field from $\boldsymbol{x}^0$ to $\boldsymbol{x}^1$. Then under this control, the positive semi-trajectory with the initial point $\boldsymbol{x}^0$ will reach U_1 at some finite time, so we can let the trajectory go into the inner side of Γ_1 in U_1. Repeating this procedure, we can prove that the trajectory will finally reach at U_2 or Γ_3.

Once the trajectory arrives at $\xi \in U_2$, we can use control $u_\xi(\boldsymbol{x})$ such that the trajectory approaches to the origin. Otherwise, if the trajectory reaches Γ_3 but not in U_2, we can use a large control such that the trajectory goes into the neighborhood U_2 of the origin. Hence, we can get the globally asymptotic controllability of the control system, and at the same time the controller can be made smooth.

3 Some examples

Example 3.1. Consider two dimensional linear systems

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + Bu, \tag{3.1}$$

where $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, (A, B) is controllable.

Since (A, B) is controllable, $\Delta = \det(B, AB) \neq 0$, i.e. $\Delta = a_{21}b_1^2 + a_{22}b_1b_2 - a_{11}b_1b_2 - a_{12}b_2^2 \neq 0$. The control curve is defined by the differential equation

$$\begin{cases} \dot{x}_1 = b_1 \\ \dot{x}_2 = b_2 \end{cases} \text{or} \quad \text{by} \begin{cases} x_1 = b_1t + c_1 \\ x_2 = b_2t + c_2 \end{cases}, \ t \in (-\infty, +\infty), \tag{3.2}$$

where c_1 and c_2 are any constants. Therefore

$$\begin{aligned} & b_2(a_{11}x_1 + a_{12}x_2) - b_1(a_{21}x_1 + a_{22}x_2) \\ = \ & b_2(a_{11}(b_1t + c_1) + a_{12}(b_2t + c_2)) - b_1(a_{21}(b_1t + c_1) + a_{22}(b_2t + c_2)) \\ = \ & (b_1b_2a_{11} + b_2^2a_{12} - a_{21}b_1^2 - a_{22}b_1b_2)t + (b_2a_{11}c_1 + b_2a_{12}c_2 - b_1a_{21}c_1 - b_1a_{22}c_2) \\ = \ & -\Delta\ t + (b_2a_{11}c_1 + b_2a_{12}c_2 - b_1a_{21}c_1 - b_1a_{22}c_2). \end{aligned}$$

Since $\Delta \neq 0$, the function $b_2(a_{11}x_1 + a_{12}x_2) - b_1(a_{21}x_1 + a_{22}x_2)$ obviously changes its sign over any control curve defined by eq. (3.2), i.e. the system (3.1) satisfies the condition of Theorem 2.1 by Proposition 2.2. Hence the system is the globally asymptotic controllability as we have already known from the linear systems theory, but this simple example shows a way how to use Theorem

2.1. Q.E.D

Example 3.2. Consider the following second-order triangular control systems

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2), \\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)u. \end{aligned} \tag{3.3}$$

Let us denote $\boldsymbol{x} = (x_1, x_2)^T$, $\boldsymbol{f}(\boldsymbol{x}) = (f_1(x_1, x_2), f_2(x_1, x_2))^T$, and assume that $\boldsymbol{f}(0) = 0$, $g_2(\boldsymbol{x}) \neq 0, \forall\ \boldsymbol{x} \in \mathbb{R}^2$.

It is easy to check that the control curves of the system (3.3) are the straight-lines which are perpendicular to the x_1-axis, i.e. straight-lines $x_1 = c_1, \quad \forall\ c_1 \in \mathbb{R}$.

By Theorem 2.1, the curve $x_1 = c_1$ is P-curve if and only if there exists an x_2' such that $f_1(c_1, x_2') < 0 (> 0)$ when $c_1 > 0 (< 0)$.

A standard assumption in the popular backstepping method[2,6], is that there is a virtual control $x_2 = \eta(x_1)$, $\eta(0) = 0$, such that $\dot{x}_1 = f_1(x_1, \eta(x_1))$ is globally stable. Therefore $f_1(x_1, \eta(x_1)) < 0$ when $x_1 > 0$, and $f_1(x_1, \eta(x_1)) > 0$ when $x_1 < 0$. Hence, the planar systems that can be treated by the backstepping method can also be treated by using our Theorem 2.1. Q.E.D.

We now give a further example where the system is not of the standard triangular structure.

Example 3.3. Consider the following planar affine nonlinear system

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) + \cos(x_1^2 + x_2^2)u, \\ \dot{x}_2 &= f_2(x_1, x_2) + \sin(x_1^2 + x_2^2)u. \end{aligned} \tag{3.4}$$

Let us denote $\boldsymbol{x} = (x_1, x_2)^T$, $\boldsymbol{g}(\boldsymbol{x}) = (\cos(x_1^2 + x_2^2), \sin(x_1^2 + x_2^2))^T$, and let

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}, \boldsymbol{M}(\boldsymbol{x}) = \begin{pmatrix} 0 \\ x_1 \end{pmatrix}, \boldsymbol{N}(\boldsymbol{x}) = \begin{pmatrix} \sin(x_1^2 + x_2^2) \\ \cos(x_1^2 + x_2^2) \end{pmatrix}.$$

Furthermore, let D_1 and D_2 be two open discs centered at the origin with radius 1 and 2 respectively. By a standard result in the differential manifold (see ref. [16] pp. 106–109), there exists a smooth function $\theta(\boldsymbol{x})$ on $\mathbb{R}^2$ which satisfies

$$0 \leqslant \theta(\boldsymbol{x}) \leqslant 1, \quad \theta(\boldsymbol{x}) = \begin{cases} 1, & \boldsymbol{x} \in D_1, \\ 0, & \boldsymbol{x} \notin D_2. \end{cases}$$

Now, let $\boldsymbol{f}(\boldsymbol{x}) = \theta(\boldsymbol{x})\boldsymbol{M}(\boldsymbol{x}) + (1 - \theta(\boldsymbol{x}))\boldsymbol{N}(\boldsymbol{x})$. Then $\boldsymbol{f}(\boldsymbol{x})$ is smooth function on $\mathbb{R}^2$ and $\boldsymbol{f}(0) = 0$. It is easy to check that $\left(\frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial \boldsymbol{x}}|_{x=0}, \boldsymbol{g}(0)\right)$ is controllable, and each solution of differential equations defining the control curve

$$\begin{aligned} \dot{x}_1 &= \cos(x_1^2 + x_2^2), \\ \dot{x}_2 &= \sin(x_1^2 + x_2^2) \end{aligned} \tag{3.5}$$

will extend to infinite. Therefore, there are at least some parts of each trajectory out of D_2.

It is easy to see that for $\boldsymbol{x}$ out of disc D_2,

$$g_1(\boldsymbol{x})f_2(\boldsymbol{x}) - g_2(\boldsymbol{x})f_1(\boldsymbol{x}) = \cos^2(x_1^2+x_2^2) - \sin^2(x_1^2+x_2^2) \\ = \cos(2(x_1^2+x_2^2)) = \cos(2r^2), \quad (3.6)$$

where $r = \sqrt{x_1^2+x_2^2}$.

Since each control curve $\gamma(t), t \in \mathbb{R}$ defined by eq. (3.5) will extend to infinite, we know that the function in (3.6) will change its sign on each control curve. Hence, by Theorem 2.1, the system is globally asymptotic controllable. Q.E.D.

Finally, we consider an example taken from ref. [17] which is not globally stabilizable.

Example 3.4. Consider the following system

$$\begin{aligned} \dot{x}_1 &= -\sin x_2 \cos x_2 + \sin x_2 \exp(-x_1)u, \\ \dot{x}_2 &= \sin^2 x_2 + \cos x_2 \exp(-x_1)u. \end{aligned} \quad (3.7)$$

It has been shown that this system is not globally linearizable (see ref. [17]), and hence a globally stabilizing control cannot be constructed by using the traditional differential geometry method. Here, we will further show that this system is actually not globally stabilizable.

First of all, it is easy to check that $\left(\frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial \boldsymbol{x}}|_{x=0}, \boldsymbol{g}(0)\right)$ is controllable, so the system (3.7) is locally stabilizable. Note that one of the control curves of the system (3.7) is

$$\begin{cases} x_1 = \ln t, t > 0, \\ x_2 = \frac{\pi}{2}, \end{cases}$$

and that on this curve, the normal vector $\boldsymbol{p}(t) = (0,1)^T$ which points to the outer side, and the function $\langle \boldsymbol{f}(\boldsymbol{x}), \boldsymbol{p}(t)\rangle = \sin^2 x_2 = 1, \forall\ t > 0$. Hence, this control curve is not a P-curve, and the system cannot be globally stabilizable by Remark 2.1. Q.E.D.

4 The proofs of main results

For the planar affine nonlinear control systems (2.1), let $\mathrm{C}^1(\mathbb{R}^2)$ denote the class of smooth functions on $\mathbb{R}^2$, and $(\varphi_u(\boldsymbol{x}^0, t), t > 0)$ be the positive semi-trajectory of the system (2.1) under control $u(\boldsymbol{x}) \in \mathrm{C}^1(\mathbb{R}^2)$ with the initial point $\boldsymbol{x}^0$. The **reachable set** of the systems (2.1) at $\boldsymbol{x}^0$ (see ref. [14]) is defined by

$$\mathcal{R}(\boldsymbol{x}^0) \triangleq \bigcup_{u \in \mathrm{C}^1(\mathbb{R}^2)} \{\varphi_u(\boldsymbol{x}^0, t) \mid t > 0\}. \quad (4.1)$$

Lemma 4.1[14]**.** Let $\boldsymbol{x}^1$, $\boldsymbol{x}^2 \in \mathbb{R}^2$ be two distinct points which lie on the same control curve $\Gamma : \gamma(t)$ of the system (2.1). Then for any small ball $U(\boldsymbol{x}^2, \varepsilon)$ centered at $\boldsymbol{x}^2$ with radius $\varepsilon > 0$, there exists a control function $u(\boldsymbol{x})$ such that the positive semi-trajectory of (2.1) with the initial point $\boldsymbol{x}^1$ reaches the set $U(\boldsymbol{x}^2, \varepsilon)$ at some finite time.

We can say that Lemma 4.1 is also valid for the negative semi-trajectory.

Lemma 4.2[14]**.** For the planar affine nonlinear control systems (2.1), if $\det(\boldsymbol{f}(\boldsymbol{x}^0), \boldsymbol{g}(\boldsymbol{x}^0)) \neq 0$, then $\mathcal{R}(\boldsymbol{x}^0)$ is an open set.

Next, we introduce the following Lemma 4.3 which is about the transitivity of the reachable points.

Lemma 4.3[14]**.** Let $\boldsymbol{x}^0, \boldsymbol{x}^1, \boldsymbol{x}^2$ be three points in $\mathbb{R}^2$. If $\boldsymbol{x}^1 \in \mathcal{R}(\boldsymbol{x}^0)$ and $\boldsymbol{x}^2 \in \mathcal{R}(\boldsymbol{x}^1)$, then $\boldsymbol{x}^2 \in \mathcal{R}(\boldsymbol{x}^0)$, where $\mathcal{R}(\boldsymbol{x}^0)$ and $\mathcal{R}(\boldsymbol{x}^1)$ are the reachable set defined by (4.1).

Proof of Theorem 2.1. We first prove the sufficiency part of the theorem by contradiction.

Assume that $(\varphi_u(\boldsymbol{x}^0, t), t > 0)$ cannot approach to the origin under any control $u(\boldsymbol{x})$, where $(\varphi_u(\boldsymbol{x}^0, t), t > 0)$ is the positive semi-trajectory of the system (2.1) under control $u(\boldsymbol{x})$ with the initial point $\boldsymbol{x}^0$.

Because the system (2.1) is locally asymptotic controllability, there exists a small neighborhood of the origin $U(0,\delta)$ such that for any initial point $\boldsymbol{x}^1 \in U(0,\delta)$, there exists a control $u(\boldsymbol{x})$ driving the state $\boldsymbol{x}(t) \to 0$ as $t \to +\infty$.

We first show that $\{\varphi_u(\boldsymbol{x}^0, t) | t > 0\} \cap U(0,\delta) = \emptyset$. In fact, if $\{\varphi_u(\boldsymbol{x}^0, t) | t > 0\} \cap U(0,\delta) \neq \emptyset$, then there exists $t_1 > 0$, such that $\varphi_u(\boldsymbol{x}^0, t_1) \in U(0,\delta)$. Therefore, by the similar proof method of Lemma 4.3 in ref. [14] and the results in refs. [18,19], we can construct a new control such that the positive semi-trajectory with the initial point $\boldsymbol{x}^0$ tends to the origin, which contradicts with our assumption. Thus we have

$$\begin{aligned}\mathcal{R}(\boldsymbol{x}^0) \cap U(0,\delta) &= \left(\bigcup_{u \in \mathrm{C}^1(\mathbb{R}^2)} \{\varphi_u(\boldsymbol{x}^0, t) \mid t > 0\} \right) \cap U(0,\delta) \\ &= \bigcup_{u \in \mathrm{C}^1(\mathbb{R}^2)} (\{\varphi_u(\boldsymbol{x}^0, t) | t > 0\} \cap U(0,\delta)) = \emptyset. \end{aligned} \tag{4.2}$$

Hence, $\mathcal{R}(\boldsymbol{x}^0)$ is not the whole plane.

Let $\overline{\mathcal{R}(\boldsymbol{x}^0)}$ be the closure of $\mathcal{R}(\boldsymbol{x}^0)$, then there must exist a point $\xi \in \overline{\mathcal{R}(\boldsymbol{x}^0)}$ such that

$$\|\xi\| = \inf\{\|\boldsymbol{x}\| : \boldsymbol{x} \in \overline{\mathcal{R}(\boldsymbol{x}^0)}\}. \tag{4.3}$$

By (4.2) we know that $\|\xi\| > 0$, and that ξ is on the boundary $\partial(\mathcal{R}(\boldsymbol{x}^0))$.

Let Γ_0 denote the trajectory of (2.2) passing through the origin. Hence Γ_0 separates the plane into two parts which are denoted by Side-A and Side-B respectively. By Lemmas 4.1, 4.2 and 4.3, it can be easily shown that ξ cannot lie on the curve Γ_0 by our contraction assumption. Without loss of generality, we suppose that ξ lies in Side-A. Now, we proceed with the following four steps:

Step 1. We prove that $\boldsymbol{x}^0$ should lie on the same side with ξ, i.e., Side-A.

First of all, by Lemma 4.1, it is obvious that the point $\boldsymbol{x}^0$ must not lie on Γ_0. Now, let $U(\xi, \varepsilon) \subseteq$ Side-A be a small neighborhood of ξ. Then there exists a control such that the trajectory $\varphi(\boldsymbol{x}^0, t), t > 0$ of (2.1) with initial point $\boldsymbol{x}^0$ can reach a point in $U(\xi, \varepsilon)$ at some time. Therefore, by the Jordan curve-like theorem, if $\boldsymbol{x}^0$ lies on Side-B, then $\varphi(\boldsymbol{x}^0, t), t > 0$ must intersect with Γ_0. By a similar argument as used before, this will contradict with our assumption. Hence $\boldsymbol{x}^0$ must lie on Side-A.

Step 2. We prove that the point $\boldsymbol{x}^0$ does not lie on the inner side of the control curve Γ_ξ which passes through ξ.

To this end, let Γ_1 denote the control curve passing through the point $\boldsymbol{x}^0$. By the uniqueness theorem of solutions and the Jordan curve-like theorem, the three curve Γ_0, Γ_1 and Γ_ξ separate Side-A into three disjoint parts as shown in Fig. 3. If $\boldsymbol{x}^0$ lies in the inner side of Γ_ξ, we then have

$$\text{the outer side of the control curve } \Gamma_1 \supset \text{the outer side of the control curve } \Gamma_\xi. \tag{4.4}$$

Therefore, we have

$$0 < \inf\{\|\boldsymbol{x}\| : \boldsymbol{x} \in \text{the outer side of } \Gamma_1\} < \inf\{\|\boldsymbol{x}\| : \boldsymbol{x} \in \text{the outer side of } \Gamma_\xi\}. \tag{4.5}$$

By Lemma 4.1, $\Gamma_1 \subseteq \overline{\mathcal{R}(\boldsymbol{x}^0)}$. Therefore, by (4.5), there exists a $\eta \in \Gamma_1$ such that $\|\eta\| < \|\xi\|$ which contradicts with the definition (4.3) of ξ.

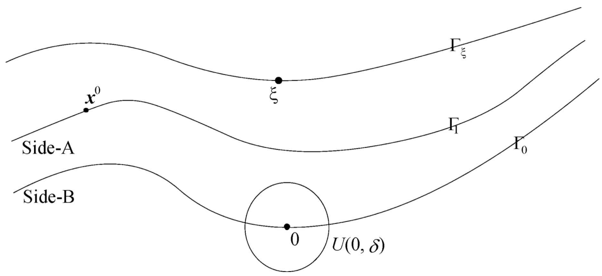

Fig. 3.

Step 3. We prove that there is no reachable point between Γ_0 and Γ_ξ. This is true, because, otherwise, by Lemmas 4.1 and 4.3, and the procedure in Step 2, we can also get a contradiction with definition (4.3) of ξ.

Step 4. Finally, we prove that the sufficiency part of Theorem 2.1 holds. Because Γ_ξ is a P-curve, there exists a point $\boldsymbol{y} \in \Gamma_\xi$ such that $\langle \boldsymbol{f}(\boldsymbol{y}), \boldsymbol{p}(\boldsymbol{y})\rangle < 0$, where $\boldsymbol{p}(\boldsymbol{y})$ is a non-zero normal vector of Γ_ξ, which points to the outer side of Γ_ξ. By Lemmas 4.1, 4.3 and the fact that ξ is on the boundary $\partial(\mathcal{R}(\boldsymbol{x}^0))$, we can construct a control such that the positive semi-trajectory of (2.1) goes to the inner side of Γ_ξ. This certainly contradicts with the conclusion of Step 3. Hence the sufficiency part of Theorem 2.1 is proven.

Next, we prove the necessity part of Theorem 2.1 also by contradiction.

We need only to show that if there exists a control curve Γ which is not a P-curve, then the system (2.1) cannot be globally asymptotic controllable. By the similar method in ref. [14], we can easily prove that the positive semi-trajectory of (2.1) starting from the outer side of Γ cannot go into its inner side under any control $u(\boldsymbol{x})$ by the assumption about Γ, and therefore it cannot approach to the origin. This completes the proof. Q.E.D.

5 Concluding remarks

In this paper, we presented a necessary and sufficient condition for the global asymptotic controllability and a necessary condition for global stabilization of the planar affine nonlinear systems. These conditions are imposed on the system nonlinear structure only, and the analysis technique is based on some basic facts in topology and in the geometric theory of ordinary differential equations, as well as on some basic results established recently in ref. [14]. For future investigation, it is desirable to extend the results of this paper to high dimensional nonlinear control systems.

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