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Convergence analysis of cautious control

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Abstract In this paper, we present a theoretical analysis on stability and convergence of the cautious control, which has advantages over the traditional certainty equivalence adaptive control, since it takes the parameter estimation error into account in the design, and is also one-step-ahead optimal in the mean square sense under Gaussian assumptions.

Keywords: adaptive control, cautious control, stability, convergence, least-squares, Kalman filter, certainty equivalence.

1 Introduction

Over the past three decades, there have been extensive efforts devoted to adaptive control of linear stochastic systems, and much progress has been made in both theory and applications (cf. refs. [1–8]).

However, most of the stochastic adaptive controllers studied so far are designed mainly based on the so-called "certainty equivalence principle". This principle means that in the design of adaptive controller, the unknown parameter is simply replaced by its on-line estimate, without taking the parameter estimation error into account. As is well-known, this kind of controller hardly has any optimality at any finite step, and may result in very poor transient performances of the underlying adaptive control systems.

In this paper, we take another view on the design of adaptive control, and investigate the so-called cautious control which may be derived from the standpoint of partially observed stochastic control systems (see ref. [3]).

Let us consider the following SISO linear discrete-time stochastic system:

$$A(z)y_t = B(z)u_{t-1} + w_t, \quad t \geqslant 0, \tag{1.1}$$

where $\{y_t\}, \{u_t\}$, and $\{w_t\}$ are the system output, input and noise processes respectively. We assume that $y_t = u_t = w_t = 0$, $\forall t < 0$, $A(z), B(z)$ are polynomials in the backward-shift operator z,

$$A(z) = 1 + a_1 z + \cdots + a_p z^p, \qquad p \geqslant 1,$$
$$B(z) = b_1 + b_2 z + \cdots + b_q z^{q-1}, \qquad q \geqslant 1,$$

where a_i, $1 \leqslant i \leqslant p$; b_j, $1 \leqslant j \leqslant q$ are unknown coefficients; p, q are the upper bounds for the true orders. Now, let the unknown parameter be denoted as

$$\theta = [-a_1 \ \cdots \ -a_p \ b_1 \ \cdots \ b_q]^T, \tag{1.2}$$

and introduce the corresponding regressor

$$\varphi_t = [y_t \ \cdots \ y_{t-p+1} \ u_t \ \cdots \ u_{t-q+1}]^T. \tag{1.3}$$

Then system (1.1) can be rewritten as

$$y_{t+1} = \theta^T \varphi_t + w_{t+1}, \quad t \geqslant 0. \tag{1.4}$$

Our control objective is, at any instant t, to construct a feedback control u_t based on the past measurements $\{y_0, \cdots, y_t, u_0, \cdots, u_{t-1}\}$ so that the following one-step-ahead tracking error is minimized:

$$I_t \triangleq E(y_{t+1} - y^*_{t+1})^2, \tag{1.5}$$

where $\{y^*_i\}$ is a known deterministic reference signal.

If the parameters of system (1.1) were known, it would be easy to determine the optimal control law as follows:

$$\theta^T \varphi_t = y^*_{t+1}.$$

From this, u_t can be expressed explicitly as

$$u_t = \frac{y^*_{t+1} - \theta^T \bar{\varphi}_t}{b_1}, \tag{1.6}$$

where $\bar{\varphi}_t = [y_t \ \cdots \ y_{t-p+1} \ 0 \ u_{t-1} \cdots \ u_{t-q+1}]^T$.

In the case where the parameter is unknown, the traditional "certainty equivalence principle" suggests that the unknown parameter θ be replaced by its online estimate θ_t, and thus we get the familiar adaptive control law as follows:

$$u_t = \frac{y^*_{t+1} - \theta_t^T \bar{\varphi}_t}{b_{1t}}, \tag{1.7}$$

where b_{1t} is the estimate of b_1. Note, however, that

$$\theta = \theta_t + \tilde{\theta}_t, \qquad \tilde{\theta}_t = \theta - \theta_t.$$

Hence, obviously, the above controller does not take the parameter estimation error $\tilde{\theta}_t$ into account, and so it is hardly to be an optimal one at any time t. Also, u_t may not be well defined, since the set $\{b_{1t} = 0\}$ may have a positive probability.

To obtain an optimal controller, let us assume that the parameter θ and the noise sequence $\{w_t\}$ are jointly Gaussian. Assume further that $\{w_t\}$ is a zero mean white noise with variance σ_w^2. Then by the Kalman filter theory (cf. ref. [5]), we have for any $u_t \in \mathcal{Y}_t$,

$$\theta_t = E[\theta|\mathcal{Y}_t], \qquad P_t = E[\tilde{\theta}_t \tilde{\theta}_t^T|\mathcal{Y}_t], \qquad \tilde{\theta}_t = \theta - \theta_t,$$

where $\mathcal{Y}_t$ is the σ-algebra generated by $\{y_0, \cdots, y_t\}$, and θ_t is the estimate of θ which may be generated by the least-square algorithm, P_t is the corresponding estimation covariance. From the facts that $E[\tilde{\theta}_t|\mathcal{Y}_t] = 0$ and $\{w_t\}$ is Gaussian and white, we can see that

$$\begin{aligned} &E[(y_{t+1} - y_{t+1}^*)^2|\mathcal{Y}_t] \\ &= E[(\theta_t^T \varphi_t - y_{t+1}^* + \tilde{\theta}_t^T \varphi_t + w_{t+1})^2)|\mathcal{Y}_t] \\ &= (\theta_t^T \varphi_t - y_{t+1}^*)^2 + \varphi_t^T P_t \varphi_t + \sigma_w^2 \\ &= (\theta_t^T \bar{\varphi}_t + b_{1t} u_t - y_{t+1}^*)^2 + \bar{\varphi}_t^T P_t \bar{\varphi}_t + u_t^2 p_{b_1}(t) + 2u_t P_l(t)\bar{\varphi}_t + \sigma_w^2, \end{aligned} \quad (1.8)$$

where $P_l(t) = l^T P_t$, $p_{b_1}(t) = l^T P_t l$ and l is the $(p+1)$th column of the $d \times d$ identity matrix with $d = p + q$.

Now, minimizing (1.8) with respect to u_t gives the one-step-ahead optimal controller

$$u_t = -\frac{[b_{1t}(\theta_t^T \bar{\varphi}_t - y_{t+1}^*) + P_l(t)\bar{\varphi}_t]}{b_{1t}^2 + p_{b_1}(t)}. \quad (1.9)$$

The above controller differs from (1.7), because the parameter estimate error measured by P_t has been taken into account. Furthermore, it is cautious since the presence of $p_{b_1}(t)$ in the denominator will reduce the magnitude of u_t, once the estimate uncertainty in b_{1t} measured by $p_{b_1}(t)$ is large (see ref. [3]).

2 Main results

Although the Gaussian assumption is used in the derivation of the cautious control in the last section, the main theorem together with the stability analysis to be given below does indeed not require this assumption.

In fact, throughout the sequel, we only need the following standard conditions:

(A.1) The noise sequence $\{w_t, \mathcal{F}_t\}$ is a martingale difference sequence i.e., $E[w_{t+1}|\mathcal{F}_t] = 0$, and satisfies

$$E\left[w_{t+1}^2|\mathcal{F}_t\right] = \sigma^2 > 0 \quad \text{a.s.} \quad (2.1)$$

$$\sup_t E\left[|w_{t+1}|^\beta|\mathcal{F}_t\right] < \infty \quad \text{a.s.} \quad \text{for some } \beta > 2. \quad (2.2)$$

(A.2) $B(z) \neq 0, \ \forall z$ with $|z| \leqslant 1$ (the minimum-phase condition).

(A.3) $\{y_t^*\}$ is a bounded reference sequence independent of $\{w_t\}$.

We remark that if $\{d_t\}$ is a nondecreasing positive deterministic sequence such that

$$w_t{}^2 = O(d_t) \quad \text{a.s.} \quad (2.3)$$

then under condition (A.1), d_t can be taken as

$$d_t = t^\gamma, \quad \forall \gamma \in \left(\frac{2}{\beta}, 1\right), \quad (2.4)$$

where β is given by (A.1) (cf. ref. [7]).

Instead of the one-step-ahead tracking error I_t, we consider the following averaged tracking error

$$J_t \triangleq \frac{1}{t}\sum_{i=1}^t (y_i - y_i^*)^2.$$

Define

$$R_t \triangleq \sum_{i=1}^{t}(y_i - y_i^* - w_i)^2. \tag{2.5}$$

Then by the condition (A.1), we know that for any adaptive sequence $\{u_t, \mathcal{F}_t\}$, the asymptotic lower bound to J_t is σ^2, and

$$R_t = o(t) \Leftrightarrow \lim_{t\to\infty} J_t = \sigma^2,$$

where σ^2 is defined by (2.1). Furthermore, once $R_t = o(t)$ is proved, the global stability, i.e., $\sum_{j=1}^{t}(y_j^2 + u_j^2) = O(t)$, can be derived easily by the conditions (A.1)–(A.3).

Now we introduce the standard least-squares (LS) algorithm for the estimation of the unknown parameter θ :

$$\theta_{t+1} = \theta_t + a_t P_t \varphi_t (y_{t+1} - \varphi_t^T \theta_t), \tag{2.6}$$

$$P_{t+1} = P_t - a_t P_t \varphi_t \varphi_t^T P_t, \tag{2.7}$$

$$a_t = (1 + \varphi_t^T P_t \varphi_t)^{-1}, \tag{2.8}$$

where the initial values $\theta_0,\ P_0 > 0$ can be chosen arbitrarily.

Let b_{1t} be the estimate for b_1 given by θ_t, and denote

$$D \triangleq \{b_{1t} \neq 0,\ \forall t;\ \liminf_{t\to\infty} \sqrt{\log(t + r_{t-1})}|b_{1t}| \neq 0\}, \tag{2.9}$$

where

$$r_t \triangleq 1 + \sum_{i=0}^{t} \|\varphi_i\|^2. \tag{2.10}$$

The main result of this paper is as follows.

Theorem 2.1. For system (1.1), let the conditions (A.1)–(A.3) be satisfied, and let the control law be defined by (1.9). Then the closed-loop system is globally stable, optimal (in the sense that $R_t = o(t)$), and has the following rate of convergence on set D:

$$R_t = O(\log t + \varepsilon_t), \tag{2.11}$$

where

$$\varepsilon_t = (\log t) \max_{1\leqslant j\leqslant t}\{\delta_j j^{\varepsilon} d_j\},\ \forall \varepsilon > 0, \tag{2.12}$$

$$\delta_t \triangleq \operatorname{tr}(P_t - P_{t+1}). \tag{2.13}$$

3 Proofs of the main results

To prove Theorem 2.1, we need to present several lemmas first.

Introduce the following notation:

$$\alpha_t \triangleq \frac{(\tilde{\theta}_t^T \varphi_t)^2}{1 + \varphi_t^T P_t \varphi_t}.$$

We then have

Lemma 3.1[6]. Under conditions (A.1) and (A.2), for any initial values (θ_0, P_0), if $\{u_t\}$ is adapted to $\mathcal{G}_t \triangleq \sigma(y_j,\ y^*_{j+1},\ j \leqslant t)$, then the estimate $\{\theta_t\}$ given by the LS-based algorithm (2.6)–(2.8) satisfies

$$\text{(H.1)} \qquad \|\theta_t\|^2 = O(\log r_{t-1}) \quad \text{a.s.}$$

$$\text{(H.2)} \qquad \sum_{i=1}^{t} \alpha_i = O(\log r_t) \quad \text{a.s.}$$

where r_t is defined by (2.10).

For simplicity, we will sometimes omit the phrase "a.s. on D" in the sequel, and all relationships should be understood to be held on D with a possible exceptional set of probability zero.

Lemma 3.2. Consider the closed-loop system (1.1) with the control law given by (1.9). If conditions (A.1)–(A.3) are satisfied, then there exists a positive random process $\{L_t\}$ such that

$$y_t^2 \leqslant L_t, \quad L_{t+1} \leqslant (\lambda + cf_t)L_t + \xi_t \quad \text{a.s.} \quad \text{on } D,$$

where the constants $\lambda \in (0,1),\ c > 0$, and

$$f_t = [\alpha_t\delta_t \log(t+r_t)]^2 + \alpha_t[\delta_t \log(t+r_t)]^3 + \alpha_t\delta_t + [\delta_t \log(t+r_t)]^4 + \delta_t^2 \log(t+r_t), \tag{3.1}$$

$$\xi_t = O(d_t \log^4(t+r_t)). \tag{3.2}$$

Proof. First of all, from (1.9), we know that

$$b_{1t}(b_{1t}u_t + \theta_t^T \bar{\varphi}_t) = -(p_{b_1}(t)u_t + P_l(t)\bar{\varphi}_t) + b_{1t}y^*_{t+1},$$

that is

$$b_{1t}(\theta_t^T\varphi_t - y^*_{t+1}) = -P_l(t)\varphi_t. \tag{3.3}$$

Next, we give an estimation of $|P_l(t)\varphi_t|^2$. From the LS algorithm(2.6)–(2.8), we have

$$\begin{aligned}
|P_l(t)\varphi_t|^2 &= l^T P_t\varphi_t\varphi_t^T P_t l \\
&= l^T \frac{(P_t - P_{t+1})}{a_t} l \\
&= (1 + \varphi_t^T P_t \varphi_t) l^T (P_t - P_{t+1}) l \\
&\leqslant [1 + \varphi_t^T (P_t - P_{t+1})\varphi_t + \varphi_t^T P_{t+1}\varphi_t]\delta_t \\
&\leqslant \delta_t(2 + \delta_t\|\varphi_t\|^2),
\end{aligned} \tag{3.4}$$

where we have used the fact that $\varphi_t^T P_{t+1}\varphi_t \leqslant 1$.

From the definition of D, we know that there exists a random variable M such that

$$\frac{1}{|b_{1t}|^2} \leqslant M \log(t + r_{t-1}) \quad \text{a.s.} \quad \text{on } D. \tag{3.5}$$

Then combining (3.3), (3.4) and (3.5), we have

$$(\theta_t^T\varphi_t - y^*_{t+1})^2 \leqslant \delta_t(2 + \delta_t\|\varphi_t\|^2)M \log(t + r_{t-1}) \quad \text{a.s.} \quad \text{on } D. \tag{3.6}$$

Now, by (1.4), we have

$$y_{t+1} = \theta^T\varphi_t + w_{t+1} = \tilde{\theta}_t^T\varphi_t + (\theta_t^T\varphi_t - y^*_{t+1}) + y^*_{t+1} + w_{t+1}. \tag{3.7}$$

From the definition of α_t, we know that

$$\begin{aligned}(\tilde{\theta}_t^T\varphi_t)^2 &= \alpha_t(1+\varphi_t^T P_t\varphi_t)\\ &= \alpha_t(1+\varphi_t^T(P_t-P_{t+1})\varphi_t+\varphi_t^T P_{t+1}\varphi_t)\\ &\leqslant \alpha_t(2+\delta_t\|\varphi_t\|^2).\end{aligned} \tag{3.8}$$

From (3.6) and the above, we know that on set D,

$$\begin{aligned}y_{t+1}^2 &\leqslant 3(\tilde{\theta}_t^T\varphi_t)^2+3(\theta_t^T\varphi_t-y_{t+1}^*)^2+3(y_{t+1}^*+w_{t+1})^2\\ &\leqslant 3\alpha_t(2+\delta_t\|\varphi_t\|^2)+3M\delta_t(2+\delta_t\|\varphi_t\|^2)\log(t+r_{t-1})+O(d_t)\\ &= (3\alpha_t+3M\delta_t\log(t+r_{t-1}))\delta_t\|\varphi_t\|^2\\ &\quad +(6\alpha_t+6M\delta_t\log(t+r_{t-1}))+O(d_t).\end{aligned} \tag{3.9}$$

By Lemma 3.1, we know that $\alpha_t=O(\log r_t)$. Also, from

$$\sum_{j=0}^{t}\delta_j=\sum_{j=0}^{t}(\operatorname{tr}P_j-\operatorname{tr}P_{j+1})\leqslant \operatorname{tr}P_0\leqslant\infty,$$

we know that $\delta_t\to 0$. Therefore,

$$y_{t+1}^2\leqslant(3\alpha_t+3M\delta_t\log(t+r_{t-1}))\delta_t\|\varphi_t\|^2+O(\log(t+r_t))+O(d_t). \tag{3.10}$$

By the stability of $B(z)$, it is seen that there exists $\lambda\in(0,1)$ such that

$$u_{t-1}^2=O\Big(\sum_{i=0}^{t}\lambda^{t-i}y_i^2\Big)+O(d_t). \tag{3.11}$$

Hence

$$\|\varphi_t\|^2-u_t^2=\sum_{i=0}^{p-1}y_{t-i}^2+\sum_{i=1}^{q-1}u_{t-i}^2=O\Big(\sum_{i=0}^{t}\lambda^{t-i}y_i^2\Big)+O(d_t). \tag{3.12}$$

By the property of (H.1), (3.5) and the fact that $P_l(t)$ is bounded, it follows from (1.9) that

$$u_t^2=O\Big([\log^2(t+r_{t-1})]\Big(\sum_{i=0}^{p-1}y_{t-i}^2+\sum_{i=1}^{q-1}u_{t-i}^2\Big)+\log(t+r_{t-1})\Big). \tag{3.13}$$

Moreover, putting (3.11) into (3.13), we have

$$u_t^2=O\Big([\log^2(t+r_{t-1})]\Big(\sum_{i=0}^{t}\lambda^{t-i}y_i^2+d_t\Big)\Big). \tag{3.14}$$

Hence, by (3.12) and the above equality, we have

$$\|\varphi_t\|^2=O(L_t\log^2(t+r_{t-1}))+O(d_t\log^2(t+r_{t-1})), \tag{3.15}$$

where $L_t\triangleq\sum_{i=0}^{t}\lambda^{t-i}y_i^2$. Note that

$$b_1u_t=\tilde{\theta}_t^T\varphi_t+(\theta_t^T\varphi_t-y_{t+1}^*)+y_{t+1}^*+(b_1u_t-\theta^T\varphi_t),$$

by (3.12), we have

$$\begin{aligned}b_1^2u_t^2 &\leqslant 4(\tilde{\theta}_t^T\varphi_t)^2+4(\theta_t^T\varphi_t-y_{t+1}^*)^2+4(y_{t+1}^*)^2+4(b_1u_t-\theta^T\varphi_t)^2\\ &= 4(\tilde{\theta}_t^T\varphi_t)^2+4(\theta_t^T\varphi_t-y_{t+1}^*)^2+O(L_t+d_t).\end{aligned}$$

Similar to the proof of (3.10), it is known that

$$u_t^2 = O([\alpha_t + \delta_t \log(t + r_{t-1})]\delta_t \|\varphi_t\|^2) + O(L_t) + O(d_t + \log(t + r_t)). \quad (3.16)$$

From (3.12) and the above, we can get

$$\|\varphi_t\|^2 = O([\alpha_t + \delta_t \log(t + r_{t-1})]\delta_t \|\varphi_t\|^2) + O(L_t) + O(d_t + \log(t + r_t)).$$

Substituting (3.15) into this, we have

$$\begin{aligned} \|\varphi_t\|^2 &= O([\alpha_t + \delta_t \log(t + r_{t-1})]\delta_t L_t \log^2(t + r_{t-1})) \\ &\quad + O([\alpha_t + \delta_t \log(t + r_{t-1})]\delta_t d_t \log^2(t + r_{t-1})) \\ &\quad + O(L_t + d_t + \log(t + r_t)) \\ &= O([\alpha_t + \delta_t \log(t + r_{t-1})]\delta_t L_t \log^2(t + r_{t-1})) \\ &\quad + O(L_t) + O(d_t \log^3(t + r_t)). \end{aligned} \quad (3.17)$$

Finally, putting the above into (3.10), we have

$$\begin{aligned} y_{t+1}^2 &= O([\alpha_t + \delta_t \log(t + r_{t-1})]^2 \delta_t^2 \log^2(t + r_{t-1})) L_t \\ &\quad + O([\alpha_t + \delta_t \log(t + r_{t-1})]\delta_t) L_t \\ &\quad + O([\alpha_t + \delta_t \log(t + r_{t-1})]\delta_t d_t \log^3(t + r_t)) \\ &\quad + O(d_t + \log(t + r_t)) \\ &= O([\alpha_t \delta_t \log(t + r_t)]^2 + \alpha_t [\delta_t \log(t + r_t)]^3 + \alpha_t \delta_t \\ &\quad + [\delta_t \log(t + r_t)]^4 + \delta_t^2 \log(t + r_t)) L_t \\ &\quad + O(d_t \log^4(t + r_t)). \end{aligned} \quad (3.18)$$

That is to say, there exists a random variable $c > 0$ such that

$$y_{t+1}^2 \leqslant c f_t L_t + \xi_t,$$

where f_t, ξ_t are defined in (3.1) and (3.2) respectively. By the definition of L_t, we know that $y_t^2 \leqslant L_t$. Furthermore,

$$L_{t+1} = y_{t+1}^2 + \lambda L_t \leqslant (\lambda + c f_t) L_t + \xi_t.$$

Hence the lemma is proved.

Lemma 3.3. Under the conditions of Lemma 3.2, we have

$$\|\varphi_t\|^2 = O((t + r_t)^\varepsilon d_t) \quad \text{a.s. on } D, \quad \forall \varepsilon > 0.$$

Proof. By Lemma 3.2, we know that

$$L_{t+1} \leqslant \lambda^{t+1} \Big[\prod_{j=0}^{t} (1 + \lambda^{-1} c f_j)\Big] L_0 + \sum_{i=0}^{t} \lambda^{t-i} \Big[\prod_{j=i+1}^{t} (1 + \lambda^{-1} c f_j)\Big] \xi_i. \quad (3.19)$$

We proceed to estimate the product $\prod_{j=i+1}^{t}(1 + \lambda^{-1} c f_j)$.

First, by Lemma 3.1, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\delta \sum_{j=0}^{t} \alpha_j \leqslant \varepsilon \log(r_t), \quad \forall\, t.$$

Also, since $\sum_{j=0}^{\infty} \delta_j < \infty$, there is an integer $i_0 > 0$ sufficiently large such that

$$\frac{4}{\delta}\left(\frac{c}{\lambda}\right)^{1/2} \sum_{j=i}^{\infty} \delta_j \leqslant \varepsilon, \quad \forall\, i \geqslant i_0.$$

Without loss of generality, we assume that $0 < \delta < 1$ and $\frac{c}{\lambda} > 1$. Note that

$$\begin{aligned}
1 + \lambda^{-1} c f_j &= 1 + \lambda^{-1} c([\alpha_j \delta_j \log(j + r_j)]^2 + \alpha_j [\delta_j \log(j + r_j)]^3 \\
&\quad + \alpha_j \delta_j + [\delta_j \log(j + r_j)]^4 + \delta_j^2 \log(j + r_j)) \\
&\leqslant (1 + \lambda^{-1} c[\alpha_j \delta_j \log(j + r_j)]^2)(1 + \lambda^{-1} c \alpha_j [\delta_j \log(j + r_j)]^3) \\
&\quad \times (1 + \lambda^{-1} c \alpha_j \delta_j)(1 + \lambda^{-1} c[\delta_j \log(j + r_j)]^4) \\
&\quad \times (1 + \lambda^{-1} c \delta_j^2 \log(j + r_j)).
\end{aligned} \tag{3.20}$$

As proved in ref. [6],

$$\prod_{j=i+1}^{t} (1 + \lambda^{-1} c[\alpha_j \delta_j \log(j + r_j)]^2) \leqslant (t + r_t)^{2\varepsilon}, \tag{3.21}$$

$$\prod_{j=i+1}^{t} (1 + \lambda^{-1} c \alpha_j \delta_j) = O(r_t^{\varepsilon}). \tag{3.22}$$

Then by the inequalities

$$1 + xy \leqslant (1 + x)(1 + y), \quad x \geqslant 0,\ y \geqslant 0,$$
$$1 + x^n \leqslant e^{nx}, \quad n \geqslant 1,\ x \geqslant 0,$$

we know that, for all $t \geqslant i \geqslant i_0$,

$$\begin{aligned}
&\prod_{j=i+1}^{t} (1 + \lambda^{-1} c[\delta_j \log(j + r_j)]^4) \\
&\leqslant \exp\left(4\left(\frac{c}{\lambda}\right)^{1/4} \sum_{j=i+1}^{t} \delta_j \log(j + r_j)\right) \\
&\leqslant \exp\left(\left[4\left(\frac{c}{\lambda}\right)^{1/4} \sum_{j=i+1}^{t} \delta_j\right] \log(t + r_t)\right) \\
&\leqslant (t + r_t)^{\varepsilon},
\end{aligned} \tag{3.23}$$

$$\begin{aligned}
&\prod_{j=i+1}^{t} (1 + \lambda^{-1} c \alpha_j [\delta_j \log(j + r_j)]^3) \\
&\leqslant \prod_{j=i+1}^{t} (1 + \delta \alpha_j) \prod_{j=i+1}^{t} \left(1 + \frac{c}{\lambda \delta} \delta_j^3 \log^3(j + r_j)\right) \\
&\leqslant \exp\left(\delta \sum_{j=i+1}^{t} \alpha_j\right) \exp\left(\frac{3}{\sqrt[3]{\delta}}\left(\frac{c}{\lambda}\right)^{1/3} \sum_{j=i+1}^{t} \delta_j \log(j + r_j)\right) \\
&\leqslant r_t^{\varepsilon} \exp\left(\left[\frac{3}{\sqrt[3]{\delta}}\left(\frac{c}{\lambda}\right)^{1/3} \sum_{j=i+1}^{t} \delta_j\right] \log(t + r_t)\right) \\
&= O((t + r_t)^{2\varepsilon}),
\end{aligned} \tag{3.24}$$

and

$$
\begin{aligned}
&\prod_{j=i+1}^{t} (1+\lambda^{-1}c\delta_j^2 \log(j+r_j)) \\
&\leqslant \prod_{j=i+1}^{t} (1+\lambda^{-1}c\delta_j) \prod_{j=i+1}^{t} (1+\delta_j \log(j+r_j)) \\
&\leqslant \exp\Big(\frac{c}{\lambda}\sum_{j=i+1}^{t}\delta_j\Big) \exp\Big(\sum_{j=i+1}^{t}\delta_j \log(j+r_j)\Big) \\
&= O\Big(\exp\Big([\log(t+r_t)]\sum_{j=i+1}^{t}\delta_j\Big)\Big) = O((t+r_t)^{\varepsilon}).
\end{aligned} \tag{3.25}
$$

Then combining (3.20)–(3.25), we have

$$
\prod_{j=i+1}^{t} (1+\lambda^{-1}cf_j) = O((t+r_t)^{7\varepsilon}), \quad i \geqslant i_0. \tag{3.26}
$$

Finally, substituting this into (3.19), and note that $0<\lambda<1$, we have

$$
L_{t+1} = O((t+r_t)^{7\varepsilon} d_t \log^4(t+r_t)), \quad \forall\, \varepsilon > 0.
$$

Then by Lemma 3.2 and the arbitrariness of ε, we know that

$$
y_{t+1}^2 \leqslant L_{t+1} = O((t+r_t)^{\varepsilon} d_t), \quad \forall\, \varepsilon > 0.
$$

From this and (3.11), we know that $u_t^2 = O((t+r_t)^{\varepsilon} d_t)$, $\forall\, \varepsilon > 0$. Finally, from (3.12), we can see the lemma is true.

Now we are in the place to prove the main result.

Proof of Theorem 2.1. By (3.7), we have

$$
\begin{aligned}
R_{t+1} &= \sum_{j=0}^{t} (y_{j+1} - y_{j+1}^* - w_{j+1})^2 \\
&= \sum_{j=0}^{t} (\tilde{\theta}_j^T \varphi_j + (\theta_j^T \varphi_j - y_{j+1}^*))^2 \\
&\leqslant 2\sum_{j=0}^{t} (\tilde{\theta}_j^T \varphi_j)^2 + 2\sum_{j=0}^{t} (\theta_j^T \varphi_j - y_{j+1}^*)^2.
\end{aligned} \tag{3.27}
$$

From (3.6) and (3.8), it follows that

$$
\begin{aligned}
R_{t+1} &\leqslant 2\sum_{j=0}^{t} \alpha_j (2+\delta_j \|\varphi_j\|^2) + 2M \sum_{j=0}^{t} [\delta_j (2+\delta_j\|\varphi_j\|^2) \log(j+r_{j-1})] \\
&= 4\sum_{j=0}^{t} \alpha_j + 4M \sum_{j=0}^{t} \delta_j \log(j+r_{j-1}) \\
&\quad + O\Big(\sum_{j=0}^{t} [\alpha_j\delta_j + \delta_j^2 \log(j+r_{j-1})](j+r_j)^{\varepsilon} d_j\Big)
\end{aligned}
$$

$$\leqslant O(\log r_t) + 4M\log(t+r_{t-1})\sum_{j=0}^{t}\delta_j$$
$$+O\Big(\max_{1\leqslant j\leqslant t}\{\delta_j(j+r_j)^{\varepsilon}d_j\}\Big[\sum_{j=0}^{t}\alpha_j+\sum_{j=0}^{t}\delta_j\log(j+r_{j-1})\Big]\Big)$$
$$=O(\log(t+r_t))+O\Big(\max_{1\leqslant j\leqslant t}\{\delta_j(j+r_j)^{\varepsilon}d_j\}\log(t+r_t)\Big). \tag{3.28}$$

Therefore, for (2.11), it suffices to prove that $r_t = O(t)$. From the above, we have $R_{t+1} = O((t+r_t)^{\varepsilon}d_t)$.

By (3.27) and the assumptions on $\{y_j^*\}$ and $\{w_j\}$, it follows that

$$\sum_{j=0}^{t+1} y_j^2 = O(R_{t+1}) + O(t)$$
$$= O((t+r_t)^{\varepsilon}d_t) + O(t), \quad \forall\, \varepsilon > 0. \tag{3.29}$$

From this and condition (A.2), it follows that

$$\sum_{j=0}^{t} u_j^2 = O\Big(\sum_{j=0}^{t+1} y_j^2\Big) + O\Big(\sum_{j=0}^{t+1} w_j^2\Big)$$
$$= O((t+r_t)^{\varepsilon}d_t) + O(t), \quad \forall\, \varepsilon > 0. \tag{3.30}$$

Then from the definition of r_t, we have for any $\varepsilon > 0$,

$$r_t = 1 + \sum_{j=0}^{t}\|\varphi_j\|^2 = O((t+r_t)^{\varepsilon}d_t) + O(t)$$
$$= O(t) + O((t+r_t)^{\varepsilon}t^{\gamma}), \quad \forall\, \gamma \in (2/\beta, 1).$$

By taking γ small enough such that $\varepsilon + \gamma < 1$, we get

$$\frac{r_t}{t} = O(1) + O\Big(\Big(\frac{t+r_t}{t}\Big)^{\varepsilon}\frac{1}{t^{1-\varepsilon-\gamma}}\Big)$$
$$= O(1) + o\Big(\Big(1+\frac{r_t}{t}\Big)^{\varepsilon}\Big).$$

From this we get $r_t = O(t)$, and hence

$$R_{t+1} = O(\log t) + O(\varepsilon_t) \quad \text{a.s. on } D,$$

where ε_t is defined by (2.12). Obviously, $R_t = o(t)$. Hence the proof of the theorem is completed.

Remark 3.1. In the above, we have discussed the SISO system. Actually the result in Theorem 2.1 can be extended to the multidimensional case. That is, if $\{y_t\}$, $\{u_t\}$, $\{w_t\}$ are all m-dimensional in (1.1), and

$$A(z) = 1 + A_1 z + \cdots + A_p z^p, \qquad p \geqslant 1,$$
$$B(z) = B_1 + B_2 z + \cdots + B_q z^{q-1}, \qquad q \geqslant 1,$$

where A_i, $1 \leqslant i \leqslant p$; B_j, $1 \leqslant j \leqslant q$ are unknown matrix coefficients and p, q are the upper bounds for the true orders, we can also obtain the cautious control under the corresponding assumptions (see ref. [7]), which has the same convergence rate as in Theorem

2.1. The analysis is similar to the SISO case, but more complicated.

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References

1 Goodwin G C, Sin, K S. Adaptive Filtering, Prediction and Control. Englewood Cliffs, NJ: Prentice Hall, 1984
2 Cains P E. Linear Stochastic Systems. New York: Wiley, 1988
3 Åström K J, Wittenmark B. Adaptive Control. Reading, MA: Addison-Wesley, 1995
4 Kumar P R, Varaiya P. Stochastic Systems: Estimation, Identification and Adaptive Control. Englewood Cliffs, NJ: Prentice Hall, 1986
5 Chen H F, Guo L. Identification and Stochastic Adaptive Control. Boston: Birkhauser, 1991
6 Guo L. Convergence and logarithm laws of self-tuning regulators. Automatica, 1995, 31(3): 435–450
7 Guo L, Chen H F. The Åström-Wittenmark self-tuning regulator revisited and ELS-based adaptive trackers. IEEE Trans Automat Contr, 1991, 36(7): 802–812
8 Guo L. Self-convergence of weighted least-squares with applications to stochastic adaptive control. IEEE Trans Automat Contr, 1996, 41(1): 79–89