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Synchronization of multi-agent systems without connectivity assumptions ⋆

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Abstract

Multi-agent systems arise from diverse fields in natural and artificial systems, such as schooling of fish, flocking of birds, coordination of autonomous agents. In multi-agent systems, a typical and basic situation is the case where each agent has the tendency to behave as other agents do in its neighborhood. Through computer simulations, Vicsek, Czirók, Ben-Jacob, Cohen and Sochet (1995) showed that such a simple local interaction rule can lead to certain kind of cooperative phenomenon (synchronization) of the overall system, if the initial states are randomly distributed and the size of the system population is large. Since this model is of fundamental importance in understanding the multi-agent systems, it has attracted much research attention in recent years. In this paper, we will present a comprehensive theoretical analysis for this class of multi-agent systems under a random framework with large population, but without imposing any connectivity assumptions as did in almost all of the previous investigations. To be precise, we will show that for any given and fixed model parameters concerning with the interaction radius r and the agents' moving speed v, the overall system will synchronize as long as the population size n is large enough. Furthermore, to keep the synchronization property as the population size n increases, both r and v can actually be allowed to decrease according to certain scaling rates.

Key words: Vicsek model, synchronization, connectivity, spectral graph theory, martingale.

1 Introduction

The collective behavior of multi-agent systems, such as swarm intelligence, consensus, coordination, is a major focus on complex systems, and it has drawn much attention from researchers in diverse fields, including biology (O'Brien, 1989; Okubo, 1986; Parrish, Viscido, Grünbaum, 2002; Shaw, 1975), physics (Vicsek et al., 1995), mathematics(Cucker & Smale, 2007), computer science (Reynolds, 1987), and control theory (Jadbabaie et al., 2003; Moreau, 2005; Olfati-Saber, 2006; Savkin, 2004; Ren & Beard, 2005). Scientifically, how locally interacting agents lead to collective behavior of the overall multi-agent systems is a basic and challenging problem to be understood.

Of course, different local rules will give rise to different collective behavior. In this paper, we will study the following basic multi-agent systems: n autonomous agents moving in the plane with the same constant speed and with heading of each agent updated according to the averaged direction of its neighbors. This model reflects a typical phenomenon in multi-agent systems: each agent has the tendency to behave as other agents do in its neighborhood (O'Brien, 1989; Vicsek et al., 1995). Vicsek et al. (1995) used this model to investigate the gathering, transport and phase transition in nonequilibrium systems, and they also pointed out its potential applications in biological systems involving clustering and migration. Through computer simulations, Vicsek *et al.* showed that the above system will synchronize when the population density is large and the noise is small. This model looks simple, but the nonlinear relationship in the model makes the theoretical analysis quite hard. In a well-known work, Jadbabaie et al. (2003) initiated a theoretical study for the synchronization of a related model, and inspired much subsequent theoretical investigations on similar problems (see, Cucker & Smale (2007), Liu & Guo (2008a),Moreau (2005), Ren & Beard (2005), Savkin (2004) among many others). What Jadbabaie et al. (2003) showed was that the system will synchronize if the associated dynamical neighbor graphs are jointly connected within some contiguous and bounded time intervals. It is worth mentioning that a similar theoretical result was presented in an earlier paper by Tsitsiklis, Bertsekas, and Athans (1986), but in a rather different context. However, how to remove or verify the troublesome connectivity condition imposed on the underlying dynamical systems turns out to be a difficult and challenging issue in theory, due to the strongly nonlinearly coupled dynam-

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ical equations describing the positions and headings of all the agents.

A preliminary step towards the above issue has been made by Liu & Guo (2008a), where a sufficient parameter condition is given for the connectivity and hence the synchronization of the system in a deterministic framework with initial headings lying in $(-\frac{\pi}{2}, \frac{\pi}{2})$. Some counterexamples are also given by Han et al. (2006) and Liu & Guo (2008a) to show that the connectivity of the associated neighbor graphs is not sufficient for synchronization if the initial headings are allowed to be in $[0, 2\pi)$. The main problem with Liu & Guo (2008a) is that the condition on the model parameters used there is rather restrictive. There are also a few other papers which establish synchronization without resorting to *a priori* connectivity conditions, by using additional information and/or constrains, for example, Cucker & Smale (2007) studied the model where each agent can interact globally, and Tahbaz-Salehi & Jadbabaie (2007) introduced a periodic boundary condition in the model.

Recently, a major advance towards the complete synchronization analysis is made by Tang & Guo (2007), where a random framework as originally considered by Vicsek et al. (1995) is used in the analysis of the linearized heading equations. They proved that the overall multi-agent system will synchronize with large probability as long as the size of the population is large enough. However, as mentioned by Jadbabaie et al. (2003) and Han et al. (2006), the linearized heading equation may give rise to some unreasonable phenomenon. It is worth pointing out that the random framework considered by Tang & Guo (2007) is just an assumption on the initial distribution of all the agents, the subsequent states together with the associated neighbor graphs, however, may well change from time to time according to the nonlinear dynamical models under consideration. This random framework is obviously different from those studied by Frasca, Carli, Fagnani and Zampieri (2009), Tahbaz-Salehi & Jadbabaie (2008) and Wu (2006), in either the problem formulations or the required assumptions, where certain connectivity conditions are essentially assumed in these papers. However, removing or verifying the troublesome connectivity condition of locally interacting nonlinear multi-agent systems appears to be a "bottleneck" problem in general.

In this paper, we will establish two synchronization theorems for the basic nonlinear model of Vicsek et al. (1995) in the random framework, without changing the locally interacting laws and without imposing any connectivity conditions. In comparison with the linearized heading equations studied by Tang & Guo (2007), a key issue now is how to deal with the difficulties arising from the nonlinear heading equations. We will give a comprehensive theoretical analysis with large population, by using some basic facts of Tang & Guo (2007), together with some estimation for multi-array martingales and with a detailed analysis for the nonlinear equations. Intuitively speaking, large population is beneficial to the connectivity in general, which in conjunction with the averaging mechanism to be given in equation (3) will ensure the topology of the dynamical network does not change too much, and hence ensure the synchronization of the system. We will give a rigorous proof for this intuition, and the main results to be established are the following:

i) For any given and fixed model parameters, *i.e.*, the interaction radius r and the agents' moving velocity v, the overall system will synchronize as long as the population size n is large enough;

ii) To keep the synchronization property as the population size n increases, both r and v can actually be allowed to decrease according to certain scaling rates to be given in the paper.

Part of the results in this paper was presented in Liu & Guo (2008b) without proof details. The rest of this paper is organized as follows: In Section 2, we will present our main results; Some notations and preliminary lemmas will be given in Section 3; The proofs of the main theorems will be given in Sections 4 and 5 respectively, and some simulation results will be given in Section 6; Finally, Some concluding remarks will be made in Section 7.

2 Main Results

The multi-agent system to be studied in this paper is composed of n autonomous agents (or subsystems or particles), labeled by $1, 2, \cdots, n$, moving in the plane with the same absolute velocity, and with each agent's heading updated according to the average of the directions of its neighbors (Vicsek et al., 1995). The neighbors of an agent $i (1 \le i \le n)$ at any discrete-time $t = 0, 1, 2, \cdots$ are those which lie within a circle of radius $r (r > 0)$ centered at the agent i's current position. Denote the neighbors of the agent i at time t as $\mathcal{N}_i(t)$, *i.e.*

$$\mathcal{N}_i(t) = \{j \,|\, d_{ij}(t) < r\}, \tag{1}$$

where $d_{ij}(t) = \sqrt{(x_i(t) - x_j(t))^2 + (y_i(t) - y_j(t))^2}$, and $(x_i(t), y_i(t))$ is the position of the agent i at time t. It is easy to see that each agent is a neighbor of itself. Each agent moves in the plane with the same constant absolute velocity $v (v > 0)$, so its position is updated according to the following equation:

$$\begin{cases} x_i(t+1) = x_i(t) + v \cos \theta_i(t+1) \\ y_i(t+1) = y_i(t) + v \sin \theta_i(t+1) \end{cases} \quad \forall i : 1 \le i \le n, \tag{2}$$

where $\theta_i(t)$ is the heading of the agent i at time t, which is updated according to the following average direction of the neighbors' velocity:

$$\theta_i(t+1) = \arctan \frac{\sum_{j \in \mathcal{N}_i(t)} \sin \theta_j(t)}{\sum_{j \in \mathcal{N}_i(t)} \cos \theta_j(t)}, \ \forall i : 1 \le i \le n. \tag{3}$$

Through computer simulations, Vicsek et al. (1995) showed that the above equations (1)-(3) can make all agents move in the same direction eventually, when the population size n is large enough. Throughout the paper, synchronization means that there exists a constant heading θ, such that

$$\lim_{t \to \infty} \theta_i(t) = \theta, \ \forall i.$$

From the description of the above mathematical model, we can see that the dynamical behavior of the overall system is determined completely by the moving velocity v, the neighborhood radius r and the initial states. Furthermore, one can observe that the neighbors of each agent are determined by the positions of other agents via (1), whereas the positions of agents are determined by the headings via (2), and moreover, the headings are influenced by the neighbors via (3) in return. So, there is a complicated nonlinear relationship between positions and headings of all agents, which makes a rigorous theoretical analysis quite involved.

The main purpose of this paper is to study the synchronization property of the multi-agent systems (1)-(3) with large population. We will conduct our analysis under the following simple and natural assumptions on the initial states of the system, which are similar to those used in the simulation study of Vicsek et al. (1995).

Assumption 1 *The initial positions and headings of all agents are mutually independent, with positions uniformly and independently distributed in the unit square $\mathcal{S}$, and headings uniformly and independently distributed in $[-\pi+\varepsilon_0, \pi-\varepsilon_0]$ with arbitrary $\varepsilon_0 \in (0,\pi)$.*

Under Assumption 1, the initial random geometric graph G_0 associated with the initial positions will have some nice properties, one of which is the connectivity studied in the celebrated paper of Gupta & Kumar (1998). Other related nice results may be found in the work of Penrose (2003) and Xue & Kumar (2004). However, in our paper, we need a deeper understanding of both the initial graph and the subsequent dynamic graphs in Sections 3 and 4, which will enable us to establish the following theorem whose proof is given in Section 4.

Theorem 2 *Under Assumption 1, for any given speed $v > 0$ and radius $r > 0$, the multi-agent system described by (1)-(3) will synchronize almost surely for all suitably large population.*

Remark 3 *In Assumption 1, the restriction $[-\pi+\varepsilon_0, \pi-\varepsilon_0]$ on the headings can be replaced by $[\alpha, \alpha+2\pi-2\varepsilon_0]$ with any constant α, on which the uniform distribution assumption of the headings can also be replaced by any other distributions, save that the synchronized direction will depend on the mean of the initial heading distributions. Moreover, the following counterexample will give us some clue why $\varepsilon_0 = 0$ may lead to difficulties in guaranteeing synchronization.*

Example 4 *Let $n = 12$, and all agents be distributed on the unit circle uniformly with the headings being symmetric as shown in Fig. 1 below. To be precise, let us assume*

$$(x_i(0), y_i(0)) = \left(\cos\frac{(i-1)\pi}{6}, \sin\frac{(i-1)\pi}{6}\right);$$
$$\theta_i(0) = \left\{[16 - i + 3\cdot(-1)^i]\frac{\pi}{6}\right\} mod(2\pi),\ 1 \le i \le 12.$$

Assume further that the absolute velocity satisfies $0 < v \le 0.1$ and the neighborhood radius is taken as $r = 0.8$. Then it can be shown that the system will never synchronize, even if the neighbor graphs are connected all the time(see Liu & Guo (2008a)).

Intuitively, when the population size n increases, the density will increase accordingly, and hence it is conceivable that the interaction radius r can be allowed to decrease with n, which may be denoted by r_n to reflect this situation. Obviously, in this case, the speed v should also decrease with n in order to ensure synchronization of the system, and again, we may denote this dependence by v_n. The following theorem shows that the multi-agent system (1)-(3) will synchronize if certain scaling rates on r_n and v_n are satisfied, in addition to the natural requirements $r_n = o(1)$ and $v_n = o(1)$ as $n \to \infty$ [1].

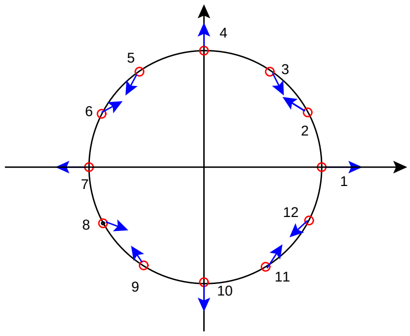

Fig. 1. The initial distribution of positions and headings of all agents in Example 4.

Theorem 5 *Let Assumption 1 hold, and let both the interaction radius r_n and the speed v_n depend on n in such a way that $r_n = o(1)$ and*

$$\sqrt[6]{\frac{\log n}{n}} = o(r_n), \quad v_n = O\left(\frac{r_n^6\sqrt{n}}{\log^{3/2} n}\right).$$

Then the multi-agent system (1)-(3) will synchronize almost surely for all large population.

The proof of Theorem 5 is similar to that of Theorem 2, and will be put in Section 5.

Remark 6 *If one wants the multi-agent system to be synchronized to a desired direction, we may apply the "soft control" idea as proposed by Han et al. (2006) or may introduce some "leaders" into the system. For the later situation, one may study this problem in two ways regardless of the independence of initial positions between the leaders and the followers. One way is to adopt a deterministic treatment by assuming a certain "uniform" conditions on the initial states as those given in Proposition 1 of Tang & Guo (2007), another way is to assume randomly distributed initial states as in the current framework, see Liu, Han, and Hu (2009) for details.*

3 Some Preliminary Results

The multi-agent system can be regarded as a dynamical network, and thus graph theory may play a role (cf., Jadbabaie et al. (2003), Liu & Guo (2008a)). Before proving our

[1] Throughout the sequel, the following standard notions will be used: for two positive sequences $\{a_n, n \ge 1\}$ and $\{b_n, n \ge 1\}$, $a_n = O(b_n)$ means that there exists a positive constant C independent of n, such that $a_n \le Cb_n$ for any $n \ge 1$; $a_n = o(b_n)$ means that $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$.

theorems, we need some basic notions and concepts from graph theory (cf., Godsil & Royle (2001), Chung (2000)).

An undirected graph $G = \{V, E\}$ is composed of a vertex (or node) set $V = \{1, 2, \cdots, n\}$ and an edge set $E = \{(i, j)\} \subseteq V \times V$, where $(i, j) \in E$ is an edge connecting vertexes i and j, and also means that j is a neighbor of i, and vice versa. For any vertex $i \in V$, if $(i, i) \in E$, then it is called a loop of G. The graphs to be studied in this paper are all undirected and contain loops. A path of length l in G that connects vertexes i and j means that there is a sequence of vertexes $i_1, i_2, \cdots, i_l$, such that $(i_m, i_{m+1}) \in E, 0 \le m \le l$, with $i_0 = i, i_{l+1} = j$. A graph G is called connected if for any two different vertexes i and j, there always exists a path that connects them.

The adjacency matrix $A = (a_{ij})_{n\times n}$ of graph G is a 0-1 matrix, where $a_{ij} = 1$ if and only if $(i, j) \in E$. The degree of vertex i is defined by $d_i = \sum_{j=1}^{n} a_{ij}$, and the minimum and maximum degrees of the graph G are defined by $d_{min} = \min_i\{d_i\}$ and $d_{max} = \max_i\{d_i\}$ respectively. The volume of the graph G are defined to be $Vol(G) = \sum_{i=1}^{n} d_i$. The degree matrix $T = (t_{ij})_{n\times n}$ is a diagonal matrix with diagonal entries $t_{ii} = d_i$, while $P^0 = T^{-1}A$ is called the average matrix of the graph G. Furthermore, the Laplacian of the graph G is defined to be $L = T - A$. The normalized Laplacian is defined to be $\mathcal{L} = T^{-1/2}LT^{-1/2}$, and it is easy to see that $\mathcal{L}$ (or L) is a nonnegative definite matrix, and 0 is the smallest eigenvalue of $\mathcal{L}$ (or L) (cf., Chung (2000)). Arrange all eigenvalues of $\mathcal{L}$ as follows: $0 = \lambda_0 \le \lambda_1 \le \cdots \le \lambda_{n-1}$, and a corresponding set of unit orthogonal eigenvectors is denoted by $\{\varphi_0, \cdots, \varphi_{n-1}\}$ with $\varphi_0 = (\sqrt{d_1}/Vol(G), \cdots, \sqrt{d_n}/Vol(G))^\tau$. Moreover, the second smallest eigenvalue λ_1 is called the algebraic connectivity of $\mathcal{L}$, since the connectivity of G is equivalent to the positivity of the algebraic connectivity λ_1. Furthermore, $\overline{\lambda} = \max\{|1 - \lambda_1|, |\lambda_{n-1} - 1|\}$ is called the spectral gap of $\mathcal{L}$, which may play a key role in the synchronization analysis of multi-agent systems (cf. Tang & Guo (2007)).

For the mathematical model (1)-(3), the neighbors of each agent will change over time, and we may use a time-dependent graph sequence $G_t = \{V, E_t\}$ to describe the evolution of the underlying system dynamics, where $V = \{1, 2, \cdots, n\}$ is the set of agents' indices (vertexes), E_t is the edge set at time t. Edges are formed in the following way: if the distance between agents i and j at time t, denoted by $d_{ij}(t)$, satisfies $d_{ij}(t) < r$, then we define an edge between i and j, denoted by $(i, j) \in E_t$. Obviously, the neighbor graphs formed in this way are undirected, and contain loops since each agent is a neighbor of itself. The degree, minimum degree, maximum degree of graph G_t are denoted by $d_i(t)(1 \le i \le n), d_{min}(t), d_{max}(t)$, respectively. The adjacency matrix, degree matrix, average matrix, and the normalized Laplacian of the graph G_t are denoted by $A(t), T(t), P^0(t)$ and $\mathcal{L}(t)$ respectively. The eigenvalues of $\mathcal{L}(t)$ are accordingly arranged in the following way: $0 = \lambda_0(t) \le \lambda_1(t) \cdots \le \lambda_{n-1}(t)$, and $\overline{\lambda}(t) = \min\{|1 - \lambda_1(t)|, |\lambda_{n-1}(t) - 1|\}$ is the spectral gap of the graph G_t.

For analyzing the heading equation (3), we rewrite it into the following equivalent form:

$$\tan\theta_i(t+1) = \sum_{j\in\mathcal{N}_i(t)} \frac{\cos\theta_j(t)}{\sum_{k\in\mathcal{N}_i(t)}\cos\theta_k(t)} \tan\theta_j(t), \quad (4)$$

and put (4) into the following compact matrix form:

$$\tan\theta(t+1) = P(t)\tan\theta(t), \quad (5)$$

where $\tan\theta(t) \triangleq (\tan\theta_1(t), \cdots, \tan\theta_n(t))^\tau$, $P(t) \triangleq (p_{ij}(t))$ is the weighted average matrix of the graph G_t:

$$p_{ij}(t) = \begin{cases} \frac{\cos\theta_j(t)}{\sum_{k\in\mathcal{N}_i(t)}\cos\theta_k(t)}, & \text{if } (i,j) \in E_t; \\ 0, & \text{otherwise.} \end{cases} \quad (6)$$

We will also use the standard average matrix $P^0(t) = (p^0_{ij}(t))$, which is actually the linearized version (around $\theta_i(t) = 0$) of the above $P(t)$, with entries defined explicitly by

$$p^0_{ij}(t) = \begin{cases} \frac{1}{d_i(t)}, & \text{if } (i,j) \in E_t; \\ 0, & \text{otherwise,} \end{cases} \quad (7)$$

where $d_i(t)$ is the degree of the agent i at time t, which equals to the cardinality of $\mathcal{N}_i(t)$.

To prove our first theorem, Theorem 2, we need to consider the following set:

$$\mathcal{R}_j = \{i : (1-\eta)r \le d_{ij}(0) \le (1+\eta)r\}, \quad (8)$$

where η can be taken as a small constant which will be specified in the paper as follows:

$$\eta = \frac{\left(\min(\sqrt{\pi r^2/4}, \sqrt{\pi/64})\right)^5}{3\cdot 2^{13}\cdot (r+\sqrt{6})^4}. \quad (9)$$

The cardinality of $\mathcal{R}_j$ will be denoted as R_j, and $R_{max} \triangleq \max_j R_j$.

To analyze R_{max} and to prove the theorem, we need to estimate some characteristics of both the initial random geometric graph G_0 and the headings at time $t = 1$. For this, we need the following multi-array martingale theorem whose proof can be carried out following the lines of Lemma 3.1 of Huang & Guo (1990), see Appendix A.

Lemma 7 *For any fixed n and $1 \le k \le n$, let $\{w_t(k,n), \mathcal{F}_t(k,n), t \ge 1\}$ be a martingale difference sequence which satisfies $|w_t(k,n)| \le 1, a.s., \forall\ 1 \le t, k \le n$. Also, for any fixed k, n, let $f_t(k,n)$ be $\mathcal{F}_t(k,n)$-measurable for all t, with $|f_t(k,n)| \le 1, a.s., 1 \le t, k \le n$. Then almost surely for large n,*

$$\max_{1\le k\le n}\left|\sum_{j=1}^{n-1} f_j(k,n) w_{j+1}(k,n)\right| \le \frac{3C_w(n)}{4} S_n + 3\log n,$$

where S_n and $C_w(n)$ are respectively the upper bounds of $\max_{1\le k\le n}\sum_{j=1}^{n} f_j^2(k,n)$ *and* $\max_{1\le k,j\le n} E\left[w_{j+1}^2(k,n)|\mathcal{F}_j(k,n)\right]$.

Throughout the sequel, all analysis will be carried out under Assumption 1 without explanations. Let the unit square $\mathcal{S}$ be divided into M_n equally small squares, labeled from left to right and from top to bottom as $S_j, j=1,\cdots,M_n$ with $M_n=\lfloor\frac{1}{a_n}\rfloor^2$, where a_n satisfying $\sqrt{\log n/n}=o(a_n)$ and $a_n=o(1)$. Denote the number of agents in $S_k(1\le k\le M_n)$ as N_k. Then, the following estimation holds almost surely for large n,

$$\max_{1\le k\le M_n} N_k = na_n^2(1+o(1)). \tag{10}$$

This result can be established by directly applying the Borel-Cantelli Lemma to Lemma 4 of Tang & Guo (2007), it can also be proved simply by using Lemma 7.

Based on (10), it is readily found that the asymptotic properties of the characteristics established by Tang & Guo (2007) for the random geometric graph G_0 actually hold almost surely, which can be summarized in the following lemma.

Lemma 8 *For the initial random geometric graph G_0, the following results hold almost surely for all large n :*

1) The maximum and minimum degrees satisfy

$$\beta^{-1}n(1+o(1))\le d_{min}(0)\le d_{max}(0)\le n,$$

where β can be taken as

$$\beta=\max\left(64/\pi, 4/\pi r^2\right). \tag{11}$$

2) The maximum number of agents in (8) satisfies

$$R_{max}\le 4n\pi\eta r^2(1+o(1)). \tag{12}$$

3) The spectral gap satisfies

$$\bar\lambda(0)\le 1-\frac{\pi r^2}{512(r+\sqrt{6})^4}(1+o(1)).$$

By using the above multi-array martingale estimation theorem, we can get the following estimation of the initial headings:

Lemma 9 *For large n, we have*

$$1)\ \max_{1\le i\le n}\left|\sum_{j\in\mathcal{N}_i(0)}\sin\theta_j(0)\right|\le C_1 b_n,\quad a.s.$$

$$2)\ \max_{1\le i\le n}\left|\sum_{j\in\mathcal{N}_i(0)}(\cos\theta_j(0)-C_2)\right|\le C_3 b_n,\quad a.s.$$

where $b_n=\sqrt{n\log n}$, and C_1, C_2, C_3 are constants whose values can be found in the proof to be given in Appendix B.

By the above lemma, we can obtain the estimation of the averaged headings as follows:

Corollary 10 *For $\theta_i(t)$ defined by (3), we have for large n*

$$1)\ \max_{1\le i\le n}|\tan\theta_i(1)|\le C_4\frac{b_n}{n},\quad a.s.;$$

$$2)\ \max_{1\le i\le n}|\cos\theta_i(t)-1|\le C_5\frac{b_n}{n},\quad a.s.,\quad \forall t\ge 1,$$

where C_4 and C_5 are constants whose values can be found in the proof to be given in Appendix C.

Remark 11 *From the above corollary, we see that for large n, $\theta_i(1)\in(-\pi/2,\pi/2),\forall i$ almost surely. We remark that some biological systems do exhibit rapid transition from disordered to highly synchronized behavior when the population density is suitably high, see, e.g., Makris et al. (2009). However, $\theta_i(1)\in(-\pi/2,\pi/2),\forall i$ does not mean that the system can reach synchronization eventually. In fact, the required proof appears to be rather complicated, as will be seen in the paper.*

To analyze the changes of the dynamical graphs associated with the multi-agent systems, we need to know the "perturbation" properties of the average matrix, which are given in the following lemma:

Lemma 12 *(Tang & Guo, 2007). Let G be an undirected graph , and let $\widehat{G}$ be another undirected graph formed by changing the neighbors of G. If for any node k, the number of its neighbors changed satisfies $R_k\le R_{\max}<d_{\min}$, then the corresponding average matrices P^0 and $\widehat{P}^0$ satisfy*

$$\|P^0-\widehat{P}^0\|\le\frac{R_{\max}}{d_{\min}}\left(\frac{d_{\max}+d_{\min}}{d_{\min}-R_{\max}}\right).$$

All the above auxiliary results will be used in the proof of the next section.

4 Proof of Theorem 2

The main purpose of this section is to prove Theorem 2 for the multi-agent system defined by (1)-(3) with the nonlinear heading equation and with any given and fixed model parameters v and r. The proof is proceeded with two lemmas.

Lemma 13 *The L_2-norm of the difference between the weighted average matrix $P(t)$ and the average matrix $P^0(t)$, defined respectively by (6) and (7), denoted by*

$$\widetilde{P}(t)=P(t)-P^0(t)\triangleq(\widetilde{p}_{ij}(t)), \tag{13}$$

is bounded by

$$\|\widetilde{P}(t)\|\le\frac{2C_5\sqrt{n\log n}}{d_{\min}(t)}(1+o(1)),\quad t\ge 1, \tag{14}$$

where C_5 is defined as in Corollary 10, and $d_{\min}(t)$ is the minimum degree of the graph G_t associated with the dynamical system.

Proof: By (6) and (7), the entries $\widetilde{p}_{ij}(t)$ of the matrix $\widetilde{P}(t)$ satisfy

$$\widetilde{p}_{ij}(t) = \begin{cases} \frac{\cos\theta_j(t)}{\sum_{k\in\mathcal{N}_i(t)}\cos\theta_k(t)} - \frac{1}{d_i(t)}, & \text{if } (i,j)\in E_t \\ 0, & \text{otherwise} \end{cases}, \quad (15)$$

where $d_i(t)$ is the degree of the agent i at time t. For any i and j such that $(i,j)\in E_t$, by 2) of Corollary 10, we have

$$|\widetilde{p}_{ij}(t)| = \Big|\frac{\sum_{k\in\mathcal{N}_i(t)}(\cos\theta_j(t)-\cos\theta_k(t))}{d_i(t)\sum_{k\in\mathcal{N}_i(t)}\cos\theta_k(t)}\Big|$$
$$\le \frac{2C_5 d_i(t)\frac{b_n}{n}}{d_i^2(t)\left(1-C_5\frac{b_n}{n}\right)} \le \frac{2C_5}{d_i(t)}\sqrt{\frac{\log n}{n}}(1+o(1)), \qquad a.s.$$

where that fact $b_n/n = (\log n/n)^{1/2}$ is used in the above inequality. Hence, we have for $t\ge 1$,

$$\|\widetilde{P}(t)\| \le n\max_{i,j}|\widetilde{p}_{ij}(t)| \le \frac{2C_5\sqrt{n\log n}}{d_{\min}(t)}(1+o(1)).$$

This completes the proof of the lemma. □

Next, we further estimate the upper bound for $\max_{1\le s\le t}\|\widetilde{P}(s)\|$, or equivalently, the lower bound to $\min_{1\le s\le t} d_{min}(s)$ by the above lemma. Obviously, this depends essentially on the estimation for the distance $d_{ij}(t)$ between any two agents.

Lemma 14 *For the multi-agent system defined by (1)-(3), the dynamical distances $d_{ij}(t)$ between any agents and the difference matrix $\widetilde{P}(t)$ defined in Lemma 13 satisfy the following properties almost surely for large n:*

1) For any agents i and j, their distance satisfies

$$|d_{ij}(t)-d_{ij}(0)| \le \eta r, \quad \forall t\ge 0, \quad (16)$$

where η is defined as in (9).

2) For any $t\ge 0$, the difference matrix $\widetilde{P}(t)$ satisfies:

$$\varepsilon_1(t) \le \frac{\pi r^2}{3\cdot 512\sqrt{\beta}(r+\sqrt{6})^4}, \quad (17)$$

where β is the constant defined by (11), and

$$\varepsilon_1(t) \triangleq \sup_{1\le s\le t} \|\,\widetilde{P}(s)\,\|, \quad \varepsilon_1(0)=0. \quad (18)$$

Proof. It is not difficult to see that the two inequalities (16) and (17) are actually coupled in the sense that proving one needs the other. Hence, in the following, we will prove that (16) and (17) hold simultaneously by induction. First, for the case where $t=0$, these two inequalities hold obviously.

Now, let us assume that for some $t\ge 0$, (16) and (17) hold for all $0\le s\le t$, *i.e.*, for any i and j, we have

$$|d_{ij}(s)-d_{ij}(0)| \le \eta r, \quad a.s., \quad (19)$$

and

$$\varepsilon_1(s) \le \frac{\pi r^2}{3\cdot 512\sqrt{\beta}(r+\sqrt{6})^4}, \quad (20)$$

We will prove that both (16) and (17) hold for $s=t+1$.

By the position update law (2), we can deduce that

$$|d_{ij}(t+1)-d_{ij}(t)| \le v\delta(t+1), \ \forall\, t\ge 0, \quad (21)$$

where

$$\delta(t) = \max_{1\le i,j\le n}\{\tan\theta_i(t)-\tan\theta_j(t)\}. \quad (22)$$

By (21), we have

$$|d_{ij}(t+1)-d_{ij}(0)| \le \sum_{s=1}^{t+1}|d_{ij}(s)-d_{ij}(s-1)|$$
$$\le v\sum_{s=1}^{t+1}\delta(s). \quad (23)$$

Note that the "linear" time-varying equation for $\tan\theta(t)$ defined by (5) has essentially the same form as that for $\theta(t)$ in Lemma 2 of Tang & Guo (2007). So, we have for $0\le s\le t$,

$$\delta(s+1)$$
$$\le \sqrt{2\beta}\left(\bar{\lambda}(0)+\sqrt{\beta}T_s\right)^s \,\|\tan\theta(1)\,\|, \quad (24)$$

where $\bar{\lambda}(0)$ is the spectral gap of the initial graph, and

$$T_s \triangleq \sup_{1\le k\le s}\|\,P(k)-P^0(0)\,\| \quad (25)$$
$$\le \sup_{1\le k\le s}\|\,P(k)-P^0(k)\,\| + \sup_{1\le k\le s}\|\,P^0(k)-P^0(0)\,\|.$$

By (18), the first term of the right-hand side is exactly $\varepsilon_1(s)$. We now proceed to estimate the second term $\sup_{1\le k\le s}\|P^0(k)-P^0(0)\|$.

By (19), one can see that the number of neighbors changed at any time $s(1\le s\le t)$ in comparison with that at time $t=0$ is bounded by $R_{\max}$ defined for (8). Furthermore, note that the graph G_t is undirected for all $t\ge 0$. Thus, by Lemmas 8 and 12 and the value of η given in (9), we have

$$\|P^0(s)-P^0(0)\| \le \frac{R_{\max}}{d_{\min}(0)}\left(\frac{d_{\max}(0)+d_{\min}(0)}{d_{\min}(0)-R_{\max}}\right)$$
$$\le \frac{4n\beta\pi\eta r^2}{n}\cdot\frac{2n}{\beta^{-1}n-4n\pi\eta r^2}(1+o(1))$$
$$\le 16\pi\beta^2 r^2\eta(1+o(1))$$
$$\le \frac{\pi r^2}{3\cdot 512\sqrt{\beta}(r+\sqrt{6})^4} \triangleq \varepsilon_2, \quad a.s., 1\le s\le t.$$

Furthermore, by this, (11), (20), (25) and Lemma 8, we can deduce that

$$\begin{aligned}&\overline{\lambda}(0)+\sqrt{\beta}T_s\\ &\le \overline{\lambda}(0)+\sqrt{\beta}\left(\varepsilon_1(s)+\varepsilon_2\right)\\ &\le 1-\frac{\pi r^2}{3\cdot 512(r+\sqrt{6})^4}(1+o(1)), \quad a.s. \end{aligned} \tag{26}$$

Substituting (26) into (24), we have for $s \le t$

$$\delta(s+1) \le \sqrt{2\beta}(\lambda')^s \|\tan\theta(1)\|, \tag{27}$$

where $\lambda' = 1 - \frac{\pi r^2}{3\cdot 512(r+\sqrt{6})^4}(1+o(1))$.

Note that by 2) of Corollary 10, we have for large n, $\theta_i(s) \in (-\pi/2, \pi/2)$ for any $s \ge 1$ and i. So, by (4), it is not difficult to see that $\{\delta(s), s \ge 1\}$ is a non-increasing sequence .

Now, we are in a position to continue the estimation of (23). Set

$$s_0 \triangleq \min\{s : \sqrt{2\beta}(\lambda')^{s-1} \parallel \tan\theta(1) \parallel \le \delta(1)\},$$

then it is not difficult to see that

$$s_0 \le \frac{-M}{\log\lambda'}+2, \ \text{ with } M = -\log\frac{\delta(1)}{\sqrt{2\beta}\parallel \tan\theta(1)\parallel}, \tag{28}$$

where $M > 0$ because $\delta(1) \le \sqrt{2\beta}\|\tan\theta(1)\|$.

Following the proof idea of Proposition 1 in Tang & Guo (2007), by (27) and (28), we have

$$\begin{aligned}&v\sum_{s=1}^{t+1}\delta(s) = v\left(\sum_{s=1}^{s_0-1}\delta(s)+\sum_{s=s_0}^{t+1}\delta(s)\right)\\ &\le v(s_0-1)\delta(1)+v\sqrt{2\beta}(\lambda')^{s_0-1}\parallel\tan\theta(1)\parallel\sum_{s=s_0}^{t+1}(\lambda')^{s-s_0}\\ &\le v\delta(1)\left(\frac{-M}{\log\lambda'}+1+\frac{1}{1-\lambda'}\right)\\ &\le \frac{v\delta(1)}{1-\lambda'}\left(2+\frac{1-\lambda'}{-\log\lambda'}M\right) \le \frac{v\delta(1)}{1-\lambda'}\left(2+M\right),\end{aligned} \tag{29}$$

where for the last inequality, we have used the fact that $\log x \le x-1 < 0,\ \forall\, 0 < x < 1$. Furthermore, to estimate $\delta(1)M$, by Corollary 10 we have for large n,

$$\delta(1) \le 2\max_i|\tan\theta_{1\le i\le n}(1)| \le 2C_4\frac{b_n}{n},\ a.s.; \tag{30}$$

$$\parallel\tan\theta(1)\parallel \le \sqrt{n}\max_{1\le i\le n}|\tan\theta_i(1)| \le C_4\frac{b_n}{\sqrt{n}},\ a.s. \tag{31}$$

Moreover, since $-x\log x$ is increasing for $0 < x < \frac{1}{e}$, by (29), (30) and (31), we have for large n

$$\begin{aligned}&v\sum_{s=1}^{t+1}\delta(s)\\ &\le \frac{2vC_4\frac{b_n}{n}}{1-\lambda'}\left(2+\log\frac{\sqrt{2\max\left(\frac{64}{\pi},\frac{4}{\pi r^2}\right)}\sqrt{n}C_4\frac{b_n}{n}}{2C_4\frac{b_n}{n}}\right)\\ &= O\left(\sqrt{\frac{\log n}{n}}\log n\right) \le \eta r, \quad a.s.,\end{aligned} \tag{32}$$

where the last inequality holds for large n. Thus by (23) and (32), we see that (16) holds for $s = t+1$.

Finally, we prove that (17) holds at time $t+1$. Since (16) holds at time $t+1$, we see that the number of each agent's neighbors changed at time $t+1$ in comparison with its neighbors at the initial time does not exceed R_{max} defined for (8). Thus, by Lemma 8 and the value of η given in (9), we see that

$$\begin{aligned}&d_{\min}(t+1) \ge d_{\min}(0) - R_{\max}\\ &\ge \frac{1}{2}\min\left(\frac{\pi r^2}{4},\frac{\pi}{64}\right)n(1+o(1))\end{aligned} \tag{33}$$

holds almost surely for large n. By substituting (33) into (14), we obtain

$$\|\widetilde{P}(t+1)\| \le \frac{2C_5\sqrt{n\log n}(1+o(1))}{d_{\min}(t+1)} = O\left(\sqrt{\frac{\log n}{n}}\right),$$

which implies that for large n we must have

$$\|\widetilde{P}(t+1)\| \le \frac{\pi r^2}{3\cdot 512\sqrt{\beta}(r+\sqrt{6})^4} \tag{34}$$

holds almost surely. Combining this with the induction assumption (20), we see that (17) holds at time $t+1$. Therefore, by induction, (16) and (17) hold almost surely for all $t \ge 0$. This completes the proof of the lemma. □

Proof of Theorem 2.

From the proof of Lemma 14, we see that (27) holds for all $t \ge 0$, i.e.,

$$\delta(t+1) \le \sqrt{2\beta}(\lambda')^t\|\tan\theta(1)\|, \tag{35}$$

where $\lambda' = 1 - \frac{\pi r^2}{3\cdot 512(r+\sqrt{6})^4}(1+o(1))$. Consequently, $\delta(t) \to 0$ exponentially fast. Moreover, by 2) of Corollary 10, we know that for large n, $\theta_i(t) \in (-\pi/2, \pi/2)$ for any $t \ge 1$ and i. So, by (4) we know that $\max_{1\le i\le n}\tan\theta_i(t)$(resp., $\min_{1\le i\le n}\tan\theta_i(t)$) is a non-increasing (resp., non-decr easing) bounded sequence, and thus has finite limit. By this and (35), we further have

$$\lim_{t\to\infty}\max_{1\le i\le n}\tan\theta_i(t) = \lim_{t\to\infty}\min_{1\le i\le n}\tan\theta_i(t).$$

Hence the system (1)-(3) will synchronize. This completes the proof of Theorem 2. □

Remark 15 *In comparison with the work of Tang & Guo (2007), we should note that the upper bound of $\varepsilon_1(t)$ defined by (18) is automatically zero for the "linearized" model studied there, whereas estimating this upper bound in Lemmas 13 and 14 is a key step here. Moreover, the synchronization property of Tang & Guo (2007) was proven by verifying the connectivity of the associated dynamical neighbor graphs G_t, whereas in our Theorem 2, the synchronization was proven by a simpler way without establishing the connectivity of G_t. However, this property does indeed hold, which can be verified along the lines of Tang & Guo (2007).*

5 Proof of Theorem 5

In this section, we will give the proof of Theorem 5. Similar to the proof of Theorem 2, we need to estimate some characteristics of the initial graph G_0 and the heading property at $t = 1$ for the case where v and r depend on n, denoted by v_n and r_n to reflect this situation. For this case, (1) defining the neighbor set of the agent i is changed to the following:

$$\mathcal{N}_i(t) = \{j \,|\, d_{ij}(t) < r_n\}. \tag{36}$$

Accordingly, the set $\mathcal{R}_j$ is changed to the following one:

$$\mathcal{R}_j = \{i : (1-\eta_n) r_n \le d_{ij}(0) \le (1+\eta_n) r_n\}. \tag{37}$$

where η_n and r_n are taken as follows:

$$\eta_n = \frac{r_n^2}{288 \times 320}, \quad \sqrt[6]{\frac{\log n}{n}} = o(r_n),\ r_n = o(1). \tag{38}$$

To prove Theorem 5, we need to consider the following new sets:

$$\mathcal{R}'_j = \{i : (2-\eta_n) r_n \le d_{ij}(0) \le (2+\eta_n) r_n\}, \tag{39}$$
$$\mathcal{N}'_j(t) = \{i : d_{ij}(t) < 2r_n\}, \quad t \ge 0. \tag{40}$$

Denote R_j, R'_j and $n'_j(t)$ as the cardinality of $\mathcal{R}_j, \mathcal{R}'_j$ and $N'_j(t)$ respectively, and $R_{max} \triangleq \max_j R_j$, $R'_{\max} \triangleq \max_j R'_j$.

For the initial graph G_0, the following lemma is given by Tang & Guo (2006), whose proof is similar to those with fixed r (see, Tang & Guo, 2007).

Lemma 16 *For the initial random geometric graph G_0, the following results hold almost surely for large n:*

1) The maximum and minimum degrees satisfy

$$d_{max}(0) = n\pi r_n^2 (1+o(1)),$$
$$d_{\min}(0) = \frac{1}{4} n\pi r_n^2 (1+o(1)). \tag{41}$$

2) The maximum number of agents in (37) satisfies

$$R_{max} = 4n\pi\eta_n r_n^2 (1+o(1)). \tag{42}$$

3) The spectral gap satisfies

$$\overline{\lambda}(0) \le 1 - \frac{r_n^2}{144}(1+o(1)). \tag{43}$$

By the similar methods as those in Lemma 16, we have the following results:

Lemma 17 *The number of agents in the set $\mathcal{N}'_i(0)$ defined by (40) satisfies*

$$\max_{1\le i\le n} n'_i(0) = 4n\pi r_n^2 (1+o(1)), \quad a.s.$$
$$\min_{1\le i\le n} n'_i(0) = n\pi r_n^2 (1+o(1)), \quad a.s. \tag{44}$$

Furthermore, the maximum number of agents in (39) satisfies

$$R'_{\max} = 8n\pi\eta_n r_n^2 (1+o(1)), \quad a.s. \tag{45}$$

Similar to the proof of Lemma 9, we can obtain the following results:

Lemma 18 *For large n, we have*

$$1)\ \max_{1\le i\le n} \left| \sum_{j\in\mathcal{N}_i(0)} \sin\theta_j(0) \right| \le C_1 f_n, \quad a.s.$$

$$2)\ \max_{1\le i\le n} \left| \sum_{j\in\mathcal{N}_i(0)} \Big(\cos\theta_j(0) - C_2\Big) \right| \le C_3 f_n, \quad a.s.$$

where $f_n = \sqrt{n\pi r_n^2 \log n}(1+o(1))$, and C_1, C_2 and C_3 take the same values as those in Lemma 9.

By the above lemma, we can derive the following corollary, whose proof is similar to that of Corollary 10.

Corollary 19 *For large n, we have*

$$1)\ \max_{1\le i\le n} |\tan\theta_i(1)| \le C_6 \frac{f_n}{nr_n^2}, \quad a.s.$$
$$2)\ \max_{1\le i\le n} |\cos\theta_i(t) - 1| \le C_7 \frac{f_n}{nr_n^2}, \quad a.s., \forall t \ge 1,$$

where C_6 and C_7 are constants independent of n.

For the case where v_n and r_n depend on n, we have the following lemma:

Lemma 20 *For $t \ge 1$, the L_2-norm of the difference between the weighted average matrix $P(t)$ and the average matrix $P^0(t)$, defined by (13), is bounded by*

$$\|\widetilde{P}(t)\|$$
$$\le \frac{2C_7(1+o(1))}{d_{min}(t)} \sqrt{\frac{\pi \log n d_{max}(t) \max_{1\le i\le n} n'_i(t)}{nr_n^2}}. \tag{46}$$

Proof. Since $\widetilde{P}(t)$ has a similar expression as in the previous case, we know that for any i and j such that $(i,j) \in E_t$,

(38) and 2) of Corollary 19 will give

$$|\widetilde{p}_{ij}(t)| = \left|\frac{\sum_{k\in\mathcal{N}_i(t)}(\cos\theta_j(t)-\cos\theta_k(t))}{d_i(t)\sum_{k\in\mathcal{N}_i(t)}\cos\theta_k(t)}\right|$$
$$\le \frac{2C_7 d_i(t)\frac{f_n}{nr_n^2}}{d_i^2(t)\left(1-C_7\frac{f_n}{nr_n^2}\right)}$$
$$= \frac{2C_7}{d_i(t)}\sqrt{\frac{\pi\log n}{nr_n^2}}(1+o(1)), \quad a.s., \tag{47}$$

where $d_i(t)$ is the degree of the agent i at time t.

To estimate $\|\widetilde{P}(t)\|$, we need to introduce the following sets:

$$\mathcal{N}_{ij}(t) = \{l : d_{il}(t) < r_n,\ d_{jl}(t) < r_n\}$$

and $\mathcal{Z}_i(t) = \{j : \mathcal{N}_{ij}(t) \ne \varnothing\}$. Moreover, denote $n_{ij}(t)$ and $z_i(t)$ as the cardinality of the sets $\mathcal{N}_{ij}(t)$ and $\mathcal{Z}_i(t)$ respectively. Obviously, we have

$$\max_{1\le i,j\le n} n_{ij}(t) \le \min\{n_i(t), n_j(t)\} \le d_{max}(t); \tag{48}$$
$$\max_{1\le i\le n} z_i(t) = \max_{1\le i\le n} n_i'(t), \tag{49}$$

where $n_i'(t)$ is the cardinality of $\mathcal{N}_i'(t)$ defined by (40). By Gěrschgorin Disk Theorem (cf., Horn & Johnson (1985)), we have

$$\|\widetilde{P}(t)\| = \max_{1\le i\le n}\sqrt{\lambda_i(\widetilde{P}(t)\widetilde{P}^\tau(t))}$$
$$\le \left(\max_{1\le i\le n}\sum_{j=1}^{n}\sum_{l=1}^{n}|\widetilde{p}_{il}(t)\widetilde{p}_{jl}(t)|\right)^{1/2}$$
$$\le \left(\max_{1\le i\le n}\sum_{j\in Z_i(t)}\sum_{l\in\mathcal{N}_{ij}(t)}|\widetilde{p}_{il}(t)\widetilde{p}_{jl}(t)|\right)^{1/2}$$
$$\le \sqrt{\max_{1\le i,j\le n} n_{ij}(t)\max_{1\le i\le n} z_i(t)\max_{ij:(i,j)\in E_t}|\widetilde{p}_{ij}(t)|^2}. \tag{50}$$

By substituting (47), (48) and (49) into (50), we have

$$\|\widetilde{P}(t)\| \le \frac{2C_7(1+o(1))}{d_{min}(t)}\sqrt{\frac{\pi\log n d_{max}(t)\max_{1\le i\le n} n_i'(t)}{nr_n^2}}.$$

This completes the proof of the lemma. □

By the above analysis, the asymptotic properties of the initial random geometric graph G_0 and the heading properties at time $t=1$, we can obtain the following lemma, whose proof is similar to that of Lemma 14.

Lemma 21 *For the multi-agent system (1)-(3), let r_n satisfy (38), and let the velocity v_n satisfy*

$$v_n \le \frac{\sqrt{n}r_n^6}{2\sqrt{\pi}\cdot 144^2\cdot 640\cdot C_6\log^{3/2} n}.$$

Then, the dynamical distance $d_{ij}(t)$ between any agents and the difference matrix $\widetilde{P}(t)$ defined in Lemma 20 satisfy the following properties almost surely for large n:

1) For any agents i and j, their distance satisfies

$$|d_{ij}(t) - d_{ij}(0)| \le \eta_n r_n(1+o(1)), \quad \forall t\ge 0,$$

with η_n defined by (38).

2) For any $t\ge 0$, the difference matrix $\widetilde{P}(t)$ satisfies

$$\varepsilon_1(t) = o(r_n^2),$$

where $\varepsilon_1(t)$ is defined as follows:

$$\varepsilon_1(t) \triangleq \sup_{1\le s\le t}\|\widetilde{P}(s)\|, \quad \varepsilon_1(0) = 0.$$

Following the proof ideas of Theorem 2 and using the above lemmas, one can establish Theorem 5 in a similar way.

6 Simulation result

In this section, we demonstrate the simulation result. Here, the initial positions and headings of all agents satisfy Assumption 1 with $\varepsilon_0 = \pi/50$, the speed of the agents and the neighborhood radius are taken as $v = 0.03$ and $r = 0.5$ respectively. Fig. 2 shows how the probability of synchronization changes with the number of agents. From this simulation, we see that the system will synchronize almost surely, when the number of agents is equal to or greater than 55.

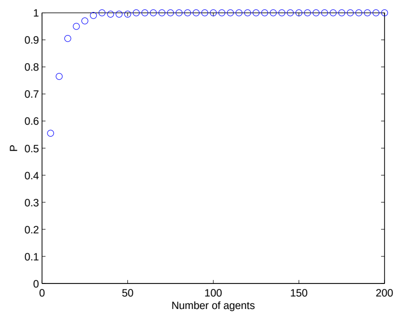

Fig. 2. Simulation result for the system with $v = 0.03$, $r = 0.5$ and $\varepsilon_0 = \pi/50$.

7 Concluding remarks

A key issue in the investigation of multi-agent systems is to understand how locally interacting agents (or subsystems) lead to the collective behavior of the overall systems. In this paper, we provided a theoretical analysis for a basic class of multi-agent systems with local interactions in a random framework with large population, without imposing any *a priori* connectivity conditions on the dynamical systems to be studied. This solves a challenging problem in this direction. We remark that similar methods may also be used to study other related problems and systems, for example, three dimensional systems, the leader-follower problems, as well as other problems with more complicated system structures.

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Appendix A. Proof of Lemma 7.

Following Huang & Guo (1990), we consider

$$S_i(k,n) = \sum_{j=1}^{i} f_j(k,n) w_{j+1}(k,n), \quad S_0(k,n) = 0,$$
$$T_i(k,n) = \mathrm{e}^{\left\{S_i(k,n) - \frac{3}{4}\sum_{j=1}^{i} E\left[w_{j+1}^2(k,n)|\mathcal{F}_j(k,n)\right] f_j^2(k,n)\right\}},$$

where $1 \le i, k \le n$.

Note that $\{T_i(k,n), \mathcal{F}_i(k,n), 1 \le i \le n\}$ is a super-martingale (cf. Stout,1974), so we have

$$P\left\{\max_{1\le k\le n} \max_{1\le i\le n} \log T_i(k,n) > 3\log n\right\}$$
$$\le \sum_{k=1}^{n} P\left\{\max_{1\le i\le n} T_i(k,n) > \exp\{3\log n\}\right\}$$
$$\le \sum_{k=1}^{n} n^{-3} = O(n^{-2}),$$

where Corollary 5.4.1 of Stout (1974) is used in the last inequality. Hence, by Borel-Cantelli Lemma , we have for large n

$$\max_{1\le k\le n} \max_{1\le i\le n} \log T_i(k,n) \le 3\log n, \quad a.s.$$

From this, it is not difficult to see that

$$\max_{1\le k\le n} \max_{1\le i\le n} S_i(k,n)$$
$$\le \max_{1\le k\le n} \frac{3}{4} C_w(n) \sum_{j=1}^{n} f_j^2(k,n) + 3\log n, \; a.s. \quad \text{(A.1)}$$

The similar result holds also for $\{-S_i(k,n)\}$. Therefore, the assertion of Lemma 7 holds almost surely for large n. □

Appendix B. Proof of Lemma 9.

1) For $1 \le i \le n$, set

$$\mathcal{F}_0(n) = \{\varnothing, \Omega\},$$
$$\mathcal{F}_i(n) = \sigma\left\{\theta_j(0), (x_q(0), y_q(0)), j \le i, 1 \le q \le n\right\}.$$

Clearly, for any n and $1 \le j \le n$, $I(j \in \mathcal{N}_i(0))$ is $\mathcal{F}_i(n)$-measurable for any i, where $I(\cdot)$ is the indicator function. Under Assumption 1, we have for any $1 \le j \le n$,

$$E[\sin\theta_j(0)|\mathcal{F}_{j-1}(n)] = E\sin\theta_j(0) = 0,$$
$$E[\sin\theta_j^2(0)|\mathcal{F}_{j-1}(n)] = E\sin\theta_j^2(0)$$
$$= \frac{1}{2} + \frac{\sin 2\varepsilon_0}{4(\pi - \varepsilon_0)} \triangleq C_0.$$

So for any positive integers n, $\{\sin\theta_j(0), \mathcal{F}_j(n), 1 \le j \le n\}$ is a martingale difference sequence with constant conditional variance. By Lemma 8, we have

$$\max_{1\le i\le n} \sum_{j=1}^{n} I(j \in \mathcal{N}_i(0)) = d_{max}(0) \le n. \quad \text{(B.1)}$$

Set $c_n = 2\sqrt{\frac{\log n}{C_0 n}}$, it is easy to see that $|c_n| \le 1$. Thus, by Lemma 7, we have for large n

$$\max_{1\le i\le n} \left|\sum_{j\in\mathcal{N}_i(0)} \sin\theta_j(0)\right|$$
$$= \frac{1}{c_n} \max_{1\le i\le n} \left|\sum_{j=1}^{n} c_n I(j \in \mathcal{N}_i(0)) \sin\theta_j(0)\right|$$
$$\le \frac{1}{c_n}\left\{\frac{3C_0 c_n^2 d_{\max(0)}}{4} + 3\log n\right\}$$
$$= 3\sqrt{\left(\frac{1}{2} + \frac{\sin 2\varepsilon_0}{4(\pi - \varepsilon_0)}\right) n\log n}, \quad a.s. \quad \text{(B.2)}$$

2) By Assumption 1, we have for any n

$$E\{\cos\theta_j(0)|\mathcal{F}_{j-1}(n)\} = \frac{\sin\varepsilon_0}{\pi - \varepsilon_0} \triangleq C_2, \quad 1 \le j \le n,$$
$$E\left\{\left(\cos\theta_j(0) - C_2\right)^2 \Big| \mathcal{F}_{j-1}(n)\right\}$$
$$= \frac{1}{2} - \frac{\sin 2\varepsilon_0}{4(\pi - \varepsilon_0)} - \left(\frac{\sin\varepsilon_0}{\pi - \varepsilon_0}\right)^2 \quad 1 \le j \le n.$$

So, for any n, $\{\cos\theta_j(0) - C_2, \mathcal{F}_j(n), 1 \le j \le n\}$ is also a martingale difference sequence with constant conditional variance. Thus, following the proof idea of (B.2), by (B.1) we have for large n

$$\max_{1\le i\le n} \left|\sum_{j\in\mathcal{N}_i(0)} (\cos\theta_j(0) - C_2)\right|$$
$$\le 3\left\{\left[\frac{1}{2} - \frac{\sin 2\varepsilon_0}{4(\pi - \varepsilon_0)} - C_2^2\right] n\log n\right\}^{1/2}, \quad a.s.$$

This completes the proof of the lemma. □

Appendix C. Proof of Corollary 10.

1) By Lemma 8, we have for $1 \le i \le n$,

$$\frac{b_n}{d_i(0)} = O\left(\frac{b_n}{n}\right) = O\left(\sqrt{\frac{\log n}{n}}\right) = o(1), \; a.s. \quad \text{(C.1)}$$

Therefore, by (3), (C.1) and Lemma 9, we have

$$\max_{1\le i\le n} |\tan\theta_i(1)| = \max_{1\le i\le n} \frac{|\sum_{j\in\mathcal{N}_i(0)} \sin\theta_j(0)|}{|\sum_{j\in\mathcal{N}_i(0)} \cos\theta_j(0)|}$$
$$\le \max_{1\le i\le n} \frac{C_1 b_n}{C_2 d_i(0) - C_3 b_n} = \frac{C_1 b_n}{C_2 d_{\min}(0) - C_3 b_n}$$
$$= \frac{C_1}{C_2}\frac{b_n}{d_{\min}(0)}(1+o(1)) = O\left(\frac{b_n}{n}\right), \quad a.s.$$

This completes the first inequality of the corollary.

2) First, we consider the asymptotic property of $\max_{1\le i\le n}(1-\cos\theta_i(1))$. By (3), for any i, we have

$$\cos\theta_i(1) = \frac{\sum_{j\in\mathcal{N}_i(0)} \cos\theta_j(0)}{\left\{\Big(\sum_{j\in\mathcal{N}_i(0)} \sin\theta_j(0)\Big)^2 + \Big(\sum_{j\in\mathcal{N}_i(0)} \cos\theta_j(0)\Big)^2\right\}^{\frac{1}{2}}}. \quad (C.2)$$

So by (C.1), Lemma 9 and the elementary inequality $\sqrt{a^2+b^2} \le a+b,\ a,b\ge 0$, we have

$$1 \ge \cos\theta_i(1) \ge \frac{C_2 d_i(0) - C_3 b_n}{\{(C_2 d_i(0) + C_3 b_n)^2 + (C_1 b_n)^2\}^{1/2}}$$
$$\ge \frac{C_2 d_i(0) - C_3 b_n}{(C_2 d_i(0) + C_3 b_n) + C_1 b_n}$$
$$= \frac{C_2 - C_3 g_{in}}{C_2 + (C_3 + C_1) g_{in}}, \quad a.s., \quad (C.3)$$

where $g_{in} = \frac{b_n}{d_i(0)}$. Furthermore, by (C.1), (C.3) and Lemma 8, we can obtain

$$\max_{1\le i\le n} |\cos\theta_i(1) - 1| \le \max_{1\le i\le n} \frac{(C_1 + 2C_3) g_{in}}{C_2 + (C_1 + C_3) g_{in}}$$
$$= O\left(\frac{b_n}{d_{min}(0)}\right) = O\left(\sqrt{\frac{\log n}{n}}\right) = o(1), \quad a.s. \quad (C.4)$$

Moreover, by this fact and 1) of Corollary 10, it is easy to see that for large n, $\theta_i(1) \in (-\pi/2, \pi/2)$, $\forall i$. Hence, by the heading update equation (3), we know that $\min_{1\le i\le n} \cos\theta_i(t)$ is non-decreasing for $t \ge 1$, so we have

$$\max_{1\le i\le n}(1 - \cos\theta_i(t)) \le \max_{1\le i\le n}(1 - \cos\theta_i(1)), \quad \forall t \ge 1,$$

which in conjunction with (C.4) yields the desired result 2). This completes the proof of Corollary 10. □

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