A New Critical Theorem for Adaptive Nonlinear Stabilization ⋆

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Abstract

It is fairly well known that there are fundamental differences between adaptive control of continuous-time and discrete-time nonlinear systems. In fact, even for the seemingly simple single-input single-output control system $y_{t+1} = \theta_1 f(y_t) + u_t + w_{t+1}$ with a scalar unknown parameter θ_1 and noise disturbance $\{w_t\}$, and with a known function $f(\cdot)$ having possible nonlinear growth rate characterized by $|f(x)| = \Theta(|x|^b)$ with $b \geq 1$, the **necessary and sufficient** condition for the system to be globally stabilizable by adaptive feedback is $b < 4$. This was first found and proved by [4] for the Gaussian white noise case, and then proved by [8] for the bounded noise case. Subsequently, a number of other type of "critical values" and "impossibility theorems" on the maximum capability of adaptive feedback were also found, mainly for systems with known control parameter as in the above model. In this paper, we will study the above basic model again but with additional unknown control parameter θ_2, i.e., u_t is replaced by $\theta_2 u_t$ in the above model. Interestingly, it turns out that the system is globally stabilizable **if and only if** $b < 3$. This is a new critical theorem for adaptive nonlinear stabilization, which has meaningful implications for the control of more general uncertain nonlinear systems.

Key words: discrete-time; nonlinear systems; globally stabilizability; stochastic imbedding

1 Introduction

It is well known that a fairly complete theory exists for adaptive control of linear systems in both continuous-time and discrete-time cases (cf. e.g., [1]-[5]). Extensions of the existing results on linear systems to nonlinear systems with nonlinearity having linear growth rate are also possible (cf. e.g. [16]). However, fundamental differences emerge between adaptive control of continuous-time and discrete-time systems when the nonlinearities are allowed to have a nonlinear growth rate. In fact, in this case, it is still possible to design globally stablizing adaptive controls for a wide class of nonlinear continuous-time systems (cf. [11]), but fundamental difficulties exist for adaptive control of nonlinear discrete-time systems, partly because the high gain or nonlinear damping methods that are so powerful in the continuous-time case are no longer effective in the discrete-time case. Similarly, for sampled-data control of nonlinear uncertain systems, the design of stabilizing sampled-data feedback is possible if the sampling rate is high enough (cf.e.g., [13] and [15]). However, if the sampling rate is a prescribed value, then difficulties again emerge in the design and analysis of globally stabilizing sampled-data feedbacks even for nonlinear systems with the nonlinearity having a linear growth rate. The fact that sampling usually destroys many helpful properties is one of the reasons why most of the existing design methods for nonlinear control remain in the continuous-time even in the nonadaptive case (cf. [12]), albeit many results in continuous-time still have their discrete-time counterparts (cf.e.g., [6]).

Knowing the above difficulties that we encountered in the adaptive control of discrete-time (or sampled-data) nonlinear systems, one may naturally ask the following question: Are the difficulties mainly caused by our incapability in designing or analyzing the adaptive control systems, or by the inherent limitations on the capability of the feedback principle? As pointed out in [19], to answer this fundamental question, we have to place ourselves in a framework that is somewhat beyond those of the classical robust control and adaptive control. We need not only to answer what adaptive control can do, but also to answer the more difficult question what adap-

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tive control cannot do. This means we need to study the maximum capability of the full feedback mechanism which includes all (nonlinear and time-varying) causal mappings from the data space to the control space, and we are not only restricted to a fixed feedback law or to a class of specific feedback laws.

A first step in this direction was made in [4], where the following basic model is considered:

$$y_{t+1} = \theta_1 f(y_t) + u_t + w_{t+1}, \tag{1}$$

where θ_1 is an unknown parameter, $\{w_t\}$ is Gaussian white noise sequence, and where $f(\cdot)$ is a known nonlinear function possibly having a nonlinear growth rate characterized by

$$|f(x)| = \Theta(|x|^b) \quad \text{with} \quad b \geq 1.$$

It was found and proved that the system is a.s. globally adaptively stabilizable if and only if $b < 4$ (see, [4]). This result is also true if the Gaussian noise is replaced by bounded noises (see, [8]). It goes without saying that this critical case on the feedback capability naturally gives an"impossibility result" on the maximum capability of feedback for the case where $b \geq 4$. It is worth pointing out that such "impossibility result" obviously holds also for any (more general) class of uncertain systems, which contains the above basic model class described by (1) as a subclass.

Later on, the above "impossibility result" was extended to systems with multiple unknown parameters and with Gaussian white noise sequence by providing a polynomial rule (see, [17]). Similar results can also be obtained for the case where the uncertain parameters lie in a bounded known region with Gaussian white noises again, but with a more general system structure (see, [19]). More recently, [9] proved that the polynomial rule of [17] does indeed gives a necessary and sufficient condition for global feedback stabilization of a wide class of nonlinear systems with multiple unknown parameters and with bounded noises.

It is worth pointing out that, for nonlinear systems with nonparametric uncertainties, fundamental limitations on the capability of adaptive feedback may still exist even for the case where the nonlinearities have a linear growth rate. For example, for the following first-order nonparametric control system:

$$y_{t+1} = f(y_t) + u_t + w_{t+1}; \quad t \geq 0; \quad y_0 \in \mathcal{R}^1,$$

where the unknown function $f(\cdot)$ belongs to the class of standard Lipschitz functions defined by:

$$\mathcal{F}(L) = \{f(\cdot) : |f(x) - f(y)| \leq L|x - y|, \quad \forall x, y\}$$

and where the noise sequence is bounded. It was found and proved by [19] that the maximum "uncertainty ball" that can be stabilized by adaptive feedback is $\mathcal{F}(L)$ with $L = \frac{3}{2} + \sqrt{2}$. This critical case again gives an "impossibility result" for the case where $f \in \mathcal{F}(L)$ with $L > \frac{3}{2} + \sqrt{2}$. A key observation for this phenomena is that the nonparametric uncertainty essentially depends on infinite number of unknown parameters. Related "impossibility results" are also found for sampled-data adaptive control of nonparametric nonlinear systems in [20].

However, all the results mentioned above assume that the parameter in front of the control law is known. A challenging problem that is important both practically and theoretically is to understand what will happen if the control parameter is also unknown. The main purpose of this paper is to answer this fundamental problem by establishing a new critical theorem for a basic class of uncertain nonlinear systems, which naturally has meaningful implications for either practical applications or for understanding more general uncertain systems.

2 Main Result

In this paper, we consider adaptive control of the following basic uncertain system

$$y_{t+1} = \theta_1 f(y_t) + \theta_2 u_t + w_{t+1}, \tag{2}$$

where $\{u_t\}$ and $\{y_t\}$ are the system input and output processes, both θ_1 and θ_2 are unknown parameters, $\{w_t\}$ is a disturbance process, and $f(\cdot) : \mathbb{R} \to \mathbb{R}$ is a known function. To study the capability of adaptive feedback, we need the following assumptions:

A1) The unknown parameter vector $\theta = (\theta_1, \theta_2)^\tau$ belongs to a bounded domain $[\underline{\theta}_1, \overline{\theta}_1] \times [\underline{\theta}_2, \overline{\theta}_2] \subset \mathbb{R} \times \mathbb{R}$, and the interval for θ_2 does not contain 0.

A2) The noise sequence $\{w_t\}$ belongs to a class of bounded sequences with an unknown bound $w > 0$, i.e.,

$$\sup_{t \geq 1} |w_t| \leq w. \tag{3}$$

A3) The nonlinear function satisfies $|f(x)| = \Theta(|x|^b)$ as $|x| \to \infty$, in the sense that there exist some constants $x' > 0$ and $c_2 > c_1 > 0$ such that

$$c_1 \leq \frac{|f(x)|}{|x|^b} \leq c_2, \quad \forall |x| > x', \tag{4}$$

where $b \geq 1$ is a constant reflecting the rate of nonlinear growth.

We are interested in designing a feedback control law which robustly stabilizes the system (2) with respect to any possible θ and $\{w_t\}$ under the assumptions **A1**)-**A2**).

First, we restate the definition of a feedback control law (cf, [19]).

Definition 2.1 *A sequence $\{u_t\}$ is called a feedback control law if at any time $t \geq 0$, u_t is a (causal) function of all the observations up to the time t: $\{y_i,\ i \leq t\}$, i.e.,*

$$u_t = h_t(y_0, \cdots, y_t) \tag{5}$$

where $h_t(\cdot) : I\!R^{t+1} \to I\!R^1$ can be any Lebesgue measurable (nonlinear) mapping.

We also need a definition of adaptive stabilizability in the sense of bounded input and bounded output.

Definition 2.2 *The system (2) under the assumptions **A1)-A3)** is said to be globally stabilizable, if there exists a feedback control law $\{u_t\}$ such that for any $y_0 \in \mathbb{R}^1$, any θ, any $\{w_t\}$ satisfying **A1)-A2)**, the outputs of the closed-loop system are bounded as follows:*

$$\sup_{t\geq 0} |y_t| < \infty. \tag{6}$$

The main result of this paper is as follows:

Theorem 2.1 *The system (2) under the assumptions **A1)-A3)** is adaptive stabilizable **if and only if** $b < 3$.*

Remark 2.1 *In comparison with the related results established in [4] and [8] as explained in the Introduction, we see that the critical nonlinear growth rate reflecting the maximum capability of the feedback mechanism is reduced from $b = 4$ to the current $b = 3$, due to the additional uncertainty in the input channel.*

3 The Proof of Sufficiency

The proof of sufficiency is constructive. We will design a simple adaptive control law, which robustly stabilizes the system (2) for any $b < 3$.

3.1 The Parameter Estimation

Without loss of generality, we can suppose that $|y_0| > x'$ is large enough. This is because we can let $u_t = 0$, $t = 0, 1, \cdots$ until there exists some $|y_{t'}|$ large enough, and then we can take $y_{t'}$ as y_0. Otherwise, if we can not find such $y_{t'}$, the sufficiency part is proven trivially.

Now, we denote $\vartheta_1 = \dfrac{\theta_1}{\theta_2}, \vartheta_2 = \dfrac{1}{\theta_2}$. Without loss of generality, suppose $\theta_2 > 0$. By **A1)**, it is easy to see that $\vartheta_1 \in [\dfrac{\underline{\theta}_1}{\overline{\theta}_2}, \dfrac{\overline{\theta}_1}{\underline{\theta}_2}]$ and $\vartheta_2 \in [\dfrac{1}{\overline{\theta}_2}, \dfrac{1}{\underline{\theta}_2}]$. For the convenience of later use, we also denote $\underline{\vartheta}_1 = \dfrac{\underline{\theta}_1}{\overline{\theta}_2}, \overline{\vartheta}_1 = \dfrac{\overline{\theta}_1}{\underline{\theta}_2}, \underline{\vartheta}_2 = \dfrac{1}{\overline{\theta}_2}, \overline{\vartheta}_2 = \dfrac{1}{\underline{\theta}_2}$. Obviously, $\underline{\vartheta}_2$ and $\overline{\vartheta}_2$ are both positive numbers.

Let us take $u_0 = 0$ and rewrite the system (2) into the following form:

$$y_{t+1} = \varepsilon_t f(y_t) + w_{t+1}, \tag{7}$$

where by definition

$$\varepsilon_t = \theta_1 - \theta_2\beta_t = \theta_2(\vartheta_1 - \beta_t) \quad \text{and} \quad \beta_t = -\frac{u_t}{f(y_t)}.$$

Now for any $t \geq 2$, let $m_t := \operatorname*{argmax}_{0\leq i\leq t-1} |f(y_i)|$, then define

$$i_t := \begin{cases} \operatorname*{argmax}_{0\leq i<m_t} |f(y_i)|, & |y_t| \leq |y_{m_t}| \\ m_t, & |y_t| > |y_{m_t}| \end{cases} \tag{8}$$

$$j_t := \operatorname*{argmax}_{0\leq i<i_t} |f(y_i)| \tag{9}$$

we can then define the parameter estimate for $(\vartheta_1, \vartheta_2)$ at time $t \geq 2$ as

$$\hat{\vartheta}_{1,t} = \frac{\begin{vmatrix} -u_{i_t} & -y_{i_t+1} \\ -u_{j_t} & -y_{j_t+1} \end{vmatrix}}{\begin{vmatrix} f(y_{i_t}) & -y_{i_t+1} \\ f(y_{j_t}) & -y_{j_t+1} \end{vmatrix}}; \quad \hat{\vartheta}_{2,t} = \frac{\begin{vmatrix} f(y_{i_t}) & -u_{i_t} \\ f(y_{j_t}) & -u_{j_t} \end{vmatrix}}{\begin{vmatrix} f(y_{i_t}) & -y_{i_t+1} \\ f(y_{j_t}) & -y_{j_t+1} \end{vmatrix}} \tag{10}$$

This estimate is defined through solving the following system equation for $(\vartheta_1, \vartheta_2)$:

$$\begin{cases} -u_{i_t} = \vartheta_1 f(y_{i_t}) + (w_{i_t+1} - y_{i_t+1})\vartheta_2 \\ -u_{j_t} = \vartheta_1 f(y_{j_t}) + (w_{j_t+1} - y_{j_t+1})\vartheta_2 \end{cases} \tag{11}$$

by setting the noise to be zero. The error of the parameter estimate at time $t \geq 2$ are denoted by

$$\tilde{\vartheta}_{1,t} = \vartheta_1 - \hat{\vartheta}_{1,t}, \qquad \tilde{\vartheta}_{2,t} = \vartheta_2 - \hat{\vartheta}_{2,t}. \tag{12}$$

Now, notice that by (7)

$$\frac{y_{j_t+1}}{f(y_{j_t})} \cdot \frac{y_{i_t+1}}{f(y_{i_t})} = \left(\varepsilon_{j_t} + \frac{w_{j_t+1}}{f(y_{j_t})}\right)\left(\varepsilon_{i_t} + \frac{w_{i_t+1}}{f(y_{i_t})}\right),$$

hence

$$\left(\varepsilon_{j_t}+\frac{w_{j_t+1}}{f(y_{j_t})}\right)\left(\varepsilon_{i_t}+\frac{w_{i_t+1}}{f(y_{i_t})}\right)<0 \tag{13}$$

will imply that $\frac{y_{j_t+1}}{f(y_{j_t})}$ and $\frac{y_{i_t+1}}{f(y_{i_t})}$ have different signs. The following lemma gives the range of the estimate error in this situation.

Lemma 3.1 *If (13) holds, then for $t \geq 2$,*

$$|\tilde{\vartheta}_{1,t}| \leq \overline{\vartheta}_2 w \frac{|y_{i_t+1}|+|y_{j_t+1}|}{|f(y_{i_t})y_{j_t+1}|}.$$

Proof. First, the equation

$$\begin{cases} y_{i_t+1}=\theta_1 f(y_{i_t})+\theta_2 u_{i_t}+w_{i_t+1} \\ y_{j_t+1}=\theta_1 f(y_{j_t})+\theta_2 u_{j_t}+w_{j_t+1} \end{cases}$$

can be rewritten as (11). Solving (11), we get

$$\begin{cases} \vartheta_1=\dfrac{\begin{vmatrix} -u_{i_t} & w_{i_t+1}-y_{i_t+1} \\ -u_{j_t} & w_{j_t+1}-y_{j_t+1} \end{vmatrix}}{\begin{vmatrix} f(y_{i_t}) & w_{i_t+1}-y_{i_t+1} \\ f(y_{j_t}) & w_{j_t+1}-y_{j_t+1} \end{vmatrix}} \\ \vartheta_2=\dfrac{\begin{vmatrix} f(y_{i_t}) & -u_{i_t} \\ f(y_{j_t}) & -u_{j_t} \end{vmatrix}}{\begin{vmatrix} f(y_{i_t}) & w_{i_t+1}-y_{i_t+1} \\ f(y_{j_t}) & w_{j_t+1}-y_{j_t+1} \end{vmatrix}} \end{cases}. \tag{14}$$

Then by (10) and (12), we can compute that

$$\begin{aligned} \tilde{\vartheta}_{1,t} &= \frac{\begin{vmatrix} -u_{i_t} & w_{i_t+1}-y_{i_t+1} \\ -u_{j_t} & w_{j_t+1}-y_{j_t+1} \end{vmatrix}}{\begin{vmatrix} f(y_{i_t}) & w_{i_t+1}-y_{i_t+1} \\ f(y_{j_t}) & w_{j_t+1}-y_{j_t+1} \end{vmatrix}} - \frac{\begin{vmatrix} -u_{i_t} & -y_{i_t+1} \\ -u_{j_t} & -y_{j_t+1} \end{vmatrix}}{\begin{vmatrix} f(y_{i_t}) & -y_{i_t+1} \\ f(y_{j_t}) & -y_{j_t+1} \end{vmatrix}} \\ &= \frac{f(y_{j_t})u_{i_t}-f(y_{i_t})u_{j_t}}{f(y_{i_t})(w_{j_t+1}-y_{j_t+1})-f(y_{j_t})(w_{i_t+1}-y_{i_t+1})} \cdot \\ &\quad \frac{y_{i_t+1}w_{j_t+1}-y_{j_t+1}w_{i_t+1}}{f(y_{i_t})y_{j_t+1}-f(y_{j_t})y_{i_t+1}} \\ &= \vartheta_2 \frac{y_{i_t+1}w_{j_t+1}-y_{j_t+1}w_{i_t+1}}{f(y_{i_t})y_{j_t+1}-f(y_{j_t})y_{i_t+1}}, \end{aligned} \tag{15}$$

where the last equality follows from the expression of ϑ_2 in (14).

Now, by (13) and the argument above, we have

$$\begin{aligned} & |f(y_{i_t})y_{j_t+1}-f(y_{j_t})y_{i_t+1}| \\ =& \left|\frac{y_{j_t+1}}{f(y_{j_t})}-\frac{y_{i_t+1}}{f(y_{i_t})}\right| \cdot |f(y_{j_t})f(y_{i_t})| \\ \geq& |f(y_{i_t})y_{j_t+1}|. \end{aligned} \tag{16}$$

Hence by (15) and (16), we have

$$|\tilde{\vartheta}_{1,t}| \leq \overline{\vartheta}_2 w \frac{|y_{i_t+1}|+|y_{j_t+1}|}{|f(y_{i_t})y_{j_t+1}|}.$$

The proof is thus completed. ■

3.2 The Design of adaptive Control

In this subsection, we will discuss the design of adaptive control and prove the sufficiency part of Theorem 2.1.

To design the control which can stabilize the system (2), we need to define a sequence of subscripts t_k for the output sequence $\{y_t\}$:

$$\begin{cases} t_0=0 \\ t_{k+1}=\inf\{t>t_k : |f(y_t)|>|f(y_{t_k})|\} \end{cases}. \tag{17}$$

then, we have

$$|f(y_t)| \leq |f(y_{t_k})| < |f(y_{t_{k+1}})|, \quad \text{for any } t_k<t<t_{k+1}.$$

Now, let $\Delta_t := \overline{\vartheta}_2 w \frac{|y_{i_t+1}|+|y_{j_t+1}|}{|f(y_{i_t})y_{j_t+1}|}$, for $k=1,2,\cdots$ where $\overline{\vartheta}_2$ and w are defined in subsection 2.1 and (3) respectively. We can define

$$\beta_t=\begin{cases} 0, & 0 \leq t < t_1 \\ 2\overline{\vartheta}_1, & t_1 \leq t < t_2 \\ \hat{\vartheta}_{1,t}-2\Delta_t, & t_{2k} \leq t < t_{2k+1},\ k \geq 1 \\ \hat{\vartheta}_{1,t}+2\Delta_t, & t_{2k+1} \leq t < t_{2(k+1)},\ k \geq 1 \end{cases} \tag{18}$$

then the control can be designed by

$$u_t=-\beta_t f(y_t). \tag{19}$$

Remark 3.1 *Notice that by (8), (9) and the definition of Δ_t, we know*

$$\hat{\vartheta}_{1,t}-2\Delta_t=\hat{\vartheta}_{1,t_k}-2\Delta_{t_k} \qquad \textit{for} \quad t_k \leq t < t_{k+1}.$$

To prove that the controller designed by (18) and (19) can stabilize the system (2), we proceed to analyze the closed-loop system.

Proposition 3.1 *For the system (2) with the controller designed by (18) and (19), the following statements hold for all $k \geq 2$ with $|y_0|$ sufficiently large:*
(i) $|y_{t_k}| \leq 2c_2|\varepsilon_{t_{k-1}}||y_{t_{k-1}}|^b$.
(ii) $|y_{t_{k-2}+1}| \geq \frac{1}{4}|y_{t_{k-1}}|$.
(iii) $\left(\varepsilon_{t_{k-1}} + \frac{w_{t_{k-1}+1}}{f(y_{t_{k-1}})}\right)\left(\varepsilon_{t_{k-2}} + \frac{w_{t_{k-2}+1}}{f(y_{t_{k-2}})}\right) < 0$.
(iv) $|\varepsilon_{t_k}| = O\left(\left|\frac{y_{t_k}}{y_{t_{k-1}}^{b+1}}\right|\right)$.

Proof. (i) For $|y_{t_k}|$ large enough, by (7) we have

$$\begin{aligned}\frac{1}{2}|y_{t_k}| &\leq |y_{t_k}| - |w_{t_k}| \leq |y_{t_k} - w_{t_k}| \\ &= |\varepsilon_{t_k-1}||f(y_{t_k-1})|. \end{aligned} \tag{20}$$

Moreover, since $t_{k-1} \leq t_k - 1 < t_k$, by (18) and Remark 3.1 we know that $\beta_{t_k-1} = \beta_{t_{(k-1)}}$ for all $k \geq 0$, which implies that $\varepsilon_{t_k-1} = \varepsilon_{t_{(k-1)}}$. Hence (20) gives

$$\begin{aligned}|y_{t_k}| &\leq 2|\varepsilon_{t_k-1}||f(y_{t_k-1})| \leq 2|\varepsilon_{t_{k-1}}||f(y_{t_{k-1}})| \\ &\leq 2c_2|\varepsilon_{t_{k-1}}||y_{t_{k-1}}|^b. \end{aligned} \tag{21}$$

(ii) By (21), we have

$$\frac{1}{2}|y_{t_{k-1}}| \leq |\varepsilon_{t_{k-2}}||f(y_{t_{k-2}})| \leq |y_{t_{k-2}+1}| + w,$$

which gives (ii) for sufficiently large $|y_{t_{k-1}}|$.

(iii) In fact, we need only to show for any $k \geq 0$,

$$\varepsilon_{t_{2k}} + \frac{w_{t_{2k}+1}}{f(y_{t_{2k}})} > 0; \tag{22}$$

$$\varepsilon_{t_{2k+1}} + \frac{w_{t_{2k+1}+1}}{f(y_{t_{2k+1}})} < 0. \tag{23}$$

We will prove it by induction. First we consider the cases where $t = t_0 = 0$ and $t = t_1$ respectively.

For $t = 0$, by (18) and the definition of ε_t, we have

$$\varepsilon_{t_0} = \theta_2(\vartheta_1 - \beta_{t_0}) \geq \underline{\theta}_2\underline{\vartheta}_1. \tag{24}$$

Then, for $|y_{t_0}|$ large enough, the above inequality gives

$$\varepsilon_{t_0} + \frac{w_{t_0+1}}{f(y_{t_0})} \geq \underline{\theta}_2\underline{\vartheta}_1 - \frac{w}{|f(y_{t_0})|} > 0.$$

For the case of $t = t_1$, it can be proven similarly that (23) holds.

Now, suppose (22) and (23) hold for some $k \geq 0$. For $t = t_{2(k+1)}$, since $i_t = t_{2k+1}$ and $j_t = t_{2k}$, by assumption, we have (13) holds. Hence from Lemma 3.1, $|\tilde{\vartheta}_{1,t_{2(k+1)}}| \leq \Delta_{t_{2(k+1)}}$. Consequently,

$$\begin{aligned}\varepsilon_{t_{2(k+1)}} &= \theta_2(\vartheta_1 - \beta_{t_{2(k+1)}}) \\ &= \theta_2(\vartheta_1 - \hat{\vartheta}_{1,t_{2(k+1)}} + 2\Delta_{t_{2(k+1)}}) \\ &= \theta_2(\tilde{\vartheta}_{1,t_{2(k+1)}} + 2\Delta_{t_{2(k+1)}}) \\ &\geq \underline{\theta}_2\Delta_{t_{2(k+1)}}. \end{aligned}$$

As a result, we have

$$\begin{aligned}&\varepsilon_{t_{2(k+1)}} + \frac{w_{t_{2(k+1)}+1}}{f(y_{t_{2(k+1)}})} \geq \underline{\theta}_2\Delta_{t_{2(k+1)}} - \frac{w}{|f(y_{t_{2(k+1)}})|} \\ &\geq w\frac{|y_{t_{2k+1}+1}| + |y_{t_{2k}+1}|}{|f(y_{t_{2k+1}})y_{t_{2k}+1}|} - \frac{w}{|f(y_{t_{2(k+1)}})|} \\ &\geq \frac{w}{|f(y_{t_{2k+1}})|} - \frac{w}{|f(y_{t_{2(k+1)}})|} > 0. \end{aligned}$$

The claim (23) also holds for $t = t_{2k+3}$ by a similar reasoning as that for $t = t_{2(k+1)}$.

Hence, by induction we know that (iii) is true.

(iv) At time $t = t_k$, it is easy to see that $i_t = t_{k-1}$ and $j_t = t_{k-2}$. Then by (ii) and (iii) ,

$$\begin{aligned}\Delta_{t_k} &= \overline{\vartheta}_2 w\frac{|y_{i_t+1}| + |y_{j_t+1}|}{|f(y_{i_t})y_{j_t+1}|} \\ &= \overline{\vartheta}_2 w\frac{|y_{t_{k-1}+1}| + |y_{t_{k-2}+1}|}{|f(y_{t_{k-1}})y_{t_{k-2}+1}|} \\ &= O\left(\left|\frac{y_{t_k}}{y_{t_{k-1}}^{b+1}}\right|\right). \end{aligned}$$

Hence, by Lemma 3.1, we have for $k \geq 2$

$$\begin{aligned}|\varepsilon_{t_k}| &= \theta_2|\vartheta_1 - \beta_t| = \theta_2|\vartheta_1 - \hat{\vartheta}_{1,t_k} \pm 2\Delta_{t_k}| \\ &\leq \theta_2(|\tilde{\theta}_{1,t_k}| + 2\Delta_{t_k}) \leq 3\theta_2\Delta_{t_k} \\ &= O\left(\left|\frac{y_{t_k}}{y_{t_{k-1}}^{b+1}}\right|\right). \end{aligned}$$

This completes the proof. ∎

The sufficiency proof of Theorem 2.1. We use a contradiction argument to prove that $\sup_{t\geq 0}|y_t| < \infty$. Suppose there exist some $y_0 \in \mathbb{R}^1$, some $\{\theta_1, \theta_2\}$ and some sequence of $\{w_t\}$, such that for the control defined in (19), $\sup_{t\geq 0}|y_t| = \infty$. Then for the subscript sequence $\{t_k\}$ defined in (17), we have $k \to \infty$.

Also note that, by Proposition 3.1 (i) and (iv), the system (2) at time t_{k+1} satisfies

$$|y_{t_{k+1}}| \leq 2c_2|\varepsilon_{t_k}||y_{t_k}|^b = O\left(\left|\frac{y_{t_k}}{y_{t_{k-1}}}\right|^{b+1}\right). \qquad (25)$$

To apply Lemma 3.5 in [19], we take $a_k = \log|y_{t_k}|$, then the outputs will be bounded when $b+1 < 4$, which contradicts to our assumption. Hence, the sufficiency is proved. ∎

4 The Proof of Necessity

We introduce a stochastic imbedding approach to the proof of necessity. Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $\theta \in \mathbb{R}^2$ be a random vector and $\{w_t\}_{t=1}^{\infty}$ be a stochastic process on this probability space respectively. (In fact, θ and $\{w_t\}_{t=1}^{\infty}$ are different from those defined in the assumptions **A1) - A2)**, we use the same notation just for convenience.) We consider the stochastic system in the form (2).

Assume that θ has a spherical p.d.f. $p(\theta)$, which satisfies

$$p(\theta) = \begin{cases} c(2^{-1}R^2 - \|\tilde{\theta}^c\|^2) & \text{if } 0 \leq \|\tilde{\theta}^c\| \leq R/2; \\ c(R - \|\tilde{\theta}^c\|)^2 & \text{if } R/2 \leq \|\tilde{\theta}^c\| \leq R; \\ 0 & \mathit{otherwise} \end{cases} \qquad (26)$$

where $\tilde{\theta}^c = \theta - \theta^c$ with $\theta^c = \left(\frac{\underline{\theta}_1 + \overline{\theta}_1}{2}, \frac{\underline{\theta}_2 + \overline{\theta}_2}{2}\right)^T$ being the center of the uncertain domain, and

$$R = \min\left\{\frac{\overline{\theta}_1 - \underline{\theta}_1}{2}, \frac{\overline{\theta}_2 - \underline{\theta}_2}{2}\right\},$$

and where c is some constant to make

$$\int_{\|\tilde{\theta}^c\| \leq R} p(\theta)d\theta = 1.$$

Also, let us take $\{w_t\}$ to be an independent sequence which is independent of θ with w_t having a Gaussian p.d.f. $q_t(z)$ defined by $N\left(0, \frac{1}{t^2}\right)$:

$$q_t(z) = \frac{t}{\sqrt{2\pi}} \exp\left(-\frac{z^2t^2}{2}\right), \qquad (27)$$

Obviously, $\{w_t\}$ satisfies A1) almost surely for large enough t, since by (27)

$$\lim_{t\to\infty} w_t = 0, \quad a.s.$$

Remark 4.1 *We need to note that* $\{\theta : \|\tilde{\theta}^c\| \leq R\} \subset [\underline{\theta}_1, \overline{\theta}_1] \times [\underline{\theta}_2, \overline{\theta}_2]$ *by (26) and the definition of* R*, see Fig 1. The distribution of the noise in (27) also shows that* $|w_t| \leq w$ *for all large enough* $t \geq 1$.

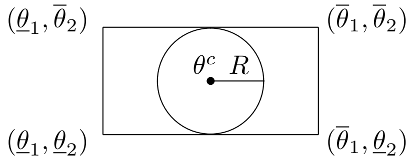

Fig.1. The area of θ

We will first show that in the above stochastic framework, if $b \geq 3$, then for any feedback control $u_t \in \mathcal{F}_t^y \triangleq \sigma\{y_i, 0 \leq i \leq t\}$, there always exists an initial condition y_0 and a set D with positive probability such that the output signal y_t of the closed-loop control system tends to infinity at a rate faster than exponential on D. Then in the last part of this subsection, we will find a point in D which corresponds to some values of θ and $\{w_t\}_{t=1}^{\infty}$, and we will see that these deterministic values are sufficient for the proof of necessity of Theorem 2.1. Thus by imbedding a random distribution, we are able to solve the problem in the original deterministic framework.

To prove the above fact, we first give a lemma which can be obtained by a little modification of the proof of [18, Theorem 3.2.2-Theorem 3.2.6 and Remark 3.2.3]. We will give the proof in Appendix A.

Lemma 4.1 *Consider the following dynamical system:*

$$y_{k+1} = \theta^\tau \phi_k + w_{k+1}, \quad k = 0, 1, \cdots,$$

where $\phi_k \triangleq (f(y_k), u_k)^\tau$, $y_0 > x'$ *is deterministic and* $y_i = 0$, $\forall i < 0$*; the unknown parameter vector* θ *with p.d.f.* $p(\theta)$ *defined in (26) is independent of* $\{w_k\}$ *which is an independent random sequence with distribution defined in (27). Then for* $t = 1, 2, \cdots$,

$$E[(\theta - \hat{\theta}_t)(\theta - \hat{\theta}_t)^\tau | \mathcal{F}_t^y] \geq \left\{t^2 \sum_{k=0}^{t-1} \phi_k \phi_k^\tau + KI\right\}^{-1},$$

where $\hat{\theta}_t \triangleq E\{\theta | \mathcal{F}_t^y\}, t = 1, 2, \cdots$ *; and* $K > 0$ *is some constant;* I *denotes the identity matrix. Furthermore, there exists some* $D \subset \Omega$ *with* $P(D) > 0$ *such that on* D *for* $t = 0, 1, \cdots$,

$$E[y_{t+1}^2 | \mathcal{F}_t^y] \leq (K_1(t+1)^4 + 4)(y_{t+1}^2 + K_2) + 1,$$

where $K_1, K_2 > 0$ *are some constants.*

In the following lemma, we will estimate the determinants of two matrices which will appear in the proof of the next proposition. It is easy to see that the two are modifications of the information matrices in Least Square-algorithm.

Lemma 4.2 *Assume that for some $\lambda > 1$ and $t \geq 1$, $|y_i| \geq |y_{i-1}|^\lambda, i = 1, 2, \cdots, t$, and that the initial condition $y_0 \geq \max\{1, x'\}$ is sufficiently large, then the determinants of the matrices*

$$P_{t+1}^{-1} \triangleq KI + (t+1)^2 \sum_{i=0}^{t} \phi_i \phi_i^\tau \tag{28}$$

$$Q_{t+1}^{-1} \triangleq P_t^{-1} + \phi_t \phi_t^\tau \tag{29}$$

satisfy

$$|P_{t+1}^{-1}| \leq M(t+1)^4 \max\left\{|y_t|^{2(b+1)}, |y_{t-1}^b y_{t+1}|^2\right\};$$
$$|Q_{t+1}^{-1}| \geq \underline{\vartheta}_2^2 \cdot (|f(y_t)y_t - f(y_{t-1})y_{t+1}| - 2w|f(y_t)|)^2,$$

where $K > 0$ is defined in Lemma 4.1 and $M > 1$ is a random variable.

Proof. See Appendix B.

Remark 4.2 *If $|f(y_{t-1})y_{t+1}| < \frac{1}{2}|f(y_t)y_t|$, then we have for large enough $|y_0|$,*

$$\begin{aligned}|Q_{t+1}^{-1}| &\geq \underline{\vartheta}_2^2 \cdot \left(\frac{1}{2}|f(y_t)y_t| - 2w|f(y_t)|\right)^2 \\ &\geq \underline{\vartheta}_2^2 \cdot \left(\frac{c_1}{2}|y_t^{b+1}| - 2wc_2|y_t|^b\right)^2 \\ &\geq \frac{\underline{\vartheta}_2^2 c_1^2}{8}|y_t|^{2(b+1)}.\end{aligned}$$

On the other hand, if $|f(y_{t-1})y_{t+1}| \geq \frac{1}{2}|f(y_t)y_t|$, then we have $c_2|y_{t-1}^b y_{t+1}| \geq \frac{c_1}{2}|y_t^{b+1}|$. Moreover, if $\lambda = \frac{b+1}{2}$, we have for large $|y_0|$ and $b \geq 3$,

$$\begin{aligned}|y_{t+1}| &\geq \frac{c_1}{2c_2}\frac{|y_t^{b+1}|}{|y_{t-1}^b|} \geq \frac{c_1}{2c_2}|y_t|^{b+1-\frac{2b}{b+1}} \\ &\geq \mu|y_t|^{b-1} \geq \mu|y_t|^{\frac{b+1}{2}},\end{aligned}$$

where μ is some constant we defined latter in the proof of Proposition 4.1.

Remark 4.3 *It is very easy to check that the upper bound of $|P_{t+1}|$ still holds for $t = -1$, where $y_i \triangleq 1$ for $i < 0$.*

Proposition 4.1 *Assume that the conditions of Lemma 4.1 hold. Then for any $u_t \in \mathcal{F}_t^y$, there always exists a y_0 and a set $D \subset \Omega$ with positive probability such that the output signal $|y_t| \nearrow \infty$ on D whenever $b \geq 3$.*

Proof. It is easy to see that $E[w_{t+1}|\mathcal{F}_t^y] = Ew_{t+1} = 0$ by (27). By (2) we know that

$$y_{t+1} = \phi_t^\tau \tilde{\theta}_t + \phi_t^\tau \hat{\theta}_t + w_{t+1}, \tag{30}$$

where $\tilde{\theta}_t \triangleq \theta_t - \hat{\theta}_t$. Consequently, by the fact $E[\tilde{\theta}_t|\mathcal{F}_t^y] = 0$ and $E[w_{t+1}|\mathcal{F}_t^y] = 0$ it follows that for any $u_t \in \mathcal{F}_t^y$,

$$\begin{aligned}E[y_{t+1}^2|\mathcal{F}_t^y] &= \phi_t^\tau E[\tilde{\theta}_t\tilde{\theta}_t^\tau|\mathcal{F}_t^y]\phi_t + (\phi_t^\tau\hat{\theta}_t)^2 + E[w_{t+1}^2|\mathcal{F}_t^y] \\ &\geq \phi_t^\tau E[\tilde{\theta}_t\tilde{\theta}_t^\tau|\mathcal{F}_t^y]\phi_t + E[w_{t+1}^2|\mathcal{F}_t^y]. \end{aligned}\tag{31}$$

By Lemma 4.1, we have on D,

$$\begin{aligned}E[y_{t+1}^2|\mathcal{F}_t^y] &\geq \phi_t^\tau \left\{t^2\sum_{k=0}^{t-1}\phi_k\phi_k^\tau + KI\right\}^{-1}\phi_t \\ &= (\phi_t^\tau P_t \phi_t + 1) - 1 \\ &= \frac{|P_t^{-1} + \phi_t\phi_t^\tau|}{|P_t^{-1}|} - 1 \qquad (32) \\ &= \frac{|Q_{t+1}^{-1}|}{|P_t^{-1}|} - 1, \qquad t \geq 1, \qquad (33)\end{aligned}$$

where P_t, Q_t are defined by (28) and (29).

Hence by Lemma 4.1 again, we have for $t \geq 0$,

$$y_{t+1}^2 \geq \frac{1}{K_1(t+1)^4 + 4} \cdot \left[\frac{|Q_{t+1}^{-1}|}{|P_t^{-1}|} - K_2(K_1(t+1)^4 + 4) - 2\right] \text{ on } D. \tag{34}$$

Now, we proceed to show that on D for sufficiently large $|y_0|$,

$$|y_t| \geq \mu|y_{t-1}|^{\frac{b+1}{2}}, \qquad t = 1, 2, \cdots, \tag{35}$$

where μ is some constant we defined latter.

We adopt the induction argument. First, we consider the initial case. Since

$$E[(\theta - \hat{\theta}_0)(\theta - \hat{\theta}_0)^\tau|\mathcal{F}_0^y] = E[(\theta - \hat{\theta}_0)(\theta - \hat{\theta}_0)^\tau] \geq \sigma_\theta^2 I,$$

where σ_θ^2 is some constant. We have by (31) that $E[y_1^2|\mathcal{F}_0^y] \geq \sigma_\theta^2\|\phi_0\|^2$. Then by (34)

$$\begin{aligned}y_1^2 &\geq \frac{1}{K_1 + 4}(\sigma_\theta^2\|\phi_0\|^2 - 2) - K_2 \\ &\geq \frac{\sigma_\theta^2}{4 + K_1}y_0^{2b} - \frac{1}{2(4 + K_1)} - K_2 \qquad \text{on } D.\end{aligned}$$

Hence (35) holds for $t = 1$ when $|y_0|$ is large enough.

Now, let us assume that for some $t \geq 1$,

$$|y_i| \geq \mu|y_{i-1}|^{\frac{b+1}{2}}, \qquad i = 1, 2, \cdots, t, \qquad \text{on } D, \tag{36}$$

then by Lemma 4.2 and Remark 4.3, it follows that for $t \geq 1$

$$|P_t^{-1}| \leq Mt^4 \max\left\{|y_{t-1}^{b+1}|^2, |y_{t-2}^b y_t|^2\right\};$$
$$|Q_{t+1}^{-1}| \geq \underline{\vartheta}_2^2 \cdot (|f(y_t)y_t - f(y_{t-1})y_{t+1}| - 2w|f(y_t)|)^2,$$

where $y_t \triangleq 1$ for any $t < 0$. By Remark 4.2, we only need to consider the case where

$$|Q_{t+1}^{-1}| \geq \frac{\underline{\vartheta}_2^2 c_1^2}{8}|y_t|^{2(b+1)}.$$

Consequently, by (34) we have

$$\begin{aligned} y_{t+1}^2 \geq & \frac{\frac{|Q_{t+1}^{-1}|}{|P_t^{-1}|} - (K_1(t+1)^4+4)K_2 - 2}{K_1(t+1)^4+4} \\ \geq & \frac{1}{K_1(t+1)^4+4}\left[\frac{\underline{\vartheta}_2^2 c_1^2}{8Mt^4}\frac{y_t^{2(b+1)}}{\max\left\{y_{t-1}^{2(b+1)}, y_{t-2}^{2b}y_t^2\right\}}\right. \\ & \left. -(K_1(t+1)^4+4)K_2 - 2\right]. \end{aligned}$$

Note that by (36), for large enough $|y_0|$, the above inequality satisfies

$$y_{t+1}^2 \geq \frac{\underline{\vartheta}_2^2 c_1^2}{16Mt^4(K_1(t+1)^4+4)} \cdot \frac{y_t^{2(b+1)}}{\max\left\{y_{t-1}^{2(b+1)}, y_{t-2}^{2b}y_t^2\right\}}. \quad (37)$$

Denote

$$x_t = \begin{cases} \sqrt{\frac{16Mt^4(K_1t^4+4)}{\underline{\vartheta}_2^2 c_1^2}}\, y_t, & t \geq 1 \\ y_0, & t = 0 \\ 1, & t < 0 \end{cases},$$

then the inequality (37) can be rewritten as

$$x_{t+1}^2 \geq \left|\frac{\max\{1, (t-1)^4(K_1(t-1)^4+4)\}}{t^4(K_1t^4+4)}\right|^{b+1}\left(\frac{x_t}{x_{t-1}}\right)^{2(b+1)}$$

or

$$x_{t+1}^2 \geq \left|\frac{\max\{1, (t-2)^4(K_1(t-2)^4+4)\}}{t^4(K_1t^4+4)}\right|^{b}\left(\frac{x_t}{x_{t-2}}\right)^{2b}$$

Since the coefficient of the R.H.S of the above two inequalities tends to 1 as $t \to \infty$, it is easy to see that there exists some constant $\nu > 0$ such that

$$|x_{t+1}| \geq \nu\left|\frac{x_t}{x_{t-1}}\right|^{b+1} \quad \text{or} \quad |x_{t+1}| \geq \nu\left|\frac{x_t}{x_{t-2}}\right|^{b}$$

Hence let $a_t = \log\frac{|x_t|}{\nu}$, we have

$$a_{t+1} \geq (b+1)(a_t - a_{t-1}) \quad \text{or} \quad a_{t+1} \geq b(a_t - a_{t-2}).$$

Now, notice that when $b \geq 3$, we have $\lambda^2 - (b+1)\lambda + (b+1) \leq 0$ for $\lambda = \frac{b+1}{2} \in (1, b+1)$, and then

$\lambda^3 - b\lambda^2 + b < 0.$

Let $P(x) = x^2 - (b+1)x + (b+1)$ or $P(x) = x^3 - bx^2 + b$, then $P(x) \leq 0$ for $x = \frac{b+1}{2}$. By [9, Lemma 3.3] and (36), we get $a_{t+1} \geq \lambda a_t$ for some $\lambda \geq 1$, which implies

$$|x_{t+1}| \geq \left|\frac{x_t}{\nu}\right|^{\lambda}.$$

Consequently, by the definition of x_t,

$$|y_{t+1}| \geq \frac{1}{\nu^\lambda}\left|\frac{16M}{\underline{\vartheta}_2^2 c_1^2}\right|^{\lambda-1}\frac{(K_1t^8+4t^4)^\lambda}{(t+1)^4(K_1(t+1)^4+4)}|y_t|^\lambda.$$

Note $t+1 \leq 2t$ for $t \geq 1$, we have

$$\begin{aligned} |y_{t+1}| \geq & \frac{1}{2^8\nu^\lambda}\left|\frac{16M(K_1t^8+4t^4)}{\underline{\vartheta}_2^2 c_1^2}\right|^{\lambda-1}|y_t|^\lambda \\ \geq & \frac{1}{2^8\nu^\lambda}\left|\frac{16M(K_1+4)}{\underline{\vartheta}_2^2 c_1^2}\right|^{\lambda-1}|y_t|^\lambda. \end{aligned}$$

So, $|y_{t+1}| \geq \mu|y_t|^\lambda$ holds if we let

$$\mu = \frac{1}{2^8\nu^\lambda}\left|\frac{16M(K_1+4)}{\underline{\vartheta}_2^2 c_1^2}\right|^{\lambda-1}.$$

By induction, we know that (35) is true. Thus, for large enough $|y_0|$, the output sequence $\{y_t\}$ diverges to infinity exponentially fast, and so we have $\sup_{t\geq 0}|y_t| = \infty$. ■

The proof of the necessity of Theorem 2.1. In the stochastic framework, note that any controller $u_t = h_t(y_0, \cdots, y_t)$ is measurable to $\mathcal{F}_t^y$. By Proposition 4.1, for any given control law $\{u_t\}$, there at least exists a sample point $\omega^* \in D \subset \Omega$ with $\theta(\omega^*) = \theta^*$ and $w_t(\omega^*) = w_t^*$ for any $t \geq 1$ such that for some y_0^*, the absolute values of the output $|y_t(\omega^*)| = |y_t^*| \nearrow \infty$. Since there exists some $t^* \geq 1$ such that $|w_t^*| \leq w$ for all large $t \geq t^*$, without loss of generality, we suppose $|w_t^*| \leq w$ for all t. Otherwise, we can take $y_0 = y_{t^*}$, and start with the time t^*.

That is, for any given Lebesgue function $h(\cdot)$, there exist some θ^* and $\{w_t^*\}$ satisfying assumptions **A1)-A2)** and a y_0^* such that the absolute values of the outputs

$$y_{t+1}^* = \theta_1^* y_t^{*b} + \theta_2^* h_t(y_0^*, \cdots, y_t^*) + w_{t+1}^*$$

monotonously increase to infinity, which gives the necessity conclusion of Theorem 2.1. ■

5 Conclusion

We have found and established a new critical theorem for global stabilization of a basic class of discrete-time nonlinear systems, with unknown parameters in both the system channel and control channel. This furthered our understanding of the maximum capability of feedback in dealing with uncertainties, especially for the case where the control channel contains uncertain parameters. For further investigation it would be interesting to get similar results for nonparametric systems and for continuous-time systems with sampled-data control. It would also be interesting to consider more general model classes with an unified treatment.

References

[1] K. J. Aström and B. Wittenmark (1995). *Adaptive Control*, 2nd ed. Reading, MA: Addison-Wesley.

[2] V. D. Blondel and A. Megretski (2004). *Unsolved Problems in Mathematical Systems and Control Theory,* Princeton University Press.

[3] L. Guo and H.F.Chen (1991) The Astrom-Wittenmark self-tuning regulator revisited and ELS based adaptive trackers, *IEEE Trans. Automatic Control*, 36(7), 802-812.

[4] L. Guo (1997). On critical stability of discrete-time adaptive nonlinear control, *IEEE Trans. Autom. Control*, 42(1997), 1488-1499.

[5] P. A. Ioannou and J. Sun (1996). *Robust Adaptive Control*, Englewood Cliffs, NJ: Prentice-Hall.

[6] Z. P. Jiang and Y. Wang (2001) Input-to-state stability for discrete-time nonlinear systems, *Automatica*, 37(6), 857-869.

[7] C. Li, L.-L. Xie. and L. Guo(2005). Robust stability of discrete-time adaptive nonlinear control, *The 16th IFAC World Congress*, Prague, July 3-8.

[8] C. Li and L.-L. Xie (2006). On robust stability of discrete-time adaptive nonlinear control, *System & Control Letters*, 55, 452-458.

[9] C. Li, L. -L. Xie and L. Guo (2006). A Polynomial Criterion for Adaptive Stabilizability of Discrete-Time Nonlinear Systems. *Communications in Information and Systems*, 6(4), 273-298.

[10] I. Kanellakopoulos (1994). A discrete-time adaptive nonlinear system,*IEEE Trans. Autom. Control*, 39, 2362-2364.

[11] M. Krstic, I. Kanellakopoulos and P.V. Kokotovic (1995). *Nonlinear and Adaptive Control Design.* New York: Wiley.

[12] P. V. Kokotovic and M. Arcak (1999). Constructive nonlinear control: Progress in the 90s, *in Proc. 14th IFACWorld Congr.*, Beijing, China, July 1999, 49-77. Plenary and Index Volume.

[13] I. M. Y. Mareels, H. B. Penfold, and R. J. Evans (1992). Controlling nonlinear time-varying systems via Euler approximations, *Automatica*, 28, 681-696.

[14] D. S, Mitrinovic. *Analytic Inequalities,* Springer-Verlag, 1970.

[15] E. Skafidas, A. Fradkov, R. J. Evans, and I. M. Y. Mareels (1998). Trajectory- approximation-based adaptive control for nonlinear systems under matching condition, *Automatica*, 34, 287-299.

[16] G. Tao and P. V. Kokotovic(1996). *Adaptive Control of Systems with Actuator and Sensor Nonlinearities*: Wiley.

[17] L.-L. Xie and L. Guo (1999), Fundamental limitations of discrete-time adaptive nonlinear control, *IEEE Trans. Autom. Control*, 44, 1777-1782.

[18] L.-L. Xie and L. Guo (2000a). Adaptive control of discrete-time nonlinear systems with structural uncertainties, *In: Lectures on Systems, Control, and Information*, AMS/IP, 49-90.

[19] L.-L. Xie and L. Guo (2000b). How much uncertainty can be dealt with by feedback?*IEEE Trans. Autom. Control*,45, 2203-2217.

[20] F. Xue and L. Guo (2002). On limitations of the sampled-data feedback for nonparametric dynamical systems, *J. Systems Science and Complexity*, 15(3), 225-250.

[21] H. L. Van Trees (1968). *Detection, Estimation, and Modulation Thorey: Part I,* New York: John Wiley & Sons.

[22] Y.S.Chow and H.Teicher (1978). *Probability Theory: Independence, Interchangeability and Martingales*, New York: Springer-Verlag.

A The Proof of Lemma 4.1

To prove Lemma 4.1, we need several lemmas as given below. The first lemma below is a standard conditional Cramer-Rao inequality (see, e.g. [18], [21]).

Lemma A.1 *Let x be random vector, and let θ be a parameter vector with p.d.f. $p(\theta)$ defined in (26). Then for any measurable vector function $g(x, \theta)$ having partial derivatives of first order w.r.t. θ, and let $E_x g(x, \theta)$ and $E_x \dfrac{\partial g(x,\theta)}{\partial \theta}$ exist, where $E_x y \triangleq E(y|x)$ for any random* *variable y. Then we have*

$$E_x[g(x,\theta)-E_xg(x,\theta)][g(x,\theta)-E_xg(x,\theta)]^\tau \geq E_x\frac{\partial g(x,\theta)}{\partial\theta}\left\{E_x\left[\frac{\partial\log p(x,\theta)}{\partial\theta}\cdot\frac{\partial^\tau\log p(x,\theta)}{\partial\theta}\right]\right\}^{-1} E_x^\tau\frac{\partial g(x,\theta)}{\partial\theta}.$$

Applying this lemma to the dynamical system defined by (2), we can further get the following result.

Lemma A.2 *Let θ be a parameter vector with p.d.f. $p(\theta)$ defined in (26), and be independent of $\{w_k\}$, which is an independent random sequence with p.d.f. $q_t(z)$ defined in (27). Then, for $t\geq 1$*

$$E_x(\theta-\hat{\theta}_t)(\theta-\hat{\theta}_t)^\tau \geq E_x^{-1}F_t(\theta), \tag{A.1}$$

where $x\triangleq\{y_1,\cdots,y_t\}$ and

$$F_t(\theta)\triangleq-\sum_{k=1}^{t}\frac{\partial^2\log q_k(y_k-f_{k-1})}{\partial\theta^2}+KI,$$

where $K>0$ is some random variable, and $f_{k-1}\triangleq\theta^\tau\phi_{k-1}, \phi_{k-1}=(f(y_{k-1}),u_{k-1})$ defined in Lemma (4.1).

Proof. Directly applying Lemma A.1, we have

$$\begin{aligned}&E_x(\theta-\hat{\theta}_t)(\theta-\hat{\theta}_t)^\tau\\ &\geq\left\{E_x\left[\frac{\partial\log p(x,\theta)}{\partial\theta}\cdot\frac{\partial^\tau\log p(x,\theta)}{\partial\theta}\right]\right\}^{-1}\\ &=-\left\{E_x\left[\frac{\partial^2\log p(x,\theta)}{\partial\theta^2}\right]\right\}^{-1},\end{aligned}$$

where the equality follows from [18]. Hence,

$$\begin{aligned}&E_x(\theta-\hat{\theta}_t)(\theta-\hat{\theta}_t)^\tau\\ &\geq-\left\{E_x\left[\frac{\partial^2[\log p(x|\theta)+\log p(\theta)]}{\partial\theta^2}\right]\right\}^{-1}.\end{aligned} \tag{A.2}$$

Note that by the Bayes rule and the dynamical equation (2), we have

$$\begin{aligned}p(x|\theta)&=p(y_1,y_2,\cdots,y_t|\theta)\\ &=p(y_1|\theta,y_0)p(y_2|\theta,y_0,y_1)\cdots p(y_t|\theta,y_0,\cdots,y_{t-1})\\ &=q_1(y_1-f_0)\cdot q_2(y_2-f_1)\cdots q_t(y_t-f_{t-1}).\end{aligned}$$

Consequently, we have

$$\frac{\partial^2\log p(x|\theta)}{\partial\theta^2}=\sum_{k=1}^{t}\frac{\partial^2\log q_k(y_k-f_{k-1})}{\partial\theta^2}.$$

So, comparing with (A.2), we only need to prove that

$$-E_x\frac{\partial^2\log p(\theta)}{\partial\theta^2}\leq KI,\quad a.s. \tag{A.3}$$

where $K>0$ is some random variable.

First, it can be shown that $\frac{\partial^2 p(\theta)}{\partial\theta^2}$ and

$$\frac{1}{p(\theta)}\left(\frac{\partial p(\theta)}{\partial\theta}\right)\left(\frac{\partial p(\theta)}{\partial\theta}\right)^\tau$$

are bounded, then with some simple manipulations we have

$$\begin{aligned}-\frac{\partial^2\log p(\theta)}{\partial\theta^2}&=\frac{1}{p^2(\theta)}\left(\frac{\partial p(\theta)}{\partial\theta}\right)\left(\frac{\partial p(\theta)}{\partial\theta}\right)^\tau\\ &\quad-\frac{1}{p(\theta)}\frac{\partial^2 p(\theta)}{\partial\theta^2}\leq\frac{C}{p(\theta)}I,\end{aligned}$$

where $C>0$ is some constant. Then

$$-E_x\frac{\partial^2\log p(\theta)}{\partial\theta^2}\leq CI\cdot E_x\frac{1}{p(\theta)}. \tag{A.4}$$

Note that $E[X|\mathcal{F}_t]$ is a.s. bounded for any integrable random variable X by [22, p.245], we have $E_x\frac{1}{p(\theta)}$ a.s. bounded since $E\frac{1}{p(\theta)}=1$, which gives (A.3). ■

Lemma A.3 *Under the conditions of Lemma A.2, we have*

$$F_t(\theta)\leq t^2\sum_{k=0}^{t-1}\phi_k\phi_k^\tau+KI$$

where $F_t(\theta)$ is defined in Lemma A.2 and $K>0$ is some constant.

Proof. Since $q_k(y_k-f_{k-1})=\frac{k}{\sqrt{2\pi}}\exp\{-\frac{k^2}{2}(y_k-f_{k-1})^2\}$, $k=1,2,\cdots,t$ we have

$$\begin{aligned}\frac{\partial^2\log q_k(y_k-f_{k-1})}{\partial\theta^2}&=\frac{\partial}{\partial\theta^2}\{-\frac{k^2}{2}(y_k-f_{k-1})^2\}\\ &=-k^2\phi_{k-1}\phi_{k-1}^\tau,\end{aligned}$$

which gives the lemma by the definition of $F_t(\theta)$. ■

By Lemmas A.2- A.3, we get the following proposition immediately.

Proposition A.1 *Under the conditions of Lemma A.2, for the dynamical equation (2) with arbitrarily deterministic initial value y_0, we have*

$$E[(\theta - \hat{\theta}_t)(\theta - \hat{\theta}_t)^\tau | \mathcal{F}_t^y] \geq \left\{ t^2 \sum_{k=0}^{t-1} \phi_k \phi_k^\tau + KI \right\}^{-1} \quad (A.5)$$

where $x \triangleq \{y_1, \cdots, y_t\}$.

The proof of Lemma 4.1. By Proposition A.1, we only need to prove the second conclusion of Lemma 4.1.

First, note that all the stochastic calculates in this paper hold almost surely. Denote Θ_0 as the corresponding domain of random variable θ on this probability 1 sampling set. Define

$$\Delta_t \triangleq \left\{ \theta \in \Theta : |\theta^\tau \phi_t| < \frac{\delta}{(t+1)^2} \|\phi_t\| \right\},$$
$$0 < \delta < \frac{1}{2SP \sum_{t=0}^{\infty} \frac{1}{(t+1)^2}}, \quad t \geq 0,$$

where $P \triangleq \sup_{\theta \in \Theta} p(\theta) = \frac{cR^2}{4}$, and $S \triangleq \sup_{L \in \mathcal{L}} V_{p-1}(L \cap \Theta)$ with $\mathcal{L}$ denotint the set of all $(p-1)$-dimensional hyperplane and $V_{p-1}(\cdot)$ denoting the Lebesgue measure on $\mathbb{R}^{p-1}$. Since Θ is bounded, we have $S < \infty$.

Recursively define $\Theta_{t+1} \triangleq \Theta_t - \Delta_t, t = 0, 1, \cdots$, where $\Theta_0 \subset \Theta$ is defined above. Let $\Theta_\infty \triangleq \lim_{t \to \infty} \Theta_t, D \triangleq \{\omega : \theta \in \Theta_\infty\}$.

Now, by almost the same proof of [18], we know that $P(\{\omega : \theta \in \Delta_t\}) \leq PS \frac{2\delta}{(t+1)^2}$. So,

$$\begin{aligned} P(\{\omega : \theta \in \bigcup_{t=0}^{\infty} \Delta_t\}) &\leq \sum_{t=0}^{\infty} P(\{\omega : \theta \in \Delta_t\}) \\ &\leq PS \sum_{t=0}^{\infty} \frac{2\delta}{(t+1)^2} < 1, \end{aligned}$$

which implies

$$P(D) \geq 1 - P(\{\omega : \theta \in \bigcup_{t=0}^{\infty} \Delta_t\}) > 0.$$

Now, let $\omega^* \in D$ be any fixed point, and let θ_t be a random variable sequence such that $|\theta_t^\tau \phi_t| = \max_{\theta \in \Theta} |\theta^\tau \phi_t|$. Then by the definitions of D and Δ_t, we have

$$\begin{aligned} [\theta_t^\tau \phi_t - \theta^\tau(\omega^*)\phi_t]^2 &\leq \|\theta_t - \theta(\omega^*)\|^2 \|\phi_t\|^2 \\ &\leq \frac{(2R)^2 (t+1)^4}{\delta^2} |\theta^\tau(\omega^*)\phi_t|^2, \quad (A.6) \end{aligned}$$

where R is defined in (26). Consequently, by noting that $w_t^2 \leq K_2$, *a.s.* for some random constant $K_2 > 0$, and the fact $\max_{\theta \in \Theta} (\theta^\tau \phi_t)^2$ is measurable $\mathcal{F}_t^y$, we have for any $\omega^* \in D$,

$$\begin{aligned} E_x y_{t+1}^2 &= E_x (\theta^\tau \phi_t)^2 + E w_{t+1}^2 \\ &\leq \max_{\theta \in \Theta} (\theta^\tau \phi_t)^2 + 1 \\ &\leq 2(\theta^\tau(\omega^*)\phi_t)^2 + 2[\theta_t^\tau \phi_t - \theta^\tau(\omega^*)\phi_t]^2 + 1 \\ &\leq \left(2 + \frac{4R^2(t+1)^4}{\delta^2}\right) (\theta^\tau(\omega^*)\phi_t)^2 + 1. \end{aligned}$$

Hence,

$$\begin{aligned} &[E_x y_{t+1}^2](\omega^*) \\ &\leq \left(2 + \frac{4R^2(t+1)^4}{\delta^2}\right) (\theta^\tau \phi_t)^2(\omega^*) + 1 \\ &= \left(2 + \frac{4R^2(t+1)^4}{\delta^2}\right) [y_{t+1}(\omega^*) - w_{t+1}(\omega^*)]^2 + 1 \\ &= \left(4 + \frac{8R^2(t+1)^4}{\delta^2}\right) (y_{t+1}^2(\omega^*) + K_2) + 1 \\ &\leq \left(4 + K_1(t+1)^4\right) (y_{t+1}^2(\omega^*) + K_2) + 1, \end{aligned}$$

where $K_1 = \frac{8R^2}{\delta^2}$ is a constant. Hence the proof is completed. ■

B The Proof of Lemma 4.2

It is not hard to find that the determinate of matrices P_{t+1}^{-1} and Q_{t+1}^{-1} are determined by the largest three elements y_{t-1}, y_t, y_{t+1} from the proof bellow.

The proof of Lemma 4.2. By (29) and the assumption $|y_t| \geq |y_{t-1}|^\lambda$, we have

$$\begin{aligned} &|Q_{t+1}^{-1}| \\ &= |KI + t^2 \sum_{i=0}^{t-1} \phi_i \phi_i^\tau + \phi_t \phi_t^\tau| \geq |\phi_t \phi_t^\tau + \phi_{t-1} \phi_{t-1}^\tau| \\ &= |f(y_t) u_{t-1} - f(y_{t-1}) u_t|^2 \quad (B.1) \\ &= \vartheta_2^2 |f(y_t)(w_t - y_t) - f(y_{t-1})(w_{t+1} - y_{t+1})|^2 \quad (B.2) \\ &\geq \underline{\vartheta}_2^2 \cdot \left(|f(y_t) y_t - f(y_{t-1}) y_{t+1}| - 2w |f(y_t)|\right)^2, \end{aligned}$$

where the last equality follows from (14) in the proof of sufficiency.

Now, we estimate $|P_{t+1}^{-1}|$. Let

$$I_{i,j} = [f(y_i)u_j - f(y_j)u_i]^2,$$

then it can be calculated that

$$\begin{aligned}
&|P_{t+1}^{-1}| \\
&= |KI + (t+1)^2 \sum_{i=0}^{t} \phi_i \phi_i^T| \\
&= \begin{vmatrix} K + (t+1)^2 \sum_{i=0}^{t} f^2(y_i) & (t+1)^2 \sum_{i=0}^{t} f(y_i)u_i \\ (t+1)^2 \sum_{i=0}^{t} f(y_i)u_i & K + (t+1)^2 \sum_{i=0}^{t} u_i^2 \end{vmatrix} \\
&= \left(K + (t+1)^2 \sum_{i=0}^{t} f^2(y_i) \right) \left(K + (t+1)^2 \sum_{i=0}^{t} u_i^2 \right) - \\
&\quad (t+1)^4 \left(\sum_{i=0}^{t} f(y_i)u_i \right)^2 \\
&= (t+1)^4 \sum_{0 \le i < j \le t} I_{i,j} + \\
&\quad K(t+1)^2 \left(\sum_{i=0}^{t} f^2(y_i) + \sum_{i=0}^{t} u_i^2 \right) + K^2. \qquad \text{(B.3)}
\end{aligned}$$

First, notice that similar to (B.1)-(B.2),

$$I_{i,j} = O(y_j^{2b} y_{i+1}^2 + y_i^{2b} y_{j+1}^2). \qquad \text{(B.4)}$$

Hence, we have

$$\begin{aligned}
\sum_{0 \le i < j \le t-1} I_{i,j} &= O\left(\sum_{0 \le i < j \le t-1} (y_j^{2b} y_{i+1}^2 + y_i^{2b} y_{j+1}^2) \right) \\
&= O\left(\frac{t(t-1)}{2} \cdot (y_{t-1}^{2(b+1)} + y_{t-2}^{2b} y_t^2) \right) \\
&= o\left(y_t^{2(b+1)} + y_{t-1}^{2b} y_{t+1}^2 \right) \qquad \text{(B.5)}
\end{aligned}$$

Moreover, by the system (2),

$$\begin{aligned}
u_i^2 &= \left(\frac{(y_{i+1} - w_{i+1}) - \theta_1 f(y_i)}{\theta_2} \right)^2 \\
&= O(y_{i+1}^2 + y_i^{2b}),
\end{aligned}$$

then by the assumption of the lemma, we have

$$\begin{aligned}
\left(\sum_{i=0}^{t} f^2(y_i) + \sum_{i=0}^{t} u_i^2 \right) &= O\left(\sum_{i=0}^{t} (y_{i+1}^2 + y_i^{2b}) \right) \\
&= O\left(y_{t+1}^2 + y_t^{2b} \right) \\
&= o\left(y_t^{2(b+1)} + y_{t-1}^{2b} y_{t+1}^2 \right). \qquad \text{(B.6)}
\end{aligned}$$

Moreover, apparently,

$$K^2 = o\left(y_t^{2(b+1)} + y_{t-1}^{2b} y_{t+1}^2 \right). \qquad \text{(B.7)}$$

Substituting (B.5)-(B.7) into (B.3), we have for some random variable $M > 0$,

$$\begin{aligned}
|P_{t+1}^{-1}| &= O\left((t+1)^4 (y_t^{2(b+1)} + y_{t-1}^{2b} y_{t+1}^2) \right) \\
&\le M(t+1)^4 \max\left\{ y_t^{2(b+1)}, y_{t-1}^{2b} y_{t+1}^2 \right\}
\end{aligned}$$

Hence, the proof is completed. ■