

Is measurement-based feedback still better for quantum control systems?

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Abstract

In this paper, we put forward a fundamental question concerning feedback control of quantum systems: Is measurement-based feedback control still better than open-loop control? In contrast to the classical control theory, the answer is far from obvious. This is because measurement-based feedback needs measurement to reduce the system uncertainty, whereas the measurement on a quantum system will inevitably increase the system uncertainty in turn. In fact, there is a complicated tradeoff between the uncertainty introduced and the information gained by the measurement on a quantum system. To investigate this fundamental problem, we will only focus on a typical model of coherent control mode with and without the decoherence term in the paper. By establishing some fundamental limits on the performances of both the open-loop and measurement-based feedback controls, we will demonstrate via simulation that the measurement-based feedback control of quantum systems is still superior to the open-loop control in some sense for the typical model under consideration.

Key words: quantum feedback control, open-loop control, capability of control, stochastic master equation, measurement

1 Introduction

In the past decades, much interest has grown around quantum information and computation [1], which further promotes the development of quantum control theory. Many interesting control problems have been analyzed for quantum systems [2], such as optimal control [3], robust control [4,5], parameter estimation [6,7,8,9] and so on. There are two types of quantum control: (i) open-loop control (OLC), and (ii) feedback control. For the OLC, we mean that during the control process, all we can use for designing the control law is the prior information of the system regardless of the systems state. For feedback control, there are two types: (i) measurement-based feedback control (MFC) [10], and (ii) coherent feedback control [5,11,12]. During the MFC process, we will perform measurement on the system to get some information of the system state and then design the control law based on the estimation of the state. Compared with the MFC, coherent feedback control does not involve measurement—the controller and the system plant can be both quantum systems and are coherently connected. Note that what we are really interested in is the state of the system, so for the coherent feedback, if we only consider the performance of the system, we have to take the partial trace over the controller from the entire connected system. Therefore, generally speaking, the controller itself will cause the quantum decoherence to the controlled system even though it coherently entangles with the system [11]. Thus, whether the coherent feedback is better than the OLC for quantum control systems need to be investigated in depth. However, in this paper, we only compare the effect of the OLC strategy with the MFC strategy.

There has been extensive research on either quantum OLC or quantum MFC. For example, the controllability of OLC has been investigated in depth for

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quantum systems including coherent [13,14,15,16] and incoherent mode [17,18]. In the quantum MFC case, since the measurement induces quantum-state-reduction, it can be regarded as a means of control itself [19], and can be optimized during the control process [20,21,22,23]. So far, to the best of the authors' knowledge, the effect of the quantum OLC and MFC has not been compared through a well defined model [24]. It is well known that in classical control systems, measurement-based feedback control is much superior to the OLC in dealing with uncertainties in initial conditions, or model parameters, or external disturbances, or all. There are some fundamental results concerning the maximum capability of feedback in dealing with uncertainties; see, e.g. [25,26]. In the control of quantum systems, a fundamental question is that whether the MFC is still superior to the OLC in dealing with uncertainties (such as the initial state being not pure or the decoherence due to spontaneous emission). Note that this question is far from trivial. In quantum MFC, the measurement required inevitably introduces another kind of uncertainty to the control system besides the initial uncertainty [27]. Hence, we must deal with these two kinds of uncertainties during the feedback control process. The measurement-introduced uncertainty we consider has nothing to do with the measurement accuracy and is inherent in the measurement-based feedback control of quantum systems. This is very different from the classical control systems where the impact of the measurement on the system state can be ignored. In this paper, we will only consider a typical coherent model and show that the MFC is still better for quantum systems under consideration, because the additional measurement-introduced uncertainty turns out to be helpful in some sense. However, if we only consider the effect of the measurement (i.e., there is no feedback control channel), it will definitely increase the system entropy. We will first demonstrate that when there is no decoherence, the MFC is superior to the OLC in dealing with uncertainties in initial states. Then, we will mainly focus on comparing the effect of the OLC and MFC in dealing with decoherence.

The paper is organized as follows. In Section 2, we briefly introduce the coherent control model and set up the control problem. We compare the OLC and MFC for the cases of dealing with the initial uncertainty and decoherence in Section 3 and 4, respectively. Section 5 concludes.

2 A standard model of coherent control

We will briefly describe the model to be used (see e.g. the Sec. 3 and 4 of Ref. [28] for more details). Consider an atomic ensemble consisting of N atoms which are placed into a leaky single mode optical cavity. We consider the (x, y, z) -configuration space and assume that the atomic transitions are far detuned from the cavity resonance, so that the interaction between the atoms and the cavity can be described as $H_{AC} = \hbar\chi F_z b^\dagger b$, where F_z is the spin- $N/2$ collective dipole moment of the ensemble, b is the annihilation operator of the cavity mode, and χ determines the coupling strength. We furthermore consider the atomic Hamiltonian $H_A = \hbar\Delta F_z + \hbar u(t) F_y$, where Δ is the atomic detuning, and $u(t)$ is the strength of a magnetic field in the y-direction and serves as the control input. Similar to F_z , F_y is also a spin- $N/2$ collective dipole moment of the ensemble.

In the OLC case, we do not perform any measurement and we only regulate $u(t)$ to adjust the system state, while in the case of MFC, there must be an additional measurement channel. Here, we adopt the following measurement methods: a probe laser is injected into the cavity (along z -direction) by a beamsplitter and the optical field is configured to good approximation so that it only interacts with the collective angular momentum degrees of all the atoms. After interacting with the system, the outgoing optical field is detected by a homodyne detection. An ideal homodyne detection measures a quadrature

of the optical field. Denote the field observable as $a_t + a_t^\dagger$ ¹, then after the optical field has interacted with the atomic ensemble, what we observed is the photocurrent $I(t) = U_t^\dagger(a_t + a_t^\dagger)U_t$, where U_t is the unitary evolution of the entire system². As usual, we prefer to deal mathematically with the integrated form, hence, we define the integrated photocurrent $y_t = U_t^\dagger(A_t + A_t^\dagger)U_t$ (equation (41) of Ref. [28]) as our observation, where A_t is the quantum Itô integral of a_t . Compared to the OLC, in the MFC, we can first estimate the system state through the observation information (i.e., the quantum filtering) and then design the real time control law to adjust the system. A rigorous treatment of the quantum stochastic calculus and quantum filtering can be found in Ref. [29].

Suppose that the initial state of the system is $\rho_0 = \sum_{i=1}^n p_i \rho_i$, where the states ρ_i has the corresponding probability p_i , $\sum_{i=1}^n p_i = 1$, $n \geq 2$. Our target is to prepare a desired eigenstate ρ_f of F_z with a high fidelity.

3 The Case Without Decoherence

For the above model, let us first consider the case where the spontaneous emission is neglected. The OLC and MFC evolution equations are described respectively as follows [28]:

$$\frac{d\rho_t}{dt} = -i[\Delta F_z + u(t)F_y, \rho_t], \quad (1)$$

$$\begin{aligned} d\rho_t = & -iu(t)[F_y, \rho_t]dt - is[F_z, \rho_t]dt + M\mathcal{D}[F_z]\rho_t dt \\ & + \sqrt{M\eta}\mathcal{H}[F_z]\rho_t dW_t, \end{aligned} \quad (2)$$

where ρ is the density operator; s is determined by some experimental parameters, such as χ , Δ and so on; M is the effective interaction strength

¹ The definition of a_t is equation (22) of Ref. [28].

² The definition of U_t is equation (39) of Ref. [28].

depending on χ , and the amplitude of the laser and so on. η is determined by the efficiency of the photodetectors. The superoperators \mathcal{D} and \mathcal{H} are defined by

$$\begin{aligned}\mathcal{D}[\Lambda]\rho &= \Lambda\rho\Lambda^\dagger - \frac{1}{2}(\Lambda^\dagger\Lambda\rho + \rho\Lambda^\dagger\Lambda), \\ \mathcal{H}[\Lambda]\rho &= \Lambda\rho + \rho\Lambda^\dagger - \text{Tr}(\Lambda\rho + \rho\Lambda^\dagger)\rho;\end{aligned}$$

The innovation process W_t satisfies

$$dW_t = dy_t - 2\sqrt{M\eta}\text{Tr}(F_z\rho_t)dt,$$

where y_t is the observation process. An important result is that the innovation process W_t is in fact a Wiener process [29,30,31].

Note that in contrast to the classical control theory, the evolution equations of the quantum system under the OLC and MFC are different. This is due to the back action effects of the measurement on the quantum systems which consist of the deterministic drift part and the uncertainty part. Specially, equation (1) describes the evolution of the OLC, there is no measurement, while in the MFC case, equation (2) describes the evolution under additional measurement. The corresponding additional terms $M\mathcal{D}[F_z]\rho_t dt$, $\sqrt{M\eta}\mathcal{H}[F_z]\rho_t dW_t$, and the change from Δ in (1) to s in (2) describe the back action effects of the measurement.

In the current case where the spontaneous emission is neglected, equations (1) and (2) have been investigated in depth; see, e.g. [32,33,34]. Here we proceed to compare the effect of OLC and MFC in dealing with the uncertainties of the initial states, by proving the following claim:

Claim 1 *It is impossible to change the entropy of system (1) by using any OLC strategy, but there exists a measurement-based feedback control law which can globally stabilize (2) around ρ_f almost surely.*

Proof. Since we know that equation (1) represents a unitary evolution, the uncertainty of the initial state will never be reduced. This can be seen from

the von Neumann entropy

$$S(\rho) = -\text{Tr}(\rho \log(\rho))$$

which has the following basic properties:

- 1) the entropy is zero if and only if the state is pure;
- 2) it is invariant under the unitary evolution.

Hence, $S(\rho_t^{\Delta, u}) \equiv S(\rho_0) > 0$ for all t , Δ , and admissible control law $u(t)$ (which makes the evolution equation has a unique solution). Since $S(\rho_f)=0$, we cannot prepare the target state no matter how we choose the admissible control law and Δ by using the OLC strategy.

On the other hand, it is known from Theorem 4.2 of Ref. [34] that there exists control law $u_t(\cdot)$ *globally* stabilizing (2) around ρ_f almost surely as $t \rightarrow \infty$. Hence, in this sense we can prepare the target state with probability 1 and say that the quantum MFC is still superior to the OLC in dealing with uncertainties of the initial states for the case we consider. \square

We remark that, if there is no feedback control besides measurement, i.e., $u(t) \equiv 0$ in equation (2), ρ_t will approximate one of the eigenstates ρ_i^z of F_z with corresponding probability $p_i = \text{Tr}(\rho_i^z \rho_0)$, $i = 1, \dots, n$ [35]. This is equivalent to a projective measurement. As we know, the projective measurement increases system entropy [1]. We conclude that if we only consider the effect of the measurement, it increases the system uncertainty. Nonetheless, if we further use the measurements for feedback control, this is no longer the case. We note that the measurement-introduced uncertainty turns out to be helpful by providing a trend of collapsing to the desired state which we can use during the control process.

4 The case with decoherence

In the following, we further consider the decoherence due to the spontaneous emission. After the decoherence term is included phenomenologically, the corresponding OLC and MFC evolution equations become (3) and (4), respectively [28],

$$\frac{d\rho_t}{dt} = -i[\Delta F_z + u(t)F_y, \rho_t] + \gamma\mathcal{D}[\sigma]\rho_t, \quad (3)$$

$$\begin{aligned} d\rho_t = & -iu(t)[F_y, \rho_t]dt - is[F_z, \rho_t]dt + M\mathcal{D}[F_z]\rho_tdt \\ & + \gamma\mathcal{D}[\sigma]\rho_tdt + \sqrt{M\eta}\mathcal{H}[F_z]\rho_t dW_t, \end{aligned} \quad (4)$$

where γ is the decoherence strength and σ is the atomic decay operator.

Similar to the case without decoherence, there is inevitably a measurement channel in the MFC, and the corresponding measurement back action effects lead to the differences between the feedback evolution equation (4) and the OLC evolution equation (3).

In the following, assume the initial state is ρ_0 , and we use $F(\rho) = \text{Tr}(\rho_f\rho)$ as the fidelity between the state ρ and the target state ρ_f .

In the current case of decoherence, we will only consider the case of a spin which may be used as a qubit for simplicity, i.e., the two-dimensional case.

Suppose $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are the two eigenvectors of F_z . Under this representation,

$$F_z = \frac{1}{2}(|1\rangle\langle 1| - |0\rangle\langle 0|) = -\frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$F_y = \frac{1}{2}(i|0\rangle\langle 1| - i|1\rangle\langle 0|) = -\frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$\sigma = |0\rangle\langle 1| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Suppose $\rho = \frac{1}{2} \begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}$. Corresponding to (3) and (4), we have the following OLC and MFC evolution equations, respectively

$$\begin{aligned} \frac{dx_t}{dt} &= -\frac{\gamma}{2}x_t - u_t z_t + \Delta y_t \\ \frac{dy_t}{dt} &= -\frac{\gamma}{2}y_t - \Delta x_t \\ \frac{dz_t}{dt} &= \gamma(1-z_t) + u_t x_t, \end{aligned} \quad (5)$$

$$\begin{aligned} dx_t &= -\frac{\gamma+M}{2}x_t dt - u_t z_t dt + s y_t dt + \sqrt{M\eta} x_t z_t dW_t \\ dy_t &= -\frac{\gamma+M}{2}y_t dt - s x_t dt + \sqrt{M\eta} y_t z_t dW_t \\ dz_t &= \gamma(1-z_t) dt + u_t x_t dt - \sqrt{M\eta}(1-z_t^2) dW_t. \end{aligned} \quad (6)$$

Assume $\rho_f = |1\rangle\langle 1|$ ³ (i.e., $z_f = -1$), under the above representation,

$$F(\rho) = \text{Tr}(\rho_f \rho) = \frac{1-z}{2}.$$

In the following sections, we will compare the OLC and MFC in dealing with decoherence.

³ It is easy to see that when $u \equiv 0$, the state of (5) and (6) will approximate to the eigenstate $|0\rangle\langle 0|$ of F_z corresponding to $z = 1$. Hence, we only consider the case of $\rho_f = |1\rangle\langle 1|$.

4.1 The Limits of Open-loop Control

At first, we consider the case of the OLC strategy. We will give some limits of preparing the target state by the OLC strategy. We have the following propositions.

Proposition 4.1 *For the model (5), suppose $|u| \leq B$, and denote $z_* = \frac{\gamma^2 - B^2}{\gamma^2 + B^2}$. If $z_0 \geq z_*$, the trajectory of z_t will be bounded from below by z_* , i.e., $\inf_{t \geq 0} z_t \geq z_*$; and if $z_0 < z_*$, for arbitrary $\varepsilon > 0$, there exists $T(\varepsilon, z_0)$, such that $z_t > z_* - \varepsilon$ for all $t > T$.*

Proof. Under the above assumptions, it is easy to know that (5) has a unique solution. It is also not difficult to see

$$x_t^2 + y_t^2 + z_t^2 \leq 1 \quad , \quad -1 \leq z_t \leq 1 \quad . \quad (7)$$

Hence, for arbitrary $t \in [0, \infty)$, we have

$$\begin{aligned} \frac{dz_t}{dt} &= \gamma - \gamma z_t + u_t x_t \\ &\geq \gamma - \gamma z_t - B \sqrt{1 - z_t^2} . \end{aligned} \quad (8)$$

Consider ordinary differential equation (ODE)

$$\begin{aligned} \frac{dx}{dt} &= \gamma - \gamma x - B \sqrt{1 - x^2} \\ x(0) &= z_0 \quad . \end{aligned} \quad (9)$$

Let us denote $f(x) = \gamma - \gamma x - B \sqrt{1 - x^2}$. Note that $f(x)$ is Lipschitz continuous for $x \in (-1, 1)$. Thus for arbitrary point $x(0) \in (-1, 1)$, the ODE (9) has a unique integral curve which can be extended to the boundary. Note that the ODE (9) has two special solutions

$$x_1 \equiv 1 \quad , \quad x_2 \equiv \frac{\gamma^2 - B^2}{\gamma^2 + B^2} \quad ,$$

and

$$\begin{cases} \gamma - \gamma x - B\sqrt{1-x^2} > 0, & 0 < x < x_2; \\ \gamma - \gamma x - B\sqrt{1-x^2} < 0, & x_2 < x < 1. \end{cases}$$

Suppose Γ is the corresponding integral curve of $x_2(t) \equiv \frac{\gamma^2 - B^2}{\gamma^2 + B^2}$. If the initial state $z_0 \in (0, x_2)$, the corresponding integral curve L_{z_0} is always below Γ and is monotonically increasing. It cannot intersect with Γ because of the uniqueness of solution. Hence, the solution starting from $x(0) = z_0$ is unique in $[0, \infty)$, and increasingly approximates to $x_2(t)$. Similarly, if $x(0) \in (x_2, 1)$, the solution is also unique in $[0, \infty)$, and decreasingly approximates to $x_2(t)$. From the comparison theorem, we also have $z_t \geq x_{t^*}$, $t \in [0, \infty)$, where x_{t^*} is the smallest solution of the ODE (9) which starts from $x(0) = z_0$. Hence, if $x_2 \leq z_0 \leq 1$, we have $\inf_{t \geq 0} z_t \geq \frac{\gamma^2 - B^2}{\gamma^2 + B^2}$; and if $0 \leq z_0 < x_2$, for arbitrary $\varepsilon > 0$, there exists $T(\varepsilon, z_0)$, such that $z_t > \frac{\gamma^2 - B^2}{\gamma^2 + B^2} - \varepsilon$ whenever $t > T$. \square

The result of Proposition 4.1 does not depend on Δ . It gives an upper bound of the fidelity ($F(\cdot)$) when we prepare the target state using the OLC strategy. Generally, a state having a high fidelity with the target state (for example, greater than 90%) is satisfactory. However, we have the following proposition for the OLC strategy.

Proposition 4.2 *For the model (5), for arbitrary initial state and arbitrary admissible control law (such that model (5) has a unique solution), we have*

$$\limsup_{t \rightarrow \infty} z_t \geq 0, \text{ i.e., } \liminf_{t \rightarrow \infty} F_t \leq 50\%$$

Proof. Denote $V_t = x_t^2 + y_t^2 + z_t^2$. From (5), it is easy to get

$$\frac{dV_t}{dt} = -\gamma V_t - \gamma z_t^2 + 2\gamma z_t.$$

Hence,

$$V_t = e^{-\gamma t} V_0 + e^{-\gamma t} \int_0^t e^{\gamma s} (-\gamma z_s^2 + 2\gamma z_s) ds \quad .$$

We use a contradiction argument. If there exists $\varepsilon > 0$ and admissible control law $u(\cdot)$, together with $T(\varepsilon, u)$, such that $z_t < -\varepsilon$ whenever $t > T(\varepsilon, u)$, we have

$$\begin{aligned} V_t &= e^{-\gamma t} V_0 + e^{-\gamma t} \int_0^t e^{\gamma s} (-\gamma z_s^2 + 2\gamma z_s) ds \\ &= e^{-\gamma t} V_0 + e^{-\gamma t} \int_0^T e^{\gamma s} (-\gamma z_s^2 + 2\gamma z_s) ds \\ &\quad + e^{-\gamma t} \int_T^t e^{\gamma s} (-\gamma z_s^2 + 2\gamma z_s) ds \\ &< \varepsilon^2 \quad \text{as } t \rightarrow \infty, \end{aligned}$$

However, when $t > T(\varepsilon, u)$, we have

$$V_t = x_t^2 + y_t^2 + z_t^2 > \varepsilon^2.$$

This is a contradiction. \square

Proposition 4.2 shows that in the decoherence case, we cannot *always* prepare the target state with a high fidelity (at least greater than 50%) after some limited time by the OLC strategy, no matter how to regulate Δ and what admissible control law is performed.

4.2 The Limits of Measurement-based Feedback Control

On the other hand, we have the following propositions when using the MFC strategy.

Proposition 4.3 *For the model (6), for arbitrary admissible control law (such that model (6) has a unique strong solution [36]), we have $x_t^2 + y_t^2 + z_t^2 \leq 1$ almost surely if $x_0^2 + y_0^2 + z_0^2 \leq 1$.*

Proof. Denote $V_t = x_t^2 + y_t^2 + z_t^2 - 1$. From (6) and Itô formula, we have

$$\begin{aligned} dV_t &= -\gamma V_t dt - M\eta(1 - z_t^2)V_t dt - \gamma(1 - z_t)^2 dt \\ &\quad - M(1 - \eta)(x_t^2 + y_t^2) dt + 2\sqrt{M\eta} z_t V_t dW_t . \end{aligned}$$

From the Lemma in the Appendix and $V_0 \leq 0$, we have

$$\begin{aligned} V_t &= \exp\{\xi_t\} V_0 \\ &\quad - \int_0^t \exp\{\xi_t - \xi_s\} (M(1 - \eta)(x_s^2 + y_s^2) + \gamma(1 - z_s)^2) ds \\ &\leq 0 \quad , \quad a.s. \end{aligned}$$

where

$$\xi_t = -(\gamma + M\eta)t - M\eta \int_0^t z_s^2 ds + 2\sqrt{M\eta} \int_0^t z_s dW_s. \quad \square$$

Proposition 4.4 *For the model (6), suppose $|u| \leq B$, and denote $z_* = \frac{\gamma^2 - B^2}{\gamma^2 + B^2}$. If $z_0 \geq z_*$, the trajectory of Ez_t will be bounded from below by z_* , i.e., $\inf_{t \geq 0} Ez_t \geq z_*$; and if $z_0 < z_*$, for arbitrary $\varepsilon > 0$, there exists $T(\varepsilon, z_0)$, such that $Ez_t > z_* - \varepsilon$ for all $t > T$.*

Proof. Under the above assumptions, it is easy to know that equation (6) has a unique strong solution. From Proposition 4.3, we have

$$\begin{aligned} \frac{dEx_t}{dt} &= -\frac{\gamma + M}{2} Ex_t - Eu_t z_t + sEy_t \\ \frac{dEy_t}{dt} &= -\frac{\gamma + M}{2} Ey_t - sEx_t \\ \frac{dEz_t}{dt} &= \gamma(1 - Ez_t) + Eu_t x_t \end{aligned} \tag{10}$$

It is easy to see that for arbitrary $t \in [0, \infty)$,

$$\begin{aligned} \frac{dEz_t}{dt} &= \gamma - \gamma Ez_t + Eu_t x_t \\ &\geq \gamma - \gamma Ez_t - B\sqrt{1 - E^2 z_t} . \end{aligned}$$

Consider ordinary differential equation (ODE)

$$\begin{aligned}\frac{dx}{dt} &= \gamma - \gamma x - B\sqrt{1-x^2} \\ x(0) &= z_0 \quad .\end{aligned}$$

Note that this is similar to (9), so Proposition 4.4 can be obtained by a similar method of Proposition 4.1. \square

Note that z_* does not depend on s and the effective interaction strength M . This proposition is different from the results in Ref. [28,34]. In Ref. [28], it points out that the effect of γ can be neglected when $\gamma \ll M$; and from Ref. [34], we further know that we can approximately prepare the target state almost surely. However, our result shows that no matter how large M is, the ability of preparing the target state is limited by the decoherence strength γ and control strength B .

In the following, we will see whether we can *always* prepare the target state with a high average fidelity (at least greater than 50%) after some limited time by using the MFC strategy ⁴. First, we give some necessary conditions. From Proposition 4.4, in order to prepare a state with an average fidelity *always* higher than 50% after some limited time, the control strength B must be greater than the decoherence strength γ (i.e., $z_* < 0$). If we further consider the effective interaction strength M , we have the following proposition. In some sense, it gives another necessary condition for preparing the target state *always* with a high average fidelity (greater than $\frac{1-z_*}{2}$) after some limited time. It depends on the mutual relations of γ , M , η , and B .

Proposition 4.5 *For the model (6), suppose $|u| \leq B$. Given arbitrary $z^* \in [z_*, 0]$, if*

- 1) $\alpha < 1$;

⁴ Note that in the OLC case, the fidelity is naturally an average fidelity because the evolution equation (5) is derived by averaging the environment degrees.

2) $\gamma(1 - z^*) > B\alpha$;

where $\alpha = C + D$, $C = \sqrt[3]{\frac{B}{M\eta} + \sqrt{Q}}$, $D = \sqrt[3]{\frac{B}{M\eta} - \sqrt{Q}}$, and $Q = \left(\frac{\gamma + M(1-\eta)}{3M\eta}\right)^3 + \left(\frac{B}{M\eta}\right)^2$, we have $\limsup_{t \rightarrow \infty} Ez_t \geq z^*$ (i.e., $\liminf_{t \rightarrow \infty} F_t \leq \frac{1-z^*}{2}$) for arbitrary initial state.

Proof. We use a contradiction argument. Suppose under conditions 1) and 2) that there exists ρ_0 , $z^* \in [z_*, 0]$, and admissible control law, such that when t is sufficient large, $Ez_t < z^*$.

Denote $V_t = x_t^2 + y_t^2$. From (6) and Itô formula, we have

$$\begin{aligned} dV_t &= -(\gamma + M)V_t dt + M\eta z_t^2 V_t dt - 2u_t x_t z_t dt \\ &\quad + 2\sqrt{M\eta} z_t V_t dW_t . \end{aligned}$$

Hence, from Proposition 4.3,

$$\begin{aligned} \frac{dEV_t}{dt} &= -(\gamma + M)EV_t + M\eta Ez_t^2 V_t dt - 2Eu_t x_t z_t \\ &\leq -(\gamma + M(1 - \eta))EV_t - M\eta E^2 V_t + 2BE^{\frac{1}{2}} V_t . \end{aligned}$$

Consider ODE

$$\begin{aligned} \frac{dx}{dt} &= -(\gamma + M(1 - \eta))x - M\eta x^2 + 2Bx^{\frac{1}{2}} \\ x(0) &= x_0 \end{aligned} \tag{11}$$

Suppose $f(x) = -(\gamma + M(1 - \eta))x - M\eta x^2 + 2Bx^{\frac{1}{2}}$. $f(x)$ is Lipschitz continuous for $x \in (0, 1)$. For arbitrary point $x_0 \in (0, 1)$, ODE (11) has a unique corresponding integral curve which can be extended to the boundary. Note that ODE (11) has two special solutions

$$x_1 \equiv 0 , \quad x_2 \equiv \alpha^2 ,$$

and

$$\begin{cases} f(t, x) > 0 , & 0 < x < \alpha^2 & ; \\ f(t, x) < 0 , & \alpha^2 < x < 1 & . \end{cases}$$

Hence, from the existence and uniqueness of the solution and the comparison theorem, for arbitrary $\varepsilon > 0$, there exists T_ε , such that $E(x_t^2 + y_t^2) < \alpha^2 + \varepsilon$, whenever $t > T_\varepsilon$.

On the other hand, from the hypothesis and equation (10), we have for $t_1 > t_2 > T_\varepsilon$,

$$\begin{aligned} Ez_{t_1} &= Ez_{t_2} + \int_{t_2}^{t_1} \gamma(1 - Ez_s) + Eu_s x_s ds \\ &\geq Ez_{t_2} + \int_{t_2}^{t_1} \gamma(1 - z^*) - BE^{\frac{1}{2}} V_s ds \\ &\geq Ez_{t_2} + [\gamma(1 - z^*) - B(\alpha^2 + \varepsilon)^{\frac{1}{2}}](t_1 - t_2) \\ &\rightarrow \infty \end{aligned}$$

when ε is sufficiently small and $t_1 - t_2 \rightarrow \infty$. This contracts with $Ez_{t_1} < z^*$.

□

The following proposition points out the limit of preparing the target state by using the MFC strategy.

Proposition 4.6 *For the model (6), for arbitrary initial state and arbitrary admissible control law, we have $\limsup_{t \rightarrow \infty} Ez_t \geq \frac{l - \sqrt{l^2 + l + 1}}{l + 1}$ (i.e., $\liminf_{t \rightarrow \infty} EF_t \leq \frac{1 + \sqrt{l^2 + l + 1}}{2(l + 1)}$), where $l = \frac{\gamma}{M\eta}$.*

Proof. Denote $V_t = x_t^2 + y_t^2 + z_t^2$. From (6) and Itô formula, we have

$$\begin{aligned} dV_t &= -\gamma V_t dt + 2\gamma z_t dt - \gamma z_t^2 dt - M\eta(1 - z_t^2)V_t dt \\ &\quad - M(1 - \eta)(x_t^2 + y_t^2)dt + M\eta(1 - z_t^2)dt \\ &\quad + 2\sqrt{M\eta}(V_t - 1)z_t dW_t \quad . \end{aligned}$$

Hence, from Proposition 4.3, we have

$$\begin{aligned}
\frac{dEV_t}{dt} &= -\gamma EV_t + 2\gamma Ez_t - \gamma Ez_t^2 - M\eta E(1 - z_t^2)V_t \\
&\quad - M(1 - \eta)E(x_t^2 + y_t^2) + M\eta E(1 - z_t^2) \\
&\leq -\gamma EV_t - M(1 - \eta)E(x_t^2 + y_t^2) + 2\gamma Ez_t \\
&\quad - \gamma E^2 z_t - M\eta E(1 - z_t^2)V_t \\
&\quad + M\eta E(1 - z_t)E(1 + z_t).
\end{aligned}$$

Consider inequality

$$\gamma x^2 - 2\gamma x - M\eta(1 - x^2) \geq 0.$$

Its solution in $[-1, 1]$ is $-1 \leq x \leq \frac{l - \sqrt{l^2 + l + 1}}{l + 1}$. We can obtain this proposition by a contradiction argument as used in Proposition 4.2. \square

Proposition 4.6 shows that no matter what admissible control law is performed and how to regulate s , we cannot prepare the target state with average fidelity *always* higher than $\frac{1 + \sqrt{l^2 + l + 1}}{2(l + 1)}$ after some limited time by using the MFC method. Note that l only depends on the decoherence strength γ and effective interaction strength M but not on the control strength B . In contrast to Proposition 4.2, which points out the limitation of the OLC in dealing with decoherence, so far, the propositions cannot tell us whether we can make $\limsup_{t \rightarrow \infty} Ez_t < 0$, (i.e., $\liminf_{t \rightarrow \infty} EF(\rho_t) > 50\%$) by using the MFC strategy. We will answer this question partly via simulation in the next subsection.

4.3 Feedback Controller Design

We first consider a simple feedback law $u_1 = -B(x + \delta)$, where δ is an adjustable parameter, which avoids the trajectory z sticking at 1⁵. The simu-

⁵ It is not difficult to see that, the “worst” case $z = 1$ implies $x = 0$ and hence $u_1 = 0$ if $\delta = 0$. Thus, z will stick at 1 since it is a stable point of equation (6).

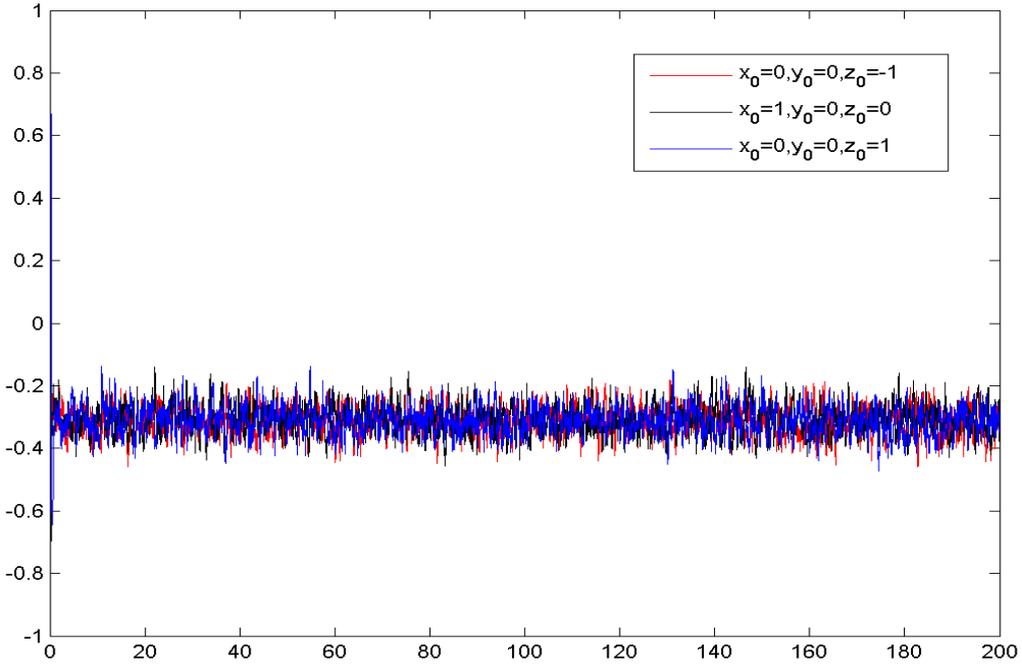


Fig. 1. (color online) Function Ez_t of $t \in [0, 200]$ with different initial states. Simulation results with different parameters are shown in Fig. 1, Fig. 2 and Fig. 3. Here we choose the parameters $s = 0$, $\delta = 0.01$, the step size of the simulations is 5×10^{-4} , and we average 300 sample paths for every curve in Fig. 1, Fig. 2, and Fig. 3. In order to compare the results of different parameters, we choose $\gamma = 1$ in Fig. 1 and Fig. 2, $\gamma = 0.1$ in Fig. 3; $\eta = 1$ in Fig.1 and Fig. 3; $M = 10$, $B = 25$ in Fig. 1, $M = 5$, $B = 15$ in Fig. 3. The other parameters for the curves are shown in the legend, where eta means η .

We find that, in contrast to the OLC case, despite the additional measurement-introduced uncertainty, we can prepare the target state (from an arbitrary initial state) *always* with a high fidelity after some limited time by a simple control law with appropriate controller parameters. In this sense, we conclude that the quantum MFC is superior to the OLC in dealing with decoherence for the case we consider.

By the simulation, we note that in average, Ez_t decreases as γ decreases or B

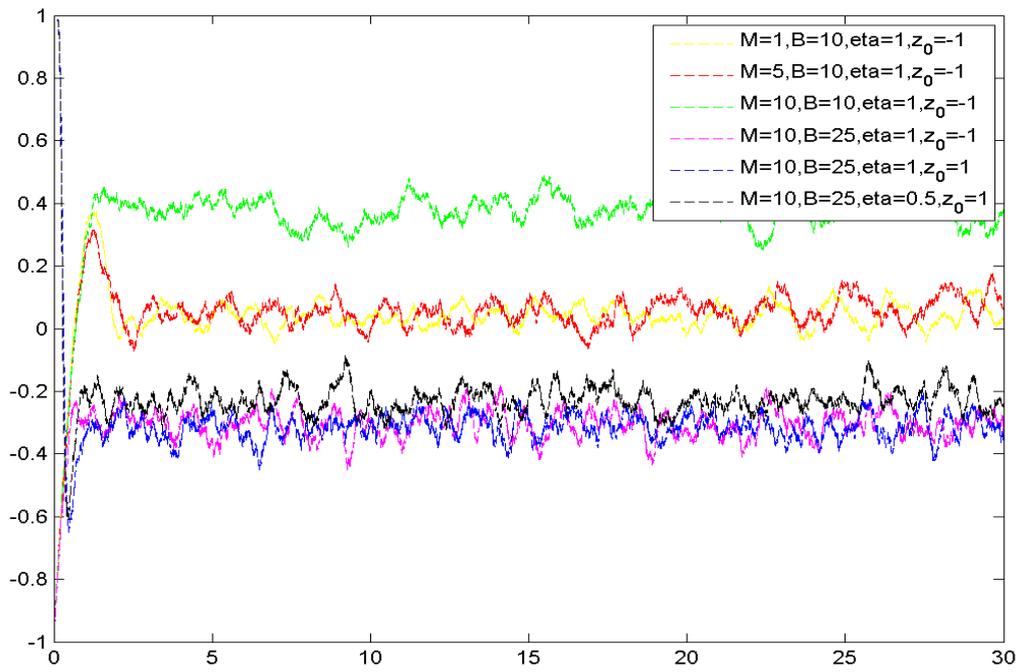


Fig. 2. (color online) Function Ez_t of $t \in [0, 30]$ with different parameters.

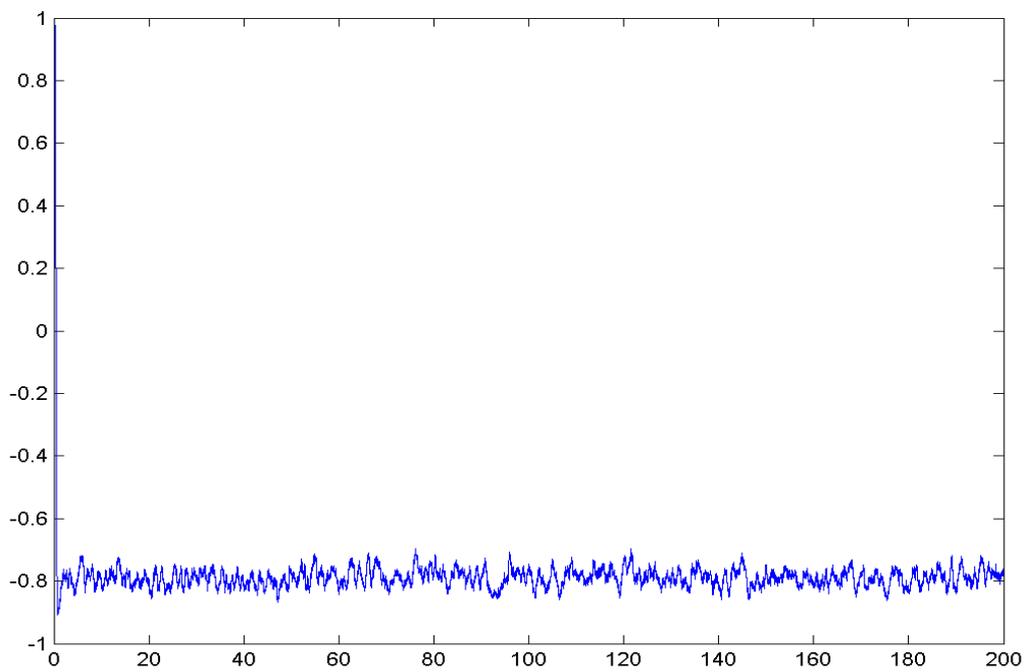


Fig. 3. Function Ez_t of $t \in [0, 200]$.

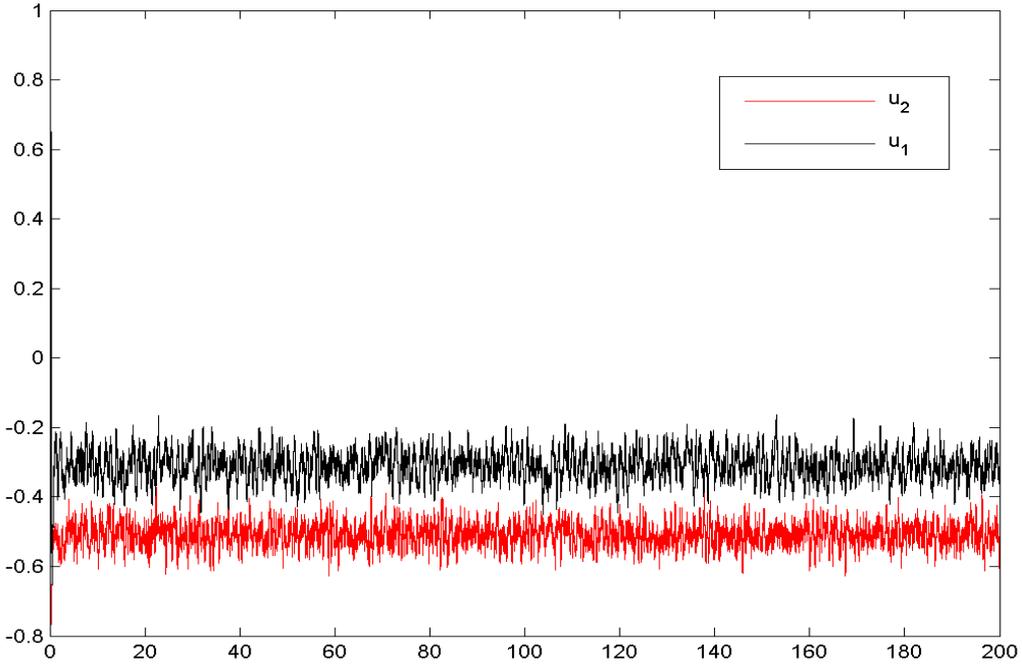


Fig. 4. (color online) Function Ez_t of $t \in [0, 200]$ with different controls.

increases, but not always decreases as M increases. This is because the larger M is, the larger the introduced uncertainty becomes. Hence, in order to deal with it, the control strength must be larger. This agrees with the result of Proposition 4.5. In fact, there are many other control laws that can prepare the target state *always* with a high average fidelity after some limited time. For example, the control law $u_2 = -B(1+z)(x+\delta)$ may do better, as the term $1+z$ adaptively adjusts the control magnitude. The simulation result is shown in Fig. 4. Here we choose $z_0 = 1$, $\gamma = 1$, $s = 0$, $M = 10$, $B = 25$, $\eta = 1$ and $\delta = 0.01$. We average 300 sample paths for every curve, and the step length is 5×10^{-4} .

We note that, on the one hand, measurement-introduced uncertainty provides a trend of collapsing to the target state (the diffusion term of dz in equation (6) equals 0 if and only if $z = \pm 1$) which we can use during the control process; On the other hand, we must deal with this inherent additional uncertainty during

the MFC process. From the simulations and Proposition 4.5, we find that the measurement-introduced uncertainty turns out to be helpful if the control strength is strong enough to overcome the negative effect of the uncertainty.

5 Conclusion

In this paper, we have compared the quantum OLC and MFC strategies through a typical model, and found that the measurement-based feedback control strategy is still better than open-loop control for quantum systems we consider in preparing a desired target state. The measurement-introduced uncertainty which is inherent in the quantum MFC turned out to be helpful in MFC, as it may soften the influence attribute to the uncertainty of the initial state as well as provide possibilities for regulating the system via feedback. It is worth emphasizing that the study of this fundamental problem is just initiated. For further investigation, it would be interesting to compare the effect of OLC and MFC in dealing with parameter or structure uncertainties, or to study other basic quantum control systems and problems. It goes without saying that such an investigation will help us to understand more about the feedback control of quantum systems.

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7 Appendix

Lemma 7.1 *Let $X = (x_t, \mathcal{F}_t)$, $0 \leq t \leq T$, be a random process having the stochastic differential*

$$dx_t = [a(t, \omega)x_t + c(t, \omega)]dt + b(t, \omega)x_t dW_t,$$

where $W = (W_t, \mathcal{F}_t)$ is a Wiener process, and the nonanticipative functions $a(t, \omega)$, $b(t, \omega)$ are such that

$$P\left\{\int_0^T |a(t, \omega)| dt < \infty\right\} = 1,$$

$$P\left\{\int_0^T |b^2(t, \omega)| dt < \infty\right\} = 1.$$

Assume that the stochastic differential has a unique strong solution, then

$$x_t = \exp\{\xi_t\} \left[x_0 + \int_0^t \exp\{-\xi_s\} c(s, \omega) ds \right],$$

where

$$\xi_t = \int_0^t a(s, \omega) ds - \frac{1}{2} \int_0^t b^2(s, \omega) ds + \int_0^t b(s, \omega) dW_s.$$

Proof. Since the stochastic differential has a unique solution, we just need to verify that x_t satisfies the differential equation. Actually, by the Itô formula,

$$\begin{aligned}
& d \exp\{\xi_t\} [x_0 + \int_0^t \exp\{-\xi_s\} c(s, \omega) ds] \\
&= \exp\{\xi_t\} d\xi_t [x_0 + \int_0^t \exp\{-\xi_s\} c(s, \omega) ds] \\
&\quad + \frac{1}{2} \exp\{\xi_t\} (d\xi_t)^2 [x_0 + \int_0^t \exp\{-\xi_s\} c(s, \omega) ds] \\
&\quad + \exp\{\xi_t\} \exp\{-\xi_t\} c(t, \omega) dt \\
&= \exp\{\xi_t\} [x_0 + \int_0^t \exp\{-\xi_s\} c(s, \omega) ds] (d\xi_t + \frac{1}{2} (d\xi_t)^2) \\
&\quad + c(t, \omega) dt \\
&= a(t, \omega) \exp\{\xi_t\} (x_0 + \int_0^t \exp\{-\xi_s\} c(s, \omega) ds) dt \\
&\quad + b(t, \omega) \exp\{\xi_t\} [x_0 + \int_0^t \exp\{-\xi_s\} c(s, \omega) ds] dW_t \\
&\quad + c(t, \omega) dt.
\end{aligned}$$

It does satisfy the equation. \square

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