On Feedback Capability for a Class of Semiparametric Uncertain Systems ⋆

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Abstract

The main purpose of this paper is to understand and characterize the maximum capability of the feedback mechanism for a basic class of scalar discrete-time semiparametric minimum-phase control systems, where both parametric and nonparametric uncertainties are included. We will demonstrate that the necessary and sufficient condition to stabilize the class of systems is $L < \frac{3}{2} + \sqrt{2}$, where L is the Lipschitz constant describing the "size" of the uncertainty of the nonparametric part. This critical value is the same as that first obtained in [11] for a class of purely nonparametric systems, which shows that the capability of feedback is not influenced by the parameterized uncertainty in the systems, as long as the corresponding parametric nonlinear function has a linear growth rate. While the necessity proof is directly obtainable from [11], the main task of of this paper is to prove the sufficiency part.

Key words: Semiparametric model, feedback capability, uncertainty, stability, nonlinear systems, parameter switching.

1 Introduction

The understanding and characterization of the capability of feedback in dealing with system uncertainties has been a central issue in control theory. Adaptive control and robust control have been two typical design methods (cf.e.g., [1]-[6]), either has its own advantages and disadvantages. However, for the purpose of understanding the maximum capability of feedback, the adaptive control design seems to be more relevant, since there is usually a learning mechanism embedded in the adaptive feedback loop, and thus adaptive control is expected to be able to deal with larger class of uncertainties than those can be dealt with by robust control.

The adaptive control of linear systems has been understood fairly well (cf.e.g., [1]-[3]), and some advances for continuous-time nonlinear systems are also available (cf.[4]). But for adaptive control of discrete-time or sampled-data nonlinear systems, essential difficulties will emerge when the growth rate of the nonlinear function involved is faster than linear. Indeed, it has been shown in a series of works (cf.e.g., [7]-[9]) that in this case, there are fundamental limitations to the capability of feedback, unless the parametric nonlinearities has a linear growth rate (cf.[10]) or the sampling rate is fast enough (cf.e.g., [16],[17]). Furthermore, if the uncertainty is characterized by a nonparametric function, feedback still has certain limitations in dealing with this kind of uncertainties, even if the nonparametric function has a linear growth rate [11].

If fact, consider the following basic nonparametric control system as studied in Xie and Guo [11]:

$$y_{t+1} = f(y_t) + u_t + w_{t+1}, \qquad t \geq 0, \tag{1}$$

where the nonlinear function $f(\cdot)$ is assumed to be unknown. To measure the system uncertainty induced by the unknown function $f(\cdot)$ quantitatively, they introduced a semi-norm $\|\cdot\|$ (called the generalized Lipschitz norm) in the space of all nonlinear functions $\mathscr{F}$. For any $L > 0$, define

$$\mathscr{F}'(L) \triangleq \{f \in \mathscr{F} : \|f\| \leq L\}. \tag{2}$$

Then, Xie and Guo [11] found and showed that the necessary and sufficient condition for the existence of a stabilizing feedback control for system (1) with any $f \in \mathscr{F}'(L)$, is $L < \frac{3}{2} + \sqrt{2}$, a number which is beyond one's intuition. This means that $L = \frac{3}{2} + \sqrt{2}$ is the critical value in characterizing the largest "size" of uncertainty for the system to be stabilized by feedback.

⋆ This work was supported by the National Natural Science Foundation of China under grant 60821091.

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Later on, [12] studied the stabilization of a class of nonlinearly parameterized uncertain systems:

$$y_{t+1} = ay_t + f(y_t) + u_t + w_{t+1}, \quad t \geq 0.$$

This model contains an additional unknown parameter a besides the nonparametric uncertainty considered in [11]. The author of [12] used the method of recurrent objective inequalities to estimate the uncertainty, and showed that $L < \frac{3}{2} + \sqrt{2}$ is again a sufficient condition for the system to be stabilized.

In the current paper, we will study uncertain systems based on and far beyond those considered previously in [11], [12] and [13], where [13] studied the adaptive stabilization of a class of nonlinearly parameterized systems. To be precise, we will study a basic class of scalar minimum-phase semiparametric systems which contain both nonparametric and parametric parts. As is well known in statistics, semiparametric models enjoy some flexibility in modeling practical nonlinear systems (cf.[14]). It turns out that the critical value for the feedback capability in this case is still $L = \frac{3}{2} + \sqrt{2}$, regardless of the increased uncertainties in the parametric part. Perhaps, the most striking point is that uncertain parameters in the input channel do not change the capability of feedback in the present case, because in the purely parametric case with nonlinear growing function, the uncertain parameter in the input channel does indeed degrade the capability of feedback (cf.e.g., [7], [8]). Related results but for a much simplified model were recently presented in [15]. While the necessity proof is directly obtainable from [11], the main task of the current paper is to prove the sufficiency part.

In the rest of the paper, we will present the main results in Section 2, and give their proofs in Section 3. Finally, Section 4 will conclude the paper with some remarks.

2 Main results

Consider the following scalar discrete-time semiparametric control model:

$$y_{t+1} = f(y_t) + g(\theta, \phi_t, u_t) + w_{t+1}, \quad t \geq 0, \tag{3}$$

where $\{y_t\}, \{u_t\}$ and $\{w_t\}$ are the system output, input and disturbance sequences, respectively, $\theta \in R^m$ is an unknown parameter vector, and $\phi_t = (y_t, \cdots y_{t-p+1}, u_{t-1}, \cdots u_{t-q+1})$ is the regression vector with $p \geq 1, q \geq 2$, and $f(\cdot)$, $g(\cdot)$ are nonlinear functions. We make the following assumptions:

A1) The unknown parameter vector $\theta = (\beta_1, \beta_2, \cdots, \beta_m)$ belongs to a bounded rectangle in R^m, denoted by $\Theta = \{\theta : |\beta_i| \leq K, 1 \leq i \leq m\}$, where $K > 0$ is a known constant.

A2) The nonlinear function $f(\cdot) : R^1 \longrightarrow R^1$ is unknown *a priori*, but belongs to a class of generalized Lipschitz functions denoted by $\mathscr{F}(L) \triangleq \{f(\cdot) : |f(x_1) - f(x_2)| \leq L|x_1 - x_2| + \gamma, L > 0, \gamma > 0, \forall\, x_1, x_2 \in R^1\}$.

A3) The nonlinear function $g(\cdot) : R^{m+p+q} \to R^1$ is known and differentiable with respect to each argument, which satisfies for any $\theta \in R^m, \phi_t \in R^{p+q-1}, u_t \in R^1$:

1) $\|\frac{\partial g(\theta,\phi_t,u_t)}{\partial \theta}\| \leq M(\|\phi_t\| + |u_t|)$, $\forall t \geq 0$, where $M > 0$ is a known constant and $\|\cdot\|$ is the Euclidean norm;

2) $\|\frac{\partial g(\theta,\phi_t,u_t)}{\partial \phi_t}\| \leq M, \forall t \geq 0$;

3) $|\frac{\partial g(\theta,\phi_t,u_t)}{\partial u_t}| \geq b, \forall t \geq 0$, where b is a positive number.

A4) The unknown disturbance sequence $\{w_t\}$ has a known upper bound w, i.e.,

$$|w_t| \leq w, \quad \forall\, t \geq 0.$$

A5) (Minimum-phase condition). There are constants $c_1 > 0$, $c_2 > 0$ and $\lambda \in (0, 1)$ such that the input sequence is bounded by the output and disturbance sequences in the sense that,

$$u_{t-1}^2 \leq c_1^2 \sum_{i=-p+1}^{t} \lambda^{2(t-i)}(y_i^2 + w_i^2) + c_2^2, \quad \forall t \geq 1. \tag{4}$$

Remark 2.1 Assumption **A1)** requires that the unknown parameter vector θ lies in a compact set; Assumption **A2)** is the generalized Lipschitz condition first introduced in [11]. Assumption **A3)** can be regarded as certain linear growth condition on the nonlinear function $g(\cdot)$. This condition is necessary in a certain sense because there is an unknown nonparametric nonlinear function in the system, and there are no constraints on the values of p and q (cf.[10]); The minimum-phase condition in Assumption **A5)** is also a basic requirement even in the case where the parametric part is linear: $y_{t+1} = f(y_t) + a_1 y_t + \cdots + a_p y_{t-p+1} + b_1 u_t + \cdots + b_q u_{t-q+1} + w_{t+1}$, for which Condition **A5)** is equivalent to the standard minimum-phase condition meaning that all the roots of the polynomial $b_1 + b_2 z + \cdots + b_q z^{q-1} = 0$ lie outside of the unit circle.

To investigate the capability and limitations of feedback, we need the following precise definition of feedback [11].

Definition 2.1 *A sequence $\{u_t\}$ is called a feedback control law if at any time $t \geq 0$, u_t is a (causal) function of all the observations up to the time t: $\{y_i, i \leq t\}$, i.e.,*

$$u_t = h_t(y_{-p+1}, \cdots, y_t), \tag{5}$$

where $h_t(\cdot) : R^{t+p} \to R^1$ can be any Lebesgue measurable (nonlinear and/or time-varying) mapping.

Then we have the following main result of the paper:

Theorem 2.1 *The **necessary and sufficient condition** to stabilize the uncertain system* (3) *with any* $(f,\theta) \in \{\mathscr{F}(L), \Theta\}$ *is* $L < \frac{3}{2} + \sqrt{2}$. *To be precise, we have the following.*

1) If $L < \frac{3}{2} + \sqrt{2}$, *then there exists a feedback control law* $\{u_t\}$ *such that for any* $(f,\theta) \in \{\mathscr{F}(L), \Theta\}$, *the corresponding closed-loop control system* (3) *is globally stable in the sense that*

$$\sup_{t\geq 0}(|y_t| + |u_t|) < \infty, \quad \forall \phi_0 \in R^{p+q-1}. \tag{6}$$

2) If $L \geq \frac{3}{2} + \sqrt{2}$, *then for any feedback control* (5), *there always exists some* $(f,\theta) \in \{\mathscr{F}(L), \Theta\}$ *such that the corresponding closed-loop system* (3) *is unstable, i.e.,*

$$\sup_{t\geq 0} |y_t| = \infty. \tag{7}$$

Remark 2.2 This result shows that the critical value on the measure of uncertainty for the system (3) to be stabilized is still $\frac{3}{2} + \sqrt{2}$, the same as that first found for a purely nonparametric model in [11], i.e., for the special case where $g(\cdot) \equiv u_t$ in (3). This is somewhat unexpected, because in the purely parametric case, an additional unknown parameter in the input channel will be bound to reduce the critical value b (which represents the growth rate of the nonlinear function) for feedback stabilization from $b = 4$ to $b = 3$ (cf.e.g., [7], [8]).

3 Proofs of the main theorem

Before proving the theorem, we present some lemmas and notations extending those in [11] which will be used in our analyses.

Lemma 3.1 *Let* $L \in (0, \frac{3}{2}+\sqrt{2})$, $d \geq 0$ *be two constants and* ϵ *be a small positive number satisfying* $\epsilon < \frac{1}{2}, L+\epsilon < \frac{3}{2} + \sqrt{2-4\epsilon}$. *If a sequence* $\{a_n, n \geq 0\}$ *satisfies*

$$a_{n+1} \leq L(a_n - a_{n-1}) + (\frac{1}{2} + \epsilon)a_n + d, \quad n \geq 1, \tag{8}$$

with $a_0 = 0$ *and* $a_1 = 1$, *then there exists some* $d_0 > 0$ *such that whenever* $d \in [0, d_0]$, *there exists some* $N \geq 1$ *such that*

$$a_n \geq a_{n-1}, \qquad 1 \leq n \leq N \quad \textit{and} \quad a_{N+1} < a_N. \tag{9}$$

Proof: Suppose that

$$a_n \geq a_{n-1}, \qquad \forall n \geq 1. \tag{10}$$

Then $a_n \geq 1$ and $x_n \triangleq \frac{a_n}{a_{n-1}} \geq 1$ for all $n \geq 2$. Dividing each side of (8) by a_n, we have

$$x_{n+1} \leq L(1 - \frac{1}{x_n}) + \frac{1}{2} + \epsilon + d. \tag{11}$$

Denote $b \triangleq \varliminf_{n\to\infty} x_n \geq 1$, then we have

$$b \leq L(1 - \frac{1}{b}) + \frac{1}{2} + \epsilon + d.$$

It is easy to see that $b \neq 1$ provided that $d \in [0, \frac{1}{2} - \epsilon)$. So

$$\begin{aligned} L &\geq \frac{b^2 - (\frac{1}{2} + \epsilon + d)b}{b-1} \\ &\geq 2\sqrt{\frac{1}{2} - \epsilon - d} + \frac{3}{2} - \epsilon - d. \end{aligned} \tag{12}$$

However, by the assumption $L + \epsilon < \frac{3}{2} + \sqrt{2-4\epsilon}$, there must exist a positive number d_0 satisfying

$$L + \epsilon < \frac{3}{2} - d_0 + \sqrt{2 - 4\epsilon - 4d_0}.$$

So, for any $d \in [0, d_0]$,

$$L + \epsilon < \frac{3}{2} - d + \sqrt{2 - 4\epsilon - 4d}, \tag{13}$$

which contradicts (12). Hence the supposition (10) is incorrect and the proof of the lemma is completed. □

Lemma 3.2 *Let* L *and* ϵ *be defined as in Lemma 3.1, and let* $d \geq 0$ *and* $n_0 \geq 0$ *be two constants. If a nonnegative sequence* $\{h_n, n \geq 0\}$ *satisfies for any* $n \geq n_0$,

$$h_{n+1} \leq \Big(L \max_{0\leq i\leq n} h_i - (\frac{1}{2} - \epsilon)\sum_{i=0}^{n} h_i + d\Big)^+, \tag{14}$$

where $(x)^+ \triangleq max\{x, 0\}$, *then* $\lim_{n\to\infty} \sum_{i=0}^{n} h_i < \infty$.

Proof: We use the contradiction argument. Suppose that

$$\sum_{i=0}^{n} h_i \to \infty. \tag{15}$$

We first show that

$$L \max_{0\leq i\leq n} h_i - (\frac{1}{2} - \epsilon)\sum_{i=0}^{n} h_i + d > 0, \quad \forall n \geq n_0. \tag{16}$$

Otherwise, if there is some $n_1 \geq n_0$ such that (16) does not hold, then we must have $h_{n_1+1} = 0$. So (16) does not hold for $n_1 + 1$, then $h_{n_1+2} = 0$. Repeating this argument, we see that $h_n = 0$ for all $n \geq n_1 + 1$, then (15) does not hold, contradicting our supposition.

Now, by (16), we rewrite (14) as

$$h_{n+1} \leq L \max_{0\leq i\leq n} h_i - (\frac{1}{2} - \epsilon)\sum_{i=0}^{n} h_i + d. \qquad (17)$$

By (15), (17) and $h_n \geq 0$, there must be $\max_{0\leq i\leq n} h_i \to \infty$. Hence, there exists some $n_1 > n_0$ such that

$$h_{n_1} > \max_{0\leq i\leq n_1-1} h_i, \quad and \quad h_{n_1} \geq \frac{d}{d_0}, \qquad (18)$$

where d_0 is defined in Lemma 3.1. Moreover, we can choose a strictly increasing subsequence $\{h_{n_j}, j \geq 1\}$ from $\{h_n, n > n_1\}$ such that $h_{n_{j+1}} > h_{n_j}$ and

$$h_n \leq h_{n_j}, \quad \forall n_j \leq n < n_{j+1}. \qquad (19)$$

Then by (17)-(19), we have for $j \geq 1$,

$$\begin{aligned} h_{n_{j+1}} &\leq L \max_{0\leq i\leq n_{j+1}-1} h_i - (\frac{1}{2} - \epsilon)\sum_{i=0}^{n_{j+1}-1} h_i + d \\ &\leq L h_{n_j} - (\frac{1}{2} - \epsilon)\sum_{k=1}^{j} h_{n_k} + d. \end{aligned} \qquad (20)$$

Let $a_0 \triangleq 0$, and $a_j \triangleq \frac{1}{h_{n_1}}\sum_{k=1}^{j} h_{n_k}, j \geq 1$, so $a_{j+1} - a_j = \frac{1}{h_{n_1}} h_{n_{j+1}}$. Then

$$a_{j+1} - a_j \leq L(a_j - a_{j-1}) - (\frac{1}{2} - \epsilon)a_j + \frac{d}{h_{n_1}}.$$

Thus we have $a_0 = 0,\ a_1 = 1$, and

$$a_{j+1} \leq L(a_j - a_{j-1}) + (\frac{1}{2} + \epsilon)a_j + \frac{d}{h_{n_1}}. \qquad (21)$$

From this, by Lemma 3.1, we have for some $J \geq 1, a_{J+1} < a_J$. Then we must have $h_{n_{J+1}} < 0$, which contradicts $h_n \geq 0, \ \forall n \geq 0$. Hence lemma 3.2 holds. □

Lemma 3.3 *Consider the following equation,*

$$F_t = f(y_t) + g(\theta, \phi_t, u_t) + w_{t+1} - y_{t+1}, \quad t \geq 0. \quad (22)$$

Under the condition of **A3)**, there exists a map $h : R^{m+p+q+1} \to R^1$ such that $u_t = h(f(y_t), \theta, \phi_t, v_t)$, where $v_t = w_{t+1} - y_{t+1}$. Furthermore, there exist two positive constants, M_1, M_2 such that for any $f(y_t)$, ϕ_t, and v_t,

$$|h(\cdot)| \leq M_1(|f(y_t)| + \|\phi_t\| + |v_t|) + M_2. \qquad (23)$$

Proof: By the system (3), $F_t \equiv 0$ for any $t \geq 0$, and $|\frac{\partial F_t}{\partial u_t}| = |\frac{\partial g(\theta,\phi_t,u_t)}{\partial u_t}| \geq b$. By the Implicit Function Theorem, there is a function $h(\cdot) : R^{m+p+q+1} \to R^1$, and $h(\cdot) = h(f(y_t), \theta, \phi_t, v_t)$ such that $u_t = h(\cdot)$. Besides, $h(\cdot)$ is differentiable and satisfies

$$\begin{aligned} \frac{\partial h}{\partial f(y_t)} &= -(\frac{\partial F_t}{\partial u_t})^{-1}\frac{\partial F_t}{\partial f(y_t)} = -(\frac{\partial F_t}{\partial u_t})^{-1}, \\ \frac{\partial h}{\partial \phi_t} &= -(\frac{\partial F_t}{\partial u_t})^{-1}\frac{\partial F_t}{\partial \phi_t} = -(\frac{\partial F_t}{\partial u_t})^{-1}\frac{\partial g_t}{\partial \phi_t}, \\ \frac{\partial h}{\partial v_t} &= -(\frac{\partial F_t}{\partial u_t})^{-1}\frac{\partial F_t}{\partial v_t} = -(\frac{\partial F_t}{\partial u_t})^{-1}. \end{aligned}$$

So, under Condition 3) of Assumption **A3)**, we know that $|\frac{\partial h}{\partial f(y_t)}| \leq \frac{1}{b}, |\frac{\partial h}{\partial \phi_t}| \leq \frac{M}{b}, |\frac{\partial h}{\partial v_t}| \leq \frac{1}{b}$. By the Mean Value Theorem, there is

$$\begin{aligned} &h(f(y_t), \theta, \phi_t, v_t) - h(0, \theta, 0, 0) \\ &= \frac{\partial h(f(y_t), \theta, \phi_t, \varphi_{1t})}{\partial v_t} v_t + \frac{\partial h(f(y_t), \theta, \varphi_{2t}, 0)}{\partial \phi_t}\phi_t \\ &\quad + \frac{\partial h(\varphi_{3t}, \theta, 0, 0)}{\partial f(y_t)} f(y_t), \end{aligned}$$

where $\varphi_{1t}, \varphi_{2t}, \varphi_{3t}$ are some vectors between $[0, v_t]$, $[0, \phi_t]$ and $[0, f(y_t)]$ respectively. If no misunderstanding exists, here the interval $[0, \phi_t]$ means the straight line between the vectors 0 and ϕ_t, etc. Because θ is bounded and $h(\cdot)$ is continuous, $h(0, \theta, 0, 0)$ is bounded by some positive constant M_2, i.e., $|h(0, \theta, 0, 0)| \leq M_2$. Setting $M_1 = \max(\frac{1}{b}, \frac{M}{b})$, we get

$$|h(\cdot)| \leq M_1(|f(y_t)| + \|\phi_t\| + |v_t|) + M_2. \qquad (24)$$

Hence the proof is completed. □

Now, we split the rectangle Θ into n small rectangles. For any given $\epsilon_1 > 0$, if n is large enough, the diameter of each rectangle can be less than ϵ_1. We will show how to choose ϵ_1 later. We label these small rectangles as S_j^n, and randomly pick n points $\{\theta_j, j = 1, 2, \cdots n\}$ from them respectively. These points are used to estimate the unknown parameter vector θ. Though the estimation is not on line and may not be so accuracy, it is still helpful to stabilize the system (3) due to the deliberate partition.

Next, we recall the notations introduced in [11], which will be used in the following proofs. Denote

$$\bar{b}_t \triangleq \max_{-p+1\leq i\leq t} y_i, \quad \underline{b}_t \triangleq \min_{-p+1\leq i\leq t} y_i, \qquad (25)$$

and

$$\begin{aligned} &i_t \triangleq \operatorname*{argmin}_{-p+1\leq i\leq t-1} |y_t - y_i|, \quad i.e., \\ &|y_t - y_{i_t}| = \min_{-p+1\leq i\leq t-1} |y_t - y_i|. \end{aligned} \qquad (26)$$

The length of the history of the trajectory is defined as

$$\begin{aligned} B_t &\triangleq [\underline{b}_t, \bar{b}_t], & \triangle B_t &\triangleq B_t - B_{t-1}, \\ |B_t| &\triangleq |\bar{b}_t - \underline{b}_t|, & |\triangle B_t| &\triangleq |B_t| - |B_{t-1}|, \end{aligned} \quad (27)$$

where $\triangle B_{-p+1} \triangleq B_{-p+1}$.

When using θ_j to replace the real parameter vector θ, the estimate of the unknown value $f(y_t)$ at time instant t is defined by the formula

$$\hat{f}_j(y_t) \triangleq y_{i_t+1} - g(\theta_j, \phi_{i_t}, u_{i_t}), \quad 1 \leq j \leq n. \quad (28)$$

Take $u_0 = 0, t_0 = 1$, for $j \geq 1$,

$$\begin{aligned} t_j \triangleq \inf \Big\{ t > t_{j-1} : \\ |y_t - \frac{\underline{b}_{t-1} + \bar{b}_{t-1}}{2}| &> L|y_{t-1} - y_{i_{t-1}}| + \gamma + 2w \\ &+ M\epsilon_1(\|\phi_{t-1}\| + \|\phi_{i_{t-1}}\|) \\ &+ M\epsilon_1(|u_{t-1}| + |u_{i_{t-1}}|), \\ \textbf{or} \\ |u_{t-1}| > M_1 \Big[L|B_{t-1}| &+ \gamma + |f(y_0)| \\ &+ M\epsilon_1(\|\phi_{i_{t-1}}\| + |u_{i_{t-1}}|) + \|\phi_{t-1}\| \\ &+ |\frac{\underline{b}_{t-1} + \bar{b}_{t-1}}{2}| + w \Big] + M_2 \Big\}. \end{aligned} \quad (29)$$

Finally, the adaptive controller is defined as follows:

$$u_t = \begin{cases} 0 & t = 0 \quad or \quad t \geq t_n \\ h(\hat{f}_1(y_t), \theta_1, \phi_t, -\frac{\underline{b}_t + \bar{b}_t}{2}) & 1 \leq t < t_1 \\ \vdots \\ h(\hat{f}_j(y_t), \theta_j, \phi_t, -\frac{\underline{b}_t + \bar{b}_t}{2}) & t_{j-1} \leq t < t_j \\ \vdots \\ h(\hat{f}_n(y_t), \theta_n, \phi_t, -\frac{\underline{b}_t + \bar{b}_t}{2}) & t_{n-1} \leq t < t_n \end{cases} \quad (30)$$

Remark 3.1 The function estimate (28) can be regarded as the nearest neighbor estimate (cf.[18]) for $f(\cdot)$ if we use the jth parameter estimate θ_j to replace the real parameter θ.

Remark 3.2 The two inequalities in (29) serve as the criteria to check whether the parameter estimate θ_j is good enough or not. As will be proved in the following paragraph, if θ_j is a good estimate of θ, then neither of the two inequalities will hold. The time t_j is defined as the first time that the jth parameter estimate θ_j is found not so good. So, we switch to the next parameter θ_{j+1} as a new estimate.

Remark 3.3 Actually, we cannot use the unknown information $|f(y_0)|$ in the second inequality in (29). However, what we only need is the upper bound of $|f(y_0)|$. If we take $u_0 = 0$, from the data y_0, y_1 and ϕ_0, we can easily calculate the bound of $|f(y_0)|$ because the function $g(\cdot)$ is continuous and θ is bounded. So, here we still use $|f(y_0)|$ for simplicity.

Lemma 3.4 *If* $t_j < \infty, \ 1 \leq j \leq n$, *then the unknown parameter vector* $\theta \notin S_j^n$.

Proof: Suppose $\theta \in S_j^n$, then we have $\|\theta - \theta_j\| \leq \epsilon_1$, and for any $t \in [t_{j-1}, t_j)$, $u_t = h(\hat{f}_j(y_t), \theta_j, \phi_t, -\frac{\underline{b}_t + \bar{b}_t}{2})$. So

$$\begin{aligned} u_t = h(f(y_{i_t}) &+ g(\theta, \phi_{i_t}, u_{i_t}) - g(\theta_j, \phi_{i_t}, u_{i_t}) \\ &+ w_{i_t+1}, \theta_j, \phi_t, -\frac{\underline{b}_t + \bar{b}_t}{2}), \end{aligned} \quad (31)$$

which means

$$\begin{aligned} g(\theta_j, \phi_t, u_t) = -f(y_{i_t}) - g(\theta, \phi_{i_t}, u_{i_t}) + g(\theta_j, \phi_{i_t}, u_{i_t}) \\ - w_{i_t+1} + \frac{\underline{b}_t + \bar{b}_t}{2}. \end{aligned}$$

Then the closed-loop system is

$$\begin{aligned} y_{t+1} &= f(y_t) + g(\theta, \phi_t, u_t) + w_{t+1} \\ &= f(y_t) - f(y_{i_t}) + [g(\theta, \phi_t, u_t) - g(\theta_j, \phi_t, u_t)] \\ &\quad - [g(\theta, \phi_{i_t}, u_{i_t}) - g(\theta_j, \phi_{i_t}, u_{i_t})] \\ &\quad + \frac{\underline{b}_t + \bar{b}_t}{2} + w_{t+1} - w_{i_{t+1}}. \end{aligned} \quad (32)$$

So, we have

$$\begin{aligned} |y_{t+1} - \frac{\underline{b}_t + \bar{b}_t}{2}| &\leq L|y_t - y_{i_t}| + M\|\theta - \theta_j\|(\|\phi_t\| + |u_t|) \\ &\quad + M\|\theta - \theta_j\|(\|\phi_{i_t}\| + |u_{i_t}|) + \gamma + 2w \\ &\leq L|y_t - y_{i_t}| + M\epsilon_1(|u_t| + |u_{i_t}|) \\ &\quad + M\epsilon_1(\|\phi_t\| + \|\phi_{i_t}\|) + \gamma + 2w. \end{aligned} \quad (33)$$

Now we calculate the bound of $f(y_t)$. From the definition of B_t, it is easy to see that for any $t \geq 0$,

$$\begin{aligned} |y_t - y_0| \leq |B_t| &\Rightarrow |y_t| \leq |B_t| + |y_0|, \\ |f(y_t) - f(y_0)| &\leq L|y_t - y_0| + \gamma \leq L|B_t| + \gamma \\ &\Rightarrow |f(y_t)| \leq L|B_t| + |f(y_0)| + \gamma. \end{aligned} \quad (34)$$

From (28), (31) and Lemma 3.3, we have

$$\begin{aligned} |u_t| &\leq M_1 \Big[|\hat{f}_i(y_t)| + \|\phi_t\| + |\frac{\underline{b}_t + \bar{b}_t}{2}| \Big] + M_2 \\ &\leq M_1 \Big[L|B_t| + \gamma + |f(y_0)| + M\epsilon_1(\|\phi_{i_t}\| + |u_{i_t}|) \\ &\quad + \|\phi_t\| + |\frac{\underline{b}_t + \bar{b}_t}{2}| + w \Big] + M_2. \end{aligned} \quad (35)$$

Taking $t = t_j - 1$, from (33), (35) and the definition of t_j, we get the contradiction. Hence the lemma is true.□

Proof of Theorem 2.1:

Sufficiency: Under the condition $L < \frac{3}{2} + \sqrt{2}$, we use the controller designed in (30). In Lemma 3.4 we have shown that if $\theta \in S_i^n$, then $t_i = \infty$. Because $\theta \in \bigcup_{i=1}^n S_i^n$, we must have some i_0, $1 \le i_0 \le n$, $t_{i_0} = \infty$. Let $j_0 \in [0, i_0 - 1]$ be the first integer that $t_{j_0+1} = \infty$, then $t_{j_0} < \infty$. This integer always exists by the fact that $t_0 < \infty$ but $t_{i_0} = \infty$. Now we have for any $t \ge t_{j_0}$,

$$
\begin{aligned}
|y_{t+1} - \frac{b_t + \bar{b}_t}{2}| \le & L|y_t - y_{i_t}| + M\epsilon_1(|u_t| + |u_{i_t}|) \\
& + M\epsilon_1(\|\phi_t\| + \|\phi_{i_t}\|) + \gamma + 2w, \quad (36)
\end{aligned}
$$

$$
\begin{aligned}
|u_t| \le M_1 \Big[& L|B_t| + \gamma + |f(y_0)| + M\epsilon_1(\|\phi_{i_t}\| + |u_{i_t}|) \\
& + \|\phi_t\| + |\frac{b_t + \bar{b}_t}{2}| + w \Big] + M_2. \quad (37)
\end{aligned}
$$

From Condition **A5)** and the inequalities in (34), we have for any $t \ge 0$,

$$
\begin{aligned}
|u_{t-1}| & \le c_1 \sum_{i=-p+1}^{t} \lambda^{t-i}(|B_t| + |y_0|) + \frac{c_1}{1-\lambda} w + c_2 \\
& \le a_1|B_t| + b_1, \quad (38)
\end{aligned}
$$

where $a_1 \triangleq \frac{c_1}{1-\lambda}$, $b_1 \triangleq \frac{c_1}{1-\lambda}(|y_0| + w) + c_2$ are both constants, and

$$
\begin{aligned}
\|\phi_t\| & = \sqrt{y_t^2 + \cdots + y_{t-p+1}^2 + u_{t-1}^2 \cdots + u_{t-q+1}^2} \\
& \le p|B_t| + p|y_0| + q(a_1|B_t| + b_1) \\
& = a_2|B_t| + b_2, \quad (39)
\end{aligned}
$$

where $a_2 \triangleq p + qa_1$, $b_2 \triangleq p|y_0| + qb_1$ are both constants. The two inequalities above imply that both $|u_{t-1}|$ and $\|\phi_t\|$ are bounded by a linear growth rate of $|B_t|$. It is easy to note that

$$
|\frac{b_t + \bar{b}_t}{2} - y_0| \le \frac{1}{2}|B_t| \Rightarrow |\frac{b_t + \bar{b}_t}{2}| \le \frac{1}{2}|B_t| + |y_0|. \quad (40)
$$

So, from inequalities (37)-(40), we get the bound of u_t:

$$
\begin{aligned}
|u_t| \le M_1 \Big[& L|B_t| + \gamma + |f(y_0)| \\
& + M\epsilon_1(a_1|B_t| + b_1 + a_2|B_t| + b_2) \\
& + a_2|B_t| + b_2 + \frac{1}{2}|B_t| + y_0 + w \Big] + M_2 \\
= & a_3|B_t| + b_3, \quad (41)
\end{aligned}
$$

where $a_3 \triangleq M_1[L + M\epsilon_1(a_1 + a_2) + a_2 + \frac{1}{2}]$, $b_3 \triangleq M_1[\gamma + |f(y_0)| + M\epsilon_1(b_1 + b_2) + b_2 + y_0 + w] + M_2$ are constants depending on ϵ_1. This means that $|u_t|$ is also bounded by a linear growth rate of $|B_t|$.

From (36)-(41), we have for any $t \ge t_{j_0}$,

$$
\begin{aligned}
|y_{t+1} - \frac{b_t + \bar{b}_t}{2} & \le L|y_t - y_{i_t}| + (a_1 + 2a_2 + a_3)M\epsilon_1|B_t| \\
& + (b_1 + 2b_2 + b_3)M\epsilon_1 + \gamma + 2w. \quad (42)
\end{aligned}
$$

Now we show how to choose ϵ_1. We can take ϵ_1 small enough that

$$
\epsilon \triangleq (a_1 + 2a_2 + a_3)M\epsilon_1 \quad (43)
$$

satisfying $\epsilon < \frac{1}{2}$, $L + \epsilon < \frac{3}{2} + \sqrt{2 - 4\epsilon}$ as in Lemma 3.1. This can always be achieved because $(a_1 + 2a_2 + a_3)M\epsilon_1$ is an increasing function of ϵ_1. So

$$
\begin{aligned}
|y_{t+1} - \frac{b_t + \bar{b}_t}{2}| & \le L|y_t - y_{i_t}| + \epsilon|B_t| + \gamma + 2w \\
& + (b_1 + 2b_2 + b_3)M\epsilon_1. \quad (44)
\end{aligned}
$$

Similar to [11], we have

$$
\begin{aligned}
& |y_t - y_{i_t}| \le \max_{-p+1 \le i \le t} |\triangle B_i|, \\
& |\triangle B_{t+1}| = \max\big\{|y_{t+1} - \frac{b_t + \bar{b}_t}{2}| - \frac{1}{2}|B_t|, 0\big\}.
\end{aligned}
$$

So, from (44), for $t \ge j_0$,

$$
\begin{aligned}
|\triangle B_{t+1}| \le \Big\{ & L \max_{-p+1 \le i \le t} |\triangle B_i| - (\frac{1}{2} - \epsilon)|B_t| \\
& + (b_1 + 2b_2 + b_3)M\epsilon_1 + \gamma + 2w \Big\}^+. \quad (45)
\end{aligned}
$$

If Denoting $h_t \triangleq |\triangle B_t|$, we can get the same form as (14) in Lemma 3.2. So, by Lemma 3.2, we have

$$
\lim_{t \to \infty} \sum_{i=-p+1}^{t} |\triangle B_i| < \infty, \quad (46)
$$

which means that $|B_t| = \sum_{i=-p+1}^{t} |\triangle B_i|$ is bounded, i.e., $\sup_{t \ge -p+1} |y_t| < \infty$. This shows that the controller designed in (30) stabilizes the system (3).

Necessity: If taking $g(\theta, \phi_t, u_t) \equiv u_t$, we have the same model as in [11]. The same method can be applied in the necessity part. So the impossible part is obvious and the proof details will not be presented here. □

4 Concluding remarks

In this paper, we have investigated the feedback capability for a basic class of semiparametric systems, and have established some interesting results. The "parameter switching" method is used to estimate the parametric part and the “nearest neighbor” idea is used to estimate the nonparametric part. The controller thus designed has shown to enjoy the maximum capability in stabilizing the uncertain systems described by a critical value $L = \frac{3}{2} + \sqrt{2}$. Although it may be desirable to simplify or to improve the designed control algorithm when necessary, it is sufficient in theory to help us to characterize the maximum capability of feedback as investigated in the paper.

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