Further Results on Stabilizing Control of Quantum Systems

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***Abstract*—It is notoriously known that in contrast to the measure of classical macro systems, the quantum measurement will in general cause quantum state collapse. In this note, we will present some further results on stabilizing control of quantum systems with measurement being involved. To be precise, we will study the effects of the open-loop control and the feedback control in preparing an arbitrarily desired eigenstate of an effective Hamiltonian, respectively. For the open-loop control case, it is shown that no matter how to select the measurement channel, control channel and the control law, one cannot prepare all the eigenstates of the effective Hamiltonian with arbitrary high fidelity for all sufficient large time. While for the feedback case, we not only demonstrate how to select the measurement channel properly to achieve the control target, but also show if not, what cannot be done by the measurement-based feedback control.**

***Index Terms*—Eigenstate preparation, master equation, quantum feedback control, quantum measurement.**

I. Introduction

Over the past decades, much progress has been made in regulating the microscopic world, such as molecules, atoms and so on (e.g., [1], [2]). The rapid development of the technology further promotes the development of the quantum control theory. According to whether or not there is measurement and whether the measurement information is used in regulating the quantum system during the control process, we may divide the control strategy into open-loop control (OLC) and feedback control (FC). For the OLC, we can further divide it into Hamiltonian and non-Hamiltonian control modes according to how the control law is introduced. For the Hamiltonian control mode (e.g., [3]–[8]), we mean the control is an adjustable parameter in the Hamiltonian. While in the non-Hamiltonian control case (e.g., [9], [10]), one regulates the quantum system by adjusting some parameters of an auxiliary system, for example, the parameter of the environment of the system. For the FC, there are three types: (i) learning control (e.g., [11], [12]), (ii) measurement-based feedback control (MFC) (e.g., [13]–[23]) and (iii) coherent feedback control (e.g., [24]–[30]). For the learning control, one begins with some initial control pulses, and then makes some measurement on the samples after the control process. A selected learning algorithm is used to adjust the control pulses and then the above steps are repeated on some new samples until the performance index does not change significantly. Hence, for this control mode, one needs different samples for different cycles. For the MFC mode, one performs some direct or indirect measurement on the quantum system and then designs a control law based on the measurement information to regulate the quantum system. In contrast to the MFC mode, the coherent feedback control does not involve any real measurement, and instead, the controller is coherently connected with the system plant and can be a quantum system itself.

In this note, we only focus on the characteristics of the control of quantum systems with measurement being involved. For classical macro systems, the back action effect of the measurement on the system state can be neglected in principle, which, however, cannot be neglected in the quantum case. It is well-known that measurement on a quantum system will generally cause the quantum state collapse, and the corresponding measurement back action effects for the state transfer consist of the deterministic drift part and the stochastic diffusion part, which will be interpreted in detail later.

Our aim is to prepare an arbitrarily desired eigenstate of an effective Hamiltonian from an arbitrary initial state. Note that the initial state of the system may generally be a mixed state owing to the inevitable interactions between the system and its bath. Furthermore, the von Neumann entropy of a mixed state is strictly greater than 0. In this sense, we say the initial state has some uncertainties. We will compare the effects of the two mentioned control strategies, OLC and MFC, in dealing with the initial state uncertainties to complete the control target.

The note is organized as follows. In Section II, we briefly introduce the typical control models to be studied, and set up the control problems specifically. The effects of the OLC and MFC in dealing with the initial state uncertainties are studied in Section III and Section IV, respectively. Some comparison remarks between the effects of the OLC and MFC are also given in Section IV. Section V concludes the note with some remarks.

II. The Control Models

We now sketch the control models to be used (see e.g., [16], [31] for more details). Let $\mathbb{H}$ be the Hilbert space of the quantum system with finite dimension, $\dim \mathbb{H} = N < \infty$. The quantum bath is modeled by the symmetric Fock space $\mathbb{F}$. Now, let $\mathbb{A}$ denote the von Neumann algebra generated by the set of all bounded operators on $\mathbb{H} \bigotimes \mathbb{F}$. Then the quantum probability space of the system and the bath is $(\mathbb{A}, \rho_{total})$, where the state $\rho_{total} = \rho \bigotimes \rho_{vacuum}$ is given by some state ρ on $\mathbb{H}$ and the vacuum state ρ_{vacuum} on the bath $\mathbb{F}$.

Under some proper approximations, the quantum dynamics can be described by the Hudson–Parthasarathy equation,[1] (e.g., [31], [32])

$$\begin{aligned} dU_t =& \bigg\{ -i(H_0 + u_t H_b)dt - \frac{\kappa}{2} L^* L dt \\ &+ \sqrt{\kappa} L dA_t^* - \sqrt{\kappa} L^* dA_t \bigg\} U_t, \\ U_0 =& I. \end{aligned} \tag{1}$$

Here, U_t describes the unitary evolution of the whole plant, H_0, H_b and L are system operators, and A_t is the annihilation operator on $\mathcal{F}$. The symbol $*$ denotes the Hilbert space adjoint as well as the scalar complex conjugate. Furthermore, H_0 and H_b are Hermitian and can be considered as the effective Hamiltonian and the control channel of the system respectively. $\{u_t\}$ is an adjustable parameter process and $\kappa > 0$ is the effective interaction strength.

In this note, for the MFC, we will measure the field observable $Y_t = U_t^*(A_t + A_t^*)U_t$. Noting the form of the observable Y_t and the quantum dynamics (1), we can name L as the measurement channel, because it is through this channel that the system signals are probed for measurement and that the back action effects of the measurement on the bath are imposed on the system..

[1]We always work in units such that $\hbar = 1$.

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Note that what we really concern is the state of the system. We can use the quantum filtering theory (see e.g., [32]–[34]) to estimate the state of the system. One can get the dynamics of the conditional density operator of the system as

$$d\rho_t = -i[H_0 + u_t H_b, \rho_t]dt + \kappa\mathcal{D}[L]\rho_t dt + \sqrt{\eta\kappa}\mathcal{H}[L]\rho_t dW_t \quad (2)$$

where $\eta \in (0, 1]$ is the detection efficiency, W_t is a Wiener process on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (e.g., [33], [35]), and the superoperators $\mathcal{D}$ and $\mathcal{H}$ are defined by

$$\mathcal{D}[\Lambda]\rho = \Lambda\rho\Lambda^* - \frac{1}{2}(\Lambda^*\Lambda\rho + \rho\Lambda^*\Lambda),$$
$$\mathcal{H}[\Lambda]\rho = \Lambda\rho + \rho\Lambda^* - \mathrm{Tr}(\Lambda\rho + \rho\Lambda^*)\rho.$$

$\mathcal{D}[L]\rho_t dt$ and $\mathcal{H}[L]\rho_t dW_t$ describe the deterministic drift part and the stochastic diffusion part of the measurement back action effects, respectively.

This form of (2) is also known as the stochastic master equation or the quantum trajectory in physics. In this note, we do not consider another MFC mode—direct feedback control, see e.g., [36] and [37]. An interesting comparison between the indirect feedback control (as the control mode in (2)) and the direct feedback control can be found in [38].

For the OLC case, as we have mentioned in the introduction, there are two cases. First, we look at the case where there is no measurement at all. This kind of OLC model is described by

$$\frac{d\rho_t}{dt} = -i\left[H_0 + u(t)H_b, \rho_t\right]. \quad (3)$$

Next, let us look at the case where there is measurement but only the prior information is used to design the control law. The corresponding OLC model is [31]

$$\frac{d\rho_t}{dt} = -i[H_0 + u_t H_b, \rho_t] + \kappa\mathcal{D}[L]\rho_t. \quad (4)$$

In contrast to (2), there is no diffusion part in (4), which depicts the measurement induced uncertainty. This is because (4) actually describes the evolution of the ensemble average. This form of evolution equation is usually called the master equation.

In the following, we suppose H_0 is nondegenerate. Our control objective is to prepare an arbitrarily desired eigenstate of H_0 asymptotically from an arbitrary initial state. We define the distance between the state ρ and one of the H_0 eigenstates ρ_d as $D(\rho, \rho_d) = 1 - \mathrm{Tr}(\rho, \rho_d)$. We remark that a desired eigenstate not only can be reached after some time, but more importantly, such a desired eigenstate can be kept forever as long as the control law is in action. Note that preparing a family of fiducial states is one of the four basic requirements for quantum computation and is the basis for subsequent manipulations, e.g., implementing a family of universal unitary operations on these fiducial states [39].

It is worth pointing out that in the special case where $H_0 = 0$, $H_b = F_y$, $L = F_z$ (F_y and F_z are spin-$N/2$ angular momentum operators, where N is the number of the atoms in the ensemble), the equation (2) corresponds to the control of quantum spins, which has been investigated in depth, see e.g., [16], [17]. While in the current note, we will focus on more general H_0, H_b, and L, and establish theorems concerning their structural relationships in completing the control task.

III. The Open-Loop Control Case

In this section we will study the effects of the quantum OLC in dealing with the initial state uncertainties. Recall when referring to the OLC strategy, we only use the prior information to design the control law.

First, we look at the case where there is no measurement at all. The OLC model is

$$\frac{d\rho_t}{dt} = -i\left[H_0 + u(t)H_b, \rho_t\right]. \quad (5)$$

For completeness we give the following well-known result, the proof of which can be find in [22].

Proposition 1: For the OLC model (5), for arbitrary control channel H_b and arbitrary control law u_t, an arbitrary eigenstate of H_0 cannot be prepared from any mixed initial state.

Next, let us look at the case where there is measurement but only the prior information is used to design the control law. The corresponding OLC model is [31]

$$\frac{d\rho_t}{dt} = -i[H_0 + u_t H_b, \rho_t] + \kappa\mathcal{D}[L]\rho_t. \quad (6)$$

Note that the Hamiltonian in [40]–[42] is time-invariant, so generally, the results there cannot be applied directly to our OLC control model (6) with time-varying Hamiltonian.

First, we consider the case where the given L and H_0 are commutative.

Proposition 2: For the OLC model (6), if H_0 is non-degenerate, and $[H_0, L] = 0$, then for arbitrary control channel H_b and arbitrary control law u_t, one cannot prepare an arbitrarily desired eigenstate of H_0 from any mixed initial state.

We omit the proof here which is similar to that in [40].

Secondly, we consider the case where the given L and H_0 are non-commutative.

Theorem 1: For the OLC model (6), if H_0 is non-degenerate, and $[H_0, L] \neq 0$, then there is at least one eigenstate of H_0 denoted as ρ_d, such that no matter how to select the control channel H_b and how to design the corresponding control law u_t, one has

$$\limsup_{t\to\infty} D(\rho_t, \rho_d) \geq \delta_d > 0$$

where $\delta_d = \mathrm{Tr}^2(\rho_d\mathcal{D}[L]\rho_d)/(2[2\mathrm{Tr}^{1/2}(L^*L)^2 + \mathrm{Tr}^{1/2}(L^*\rho_d L)^2 + \mathrm{Tr}^{1/2}(L^*LL^*L\rho_d)]^2)$.

a) Proof: First of all, let us take the N eigenstates $\psi_n, n = 1, \ldots, N$, of H_0 as the basis of the matrix representation of the system operators in question, where ψ_n is the vector with only a nonzero element 1 in the n-th row. Since H_0 is diagonal, non-degenerate and $[H_0, L] \neq 0$, we know that there exists at least one non-diagonal entry of L which is not zero. Without loss of generality, suppose it is in the d-th column of L. Hence one has

$$\mathrm{Tr}\left(\rho_d\mathcal{D}[L]\rho_d\right) = -\sum_{i\neq d}^{N} L_{id}^* L_{id} < 0. \quad (7)$$

Next we use a contradiction argument to give the proof of this theorem. Suppose that for any eigenstate ρ_i of H_0, their exist a corresponding control channel H_b and a corresponding control law u_t, such that $\limsup_{t\to\infty} D(\rho_t, \rho_i) < \delta_i$. Then there exist some $\epsilon > 0, T > 0$, such that for any $t > T$, $D(\rho_t, \rho_i) < \delta_i - \epsilon$. Therefore for $t > T$

$$\|\rho_t - \rho_i\| = \mathrm{Tr}^{\frac{1}{2}}\left((\rho_t - \rho_i)^2\right)$$
$$\leq (2D(\rho_t, \rho_i))^{\frac{1}{2}} < \sqrt{2(\delta_i - \epsilon)}. \quad (8)$$

Then by (8) one has

$$\begin{aligned}
&\kappa\left(\mathrm{Tr}\left(\rho_t\mathcal{D}[L]\rho_t\right)-\mathrm{Tr}\left(\rho_d\mathcal{D}[L]\rho_d\right)\right)\\
&=\kappa\left(\mathrm{Tr}\left(L\rho_tL^*(\rho_t-\rho_d)\right)+\mathrm{Tr}\left((\rho_t-\rho_d)L^*\rho_dL\right)\right.\\
&\quad\left.+\mathrm{Tr}\left(L^*L\rho_d(\rho_d-\rho_t)\right)+\mathrm{Tr}\left(\rho_tL^*L(\rho_d-\rho_t)\right)\right)\\
&\le\kappa\left(\mathrm{Tr}^{\frac12}\left(L\rho_tL^*\right)^2+\mathrm{Tr}^{\frac12}\left(L^*\rho_dL\right)^2+\mathrm{Tr}^{\frac12}\left(L^*LL^*L\rho_d\right)\right.\\
&\quad\left.+\mathrm{Tr}^{\frac12}\left(L^*LL^*L\rho_t^2\right)\right)\|\rho_t-\rho_d\|\\
&<\sqrt{2(\delta_d-\epsilon)}\kappa\left(2\mathrm{Tr}^{\frac12}(L^*L)^2+\mathrm{Tr}^{\frac12}\left(L^*\rho_dL\right)^2\right.\\
&\quad\left.+\mathrm{Tr}^{\frac12}\left(L^*LL^*L\rho_d\right)\right).
\end{aligned}\tag{9}$$

Then by (7) and (9), one has for $t>T$

$$\begin{aligned}
\frac{d}{dt}\mathrm{Tr}\left(\rho_t^2\right)=&2\kappa\mathrm{Tr}\left(\rho_d\mathcal{D}[L]\rho_d\right)\\
&+2\kappa\left(\mathrm{Tr}\left(\rho_t\mathcal{D}[L]\rho_t\right)-\mathrm{Tr}\left(\rho_d\mathcal{D}[L]\rho_d\right)\right)\\
<&2\kappa\mathrm{Tr}\left(\rho_d\mathcal{D}[L]\rho_d\right)\\
&+2\sqrt{2(\delta_d-\epsilon)}\kappa\left(2\mathrm{Tr}^{\frac12}(L^*L)^2+\mathrm{Tr}^{\frac12}\left(L^*\rho_dL\right)^2\right.\\
&\left.+\mathrm{Tr}^{\frac12}\left(L^*LL^*L\rho_d\right)\right)\\
\triangleq&\alpha(\epsilon).
\end{aligned}\tag{10}$$

By substituting the definition of δ_d into $\alpha(\epsilon)$ defined above, we can find that $\alpha(\epsilon)<0$. Hence, from the above inequality, we have a contradiction $0\le\mathrm{Tr}(\rho_t^2)\to-\infty$, as $t\to\infty$, and the proof is completed.□

Generally speaking, a state having a high fidelity[2] with the target state (e.g., greater than 90%) is satisfactory. However, Theorem 1 points out that under some conditions, there is at least one eigenstate such that no matter how to choose the control channel and how to design the control law, one cannot prepare this target state with arbitrary high fidelity for all sufficient large t.

IV. The Measurement-Based Feedback Control Case

As we have mentioned in Section II, L can be referred to as the measurement channel. Note that for the quantum MFC, the measurement channel L should be chosen appropriately, since the measurement on a quantum state gains some information of the state as well as introduces a state collapse in general.

In this section, we will first show how to choose the measurement channel L appropriately to achieve the control target. We have the following selection theorem.

Theorem 2: For the MFC model (2), suppose that H_0 and L are non-degenerate, $L=L^*$ and $[H_0,L]=0$. Then every eigenstate of H_0 can be prepared asymptotically with probability 1 under L.

The proof of the above theorem is inspired by that of Theorem 4.2 in [17].

First, we define

$$\begin{aligned}
\mathcal{S}&\triangleq\left\{\rho\in\mathbb{C}^{N\times N}:\rho=\rho^*\ge0,\mathrm{Tr}(\rho)=1\right\},\\
\mathcal{S}_{>\beta}&\triangleq\left\{\rho\in\mathcal{S}:\beta<D(\rho,\rho_d)\le1\right\},\\
\mathcal{S}_{\ge\beta}&\triangleq\left\{\rho\in\mathcal{S}:\beta\le D(\rho,\rho_d)\le1\right\}.
\end{aligned}$$

Footnote to above equation.[3] Secondly, we give the following three lemmas.

[2]Here, the fidelity between the state ρ and the target state ρ_d is defined by $F(\rho,\rho_d)\triangleq1-D(\rho,\rho_d)$.

[3]Similarly, we define $\mathcal{S}_{\le\beta}\triangleq\{\rho\in\mathcal{S}:0\le D(\rho,\rho_d)\le\beta\}$, $\mathcal{S}_{<\beta}\triangleq\{\rho\in\mathcal{S}:0\le D(\rho,\rho_d)<\beta\}$.

Lemma 1: For the control model (2) with $u_t\equiv1$, denote $A\triangleq-i(H_0+H_b)-\kappa L^*L+\alpha\sqrt{\kappa}L$, where α is a real number. If there exist an $\alpha\in\mathbb{R}$ and an eigenstate $\rho_d=\psi_d\psi_d^*$ of H_0 such that

$$\mathrm{rank}\begin{pmatrix}\psi_d & A^*\psi_d & \ldots & A^{*N-1}\psi_d\end{pmatrix}^*=N\tag{11}$$

then there exists $\gamma>0$ such that for any initial state $\rho_0\in\mathcal{S}_{>1-\gamma}$, ρ_t will exit $\mathcal{S}_{>1-\gamma}$ in finite time with probability 1.

Lemma 2: For the feedback control model (2) with $u(\rho_t)=-\mathrm{Tr}(i[H_b,\rho_t]\rho_d)$, denote by $\Phi_t(\rho,u)$ the solution of the model with an initial state ρ and a control u. Then one has for all $\rho\in\mathcal{S}_{\le1-\gamma}$

$$\mathbb{P}\left\{\sup_{0\le t<\infty}D\left(\Phi_t(\rho,u),\rho_d\right)\ge1-\gamma/2\right\}\le\frac{1-\gamma}{1-\gamma/2}=1-p<1.$$

Lemma 3: For the feedback control model (2) with $u(\rho_t)=-\mathrm{Tr}(i[H_b,\rho_t]\rho_d)$, if H_0, L are non-degenerate, $L=L^*$ and $[H_0,L]=0$, then the trajectories of ρ_t which never exit $\mathcal{S}_{<1-\gamma/2}$ will converge almost surely to ρ_d as $t\to\infty$.

The corresponding proofs of the above lemmas are omitted and can be gotten in a similar way as the proof of Theorem 4.2 in [17], but for the more general model (2).

In the following, we just show how to select the control channel H_b and to construct the corresponding control law u_t according to the arbitrarily desired eigenstate of H_0.

The control channel H_b should be chosen such that there exists a real number α satisfying the following **rank condition**:

$$\mathrm{rank}\begin{pmatrix}\psi_d & A^*\psi_d & \ldots & A^{*N-1}\psi_d\end{pmatrix}^*=N\tag{12}$$

for all the eigenstates $\rho_d=\psi_d\psi_d^*$ $(d=1,\cdots,N)$ of H_0, where A is defined as in Lemma 1.

Note that this is similar to the observability condition in the linear control system theory. Now, we show how to choose the control channel H_b to meet the above rank condition.

First we associate $(A;H_0,H_b,L)$ with a non-oriented graph $G(A;H_0,H_b,L)=(V,E)$, where the vertices set V corresponds to the eigenstates of H_0, i.e., $V=\{\psi_d,d=1,\cdots,N\}$. There is an edge between ψ_i and ψ_j iff $(H_b)_{i,j}\ne0$ for $i\ne j$. Hence, $E=\{(\psi_i,\psi_j):(H_b)_{i,j}\ne0\text{ for }i\ne j\}$ does not depend on L.

For the feedback control model (2), a necessary condition for A to meet the rank condition (12) is that the graph $G(A;H_0,H_b,L)$ is connected.

Actually, Since H_0 is diagonal, non-degenerate and $[H_0,L]=0$, we know that L is also diagonal. Thus the off-diagonal elements of A are those of $-iH_b$. Hence, if $G(A;H_0,H_b,L)$ is not connected, then we can find a permutation matrix P, such that $A=P\begin{pmatrix}A_1 & 0\\ 0 & A_2\end{pmatrix}P^T$ [43].

Moreover, note that ψ_i, $i=1,\cdots,N$, are the basis of the matrix representation, and P is a permutation matrix, thus for arbitrary $d\in\{1,\cdots,N\}$, there exits some integer $k\in\{1,\cdots,N\}$, which may depend on d and P, such that

$$\begin{aligned}
&\mathrm{rank}\left(\psi_d\quad A^*\psi_d\quad\ldots\quad A^{*N-1}\psi_d\right)^*\\
&=\mathrm{rank}\begin{pmatrix}\psi_k^*\\ \psi_k^*\begin{pmatrix}A_1 & 0\\ 0 & A_2\end{pmatrix}\\ \vdots\\ \psi_k^*\begin{pmatrix}A_1^{N-1} & 0\\ 0 & A_2^{N-1}\end{pmatrix}\end{pmatrix}<N.
\end{aligned}$$

Therefore, the connectivity of graph $G(A; H_0, H_b, L)$ is necessary for A to meet the rank condition (12).

Intuitively, a connected graph $G(A; H_0, H_b, L)$ implies that all the eigenstates of H_0 (the corresponding energy levels in physical meaning) can reach each other under the control law u_t.

For the feedback control model (2), a sufficient condition for A to meet the rank condition (12) is to choose the control channel H_b such that $G(A; H_0, H_b, L)$ is actually a path[4] of length $N-1$. This can be proved from the proof of Lemma 4.4 in [17] and the property of permutation matrix.

Next, we give the feedback control law u_t.

In order to globally prepare an arbitrary eigenstate ρ_d of H_0 with probability 1, let $\gamma > 0$ be as defined in Lemma 1, and define $\mathcal{B} \triangleq \{\rho : \gamma/2 < \mathrm{Tr}(\rho\rho_d) < \gamma\}$, we construct the feedback control law as follows:

1) If $\mathrm{Tr}(\rho_t\rho_d) \geq \gamma$, $u_t = -\mathrm{Tr}(i[H_b, \rho_t]\rho_d)$;
2) If $\mathrm{Tr}(\rho_t\rho_d) \leq \gamma/2$, $u_t = 1$;
3) If $\rho_t \in \mathcal{B}$, then $u_t = -\mathrm{Tr}(i[H_b, \rho_t]\rho_d)$ if ρ_t enters into $\mathcal{B}$ through the boundary $\mathrm{Tr}(\rho_t\rho_d) = \gamma$; $u_t = 1$ otherwise.

The proof of Theorem 2 is omitted here and can be proceeded by the above three lemmas.

Furthermore, we consider the case where the given measurement channel L and the effective Hamiltonian H_0 are not commutative. We have the following theorem on the limitation of MFC.

Theorem 3: For the MFC model (2), if H_0 is non-degenerate, $[H_0, L] \neq 0$ and $\eta \in [0, 1)$, then there is at least one eigenstate of H_0 denoted as ρ_d, such that no matter how to select the control channel H_b and how to design the corresponding control law u_t, $\limsup_{t\to\infty} D(\rho_t, \rho_d) \geq \Delta_d > 0$, $a.s.$, where $\Delta_d = [2\mathrm{Tr}(\rho_d\mathcal{D}[L]\rho_d) + \eta\mathrm{Tr}(\mathcal{H}[L]\rho_d)^2]^2/2[2\varphi_1(L,\rho_d) + \eta\varphi_2(L,\rho_d)]^2$

$$\varphi_1(L,\rho_d) = 2\mathrm{Tr}^{\frac{1}{2}}(L^*L)^2 + \mathrm{Tr}^{\frac{1}{2}}(L^*\rho_d L)^2 + \mathrm{Tr}^{\frac{1}{2}}\left((L^*L)^2\rho_d\right),$$
$$\begin{aligned}\varphi_2(L,\rho_d) =& 2\mathrm{Tr}^{\frac{1}{2}}(L^*L)^2 + 2\mathrm{Tr}^{\frac{1}{2}}(L^*LL^*L\rho_d) + 2\mathrm{Tr}(L^*L)\\ &+ 2\mathrm{Tr}^{\frac{1}{2}}(LL^*\rho_d L^*L\rho_d)\\ &+ 3\mathrm{Tr}^{\frac{1}{2}}(L+L^*)^2\mathrm{Tr}((L+L^*)\rho_d)\\ &+ 3\mathrm{Tr}^{\frac{1}{2}}(L+L^*)^2\mathrm{Tr}^{\frac{1}{2}}(L+L^*)^2\\ &+ 2\mathrm{Tr}^{\frac{1}{2}}(L+L^*)^2\mathrm{Tr}^{\frac{1}{2}}\left((L+L^*)^2\rho_d\right).\end{aligned}$$

a) Proof: A contradiction argument will be used. First, since H_0 is diagonal, non-degenerate and $[H_0, L] \neq 0$, we know that L has at least one non-zero non-diagonal entry. For simplicity, we suppose it is in the d-th column of L. Thus for the eigenstate ρ_d, one has

$$\begin{aligned}&2\kappa\mathrm{Tr}(\rho_d\mathcal{D}[L]\rho_d) + \eta\kappa\mathrm{Tr}(\mathcal{H}[L]\rho_d)^2\\ &= -2\kappa(1-\eta)\sum_{i\neq d}^{N} L_{id}^* L_{id}\\ &\triangleq -2\kappa(1-\eta)\alpha < 0.\end{aligned} \tag{13}$$

Suppose that for any eigenstate ρ_i of H_0, there exist corresponding control channel H_b and control law u_t, such that $\limsup_{t\to\infty} D(\rho_t, \rho_i) < \Delta_i$ on A, where A is a set with positive probability $\mathbb{P}\{A\} > 0$. Then for each sample point $\omega \in A$, there exist $\epsilon > 0$ and $T(\omega) > 0$, such that for any $t > T(\omega)$, one has $D(\rho_t(\omega), \rho_i) < \Delta_i - \epsilon$. Therefore, for $t > T(\omega)$ with $\omega \in A$

$$\|\rho_t(\omega) - \rho_i\|^2 \leq 2D(\rho_t(\omega), \rho_i) < 2(\Delta_i - \epsilon). \tag{14}$$

[4]A path in $G(A; H_0, H_b, L)$ of length r is a sequence $[\psi_{i_0}, \cdots, \psi_{i_r}]$ of distinct vertices such that $(\psi_{i_{j-1}}, \psi_{i_j}) \in E$.

Then for each $\omega \in A$, by (14) one has

$$\begin{aligned}&\kappa(\mathrm{Tr}(\rho_t\mathcal{D}[L]\rho_t) - \mathrm{Tr}(\rho_d\mathcal{D}[L]\rho_d))\\ &= \kappa(\mathrm{Tr}(L\rho_t L^*(\rho_t - \rho_d)) + \mathrm{Tr}((\rho_t - \rho_d)L^*\rho_d L)\\ &\quad + \mathrm{Tr}(L^*L\rho_d(\rho_d - \rho_t)) + \mathrm{Tr}(\rho_t L^*L(\rho_d - \rho_t)))\\ &\leq \kappa\Big(\mathrm{Tr}^{\frac{1}{2}}(L\rho_t L^*)^2 + \mathrm{Tr}^{\frac{1}{2}}(L^*\rho_d L)^2 + \mathrm{Tr}^{\frac{1}{2}}(L^*LL^*L\rho_d)\\ &\quad + \mathrm{Tr}^{\frac{1}{2}}\left(L^*LL^*L\rho_t^2\right)\Big)\|\rho_t - \rho_d\|\\ &< \sqrt{2(\Delta_d - \epsilon)}\kappa\Big(2\mathrm{Tr}^{\frac{1}{2}}(L^*L)^2 + \mathrm{Tr}^{\frac{1}{2}}(L^*\rho_d L)^2\\ &\quad + \mathrm{Tr}^{\frac{1}{2}}(L^*LL^*L\rho_d)\Big)\\ &= \kappa\varphi_1(L,\rho_d)\sqrt{2(\Delta_d - \epsilon)}\end{aligned} \tag{15}$$

and similarly for $\omega \in A$

$$\begin{aligned}&\mathrm{Tr}(\mathcal{H}[L]\rho_t)^2 - \mathrm{Tr}(\mathcal{H}[L]\rho_d)^2\\ &= 2\mathrm{Tr}\left(L^*L\rho_t^2\right) - 2\mathrm{Tr}\left(L^*L\rho_d^2\right) + \mathrm{Tr}(L\rho_t L\rho_t)\\ &\quad - \mathrm{Tr}(L\rho_d L\rho_d) + \mathrm{Tr}(L^*\rho_t L^*\rho_t) - \mathrm{Tr}(L^*\rho_d L^*\rho_d)\\ &\quad - 2\mathrm{Tr}((L+L^*)\rho_t)\,\mathrm{Tr}\left((L+L^*)\rho_t^2\right)\\ &\quad + 2\mathrm{Tr}((L+L^*)\rho_d)\,\mathrm{Tr}\left((L+L^*)\rho_d^2\right)\\ &\quad + \mathrm{Tr}^2((L+L^*)\rho_t)\,\mathrm{Tr}\left(\rho_t^2\right) - \mathrm{Tr}^2((L+L^*)\rho_d)\,\mathrm{Tr}\left(\rho_d^2\right)\\ &< 2\left(\mathrm{Tr}^{\frac{1}{2}}(L^*L)^2 + \mathrm{Tr}^{\frac{1}{2}}(L^*LL^*L\rho_d)\right)\|\rho_t - \rho_d\|\\ &\quad + \mathrm{Tr}(L\rho_t L(\rho_t - \rho_d)) + \mathrm{Tr}((\rho_t - \rho_d)L\rho_d L)\\ &\quad + \mathrm{Tr}(L^*\rho_t L^*(\rho_t - \rho_d)) + \mathrm{Tr}((\rho_t - \rho_d)L^*\rho_d L^*)\\ &\quad + 2\mathrm{Tr}((L+L^*)\rho_d)\,\mathrm{Tr}((L+L^*)(\rho_d - \rho_t))\\ &\quad + 2\mathrm{Tr}((L+L^*)\rho_t)\,\mathrm{Tr}((L+L^*)\rho_d(\rho_d - \rho_t))\\ &\quad + 2\mathrm{Tr}((L+L^*)\rho_t)\,\mathrm{Tr}((L+L^*)(\rho_d - \rho_t)\rho_t)\\ &\quad + \mathrm{Tr}((L+L^*)(\rho_d + \rho_t))\,\mathrm{Tr}((L+L^*)(\rho_t - \rho_d))\\ &< \Big\{2\mathrm{Tr}^{\frac{1}{2}}(L^*L)^2 + 2\mathrm{Tr}^{\frac{1}{2}}(L^*LL^*L\rho_d) + 2\mathrm{Tr}(L^*L)\\ &\qquad + 2\mathrm{Tr}^{\frac{1}{2}}(LL^*\rho_d L^*L\rho_d)\\ &\qquad + 3\mathrm{Tr}^{\frac{1}{2}}(L+L^*)^2\mathrm{Tr}((L+L^*)\rho_d)\\ &\qquad + 3\mathrm{Tr}^{\frac{1}{2}}(L+L^*)^2\mathrm{Tr}^{\frac{1}{2}}(L+L^*)^2\\ &\qquad + 2\mathrm{Tr}^{\frac{1}{2}}(L+L^*)^2\mathrm{Tr}^{\frac{1}{2}}\left((L+L^*)^2\rho_d\right)\Big\}\|\rho_t - \rho_d\|\\ &= \varphi_2(L,\rho_d)\|\rho_t - \rho_d\|\\ &< \varphi_2(L,\rho_d)\sqrt{2(\Delta_d - \epsilon)}.\end{aligned} \tag{16}$$

By the Itô formula, one has

$$\begin{aligned}&d\mathrm{Tr}(\rho_t^2)\\ &= 2\kappa\mathrm{Tr}(\rho_t\mathcal{D}[L]\rho_t)\,dt + \eta\kappa\mathrm{Tr}(\mathcal{H}[L]\rho_t)^2\,dt\\ &\quad + 2\sqrt{\eta\kappa}\mathrm{Tr}(\rho_t\mathcal{H}[L]\rho_t)\,dW_t\\ &= \left(2\kappa\mathrm{Tr}(\rho_d\mathcal{D}[L]\rho_d) + \eta\kappa\mathrm{Tr}(\mathcal{H}[L]\rho_d)^2\right)dt\\ &\quad + \big(2\kappa\mathrm{Tr}(\rho_t\mathcal{D}[L]\rho_t) + \eta\kappa\mathrm{Tr}(\mathcal{H}[L]\rho_t)^2\\ &\qquad - 2\kappa\mathrm{Tr}(\rho_d\mathcal{D}[L]\rho_d) - \eta\kappa\mathrm{Tr}(\mathcal{H}[L]\rho_d)^2\big)\,dt\\ &\quad + 2\sqrt{\eta\kappa}\mathrm{Tr}(\rho_t\mathcal{H}[L]\rho_t)\,dW_t.\end{aligned} \tag{17}$$

Then by (13), (15), (16) and (17), for $t_0 > T(\omega)$ with $\omega \in A$, one has

$$\begin{aligned}&\mathrm{Tr}\left(\rho_t^2\right)\\ &= \mathrm{Tr}\left(\rho_{t_0}^2\right) + \int_{t_0}^{t} 2\kappa\mathrm{Tr}(\rho_d\mathcal{D}[L]\rho_d) + \eta\kappa\mathrm{Tr}(\mathcal{H}[L]\rho_d)^2\,ds\end{aligned}$$

$$
\begin{aligned}
&+ \int_{t_0}^{t} \Big[2\kappa \mathrm{Tr}\left(\rho_s \mathcal{D}[L]\rho_s\right) - 2\kappa \mathrm{Tr}\left(\rho_d \mathcal{D}[L]\rho_d\right) \\
&\qquad - \eta\kappa \mathrm{Tr}\left(\mathcal{H}[L]\rho_d\right)^2 + \eta\kappa \mathrm{Tr}\left(\mathcal{H}[L]\rho_s\right)^2 \Big] ds \\
&+ 2\sqrt{\eta\kappa} \int_{t_0}^{t} \mathrm{Tr}\left(\rho_s \mathcal{H}[L]\rho_s\right) dW_s \\
&< \mathrm{Tr}\left(\rho_{t_0}^2\right) + \Big\{ -2\kappa(1-\eta)\alpha + 2\kappa\varphi_1(L,\rho_d)\sqrt{2(\Delta_d - \epsilon)} \\
&\qquad + \eta\kappa\varphi_2(L,\rho_d)\sqrt{2(\Delta_d - \epsilon)} \Big\} (t - t_0) \\
&+ 2\sqrt{\eta\kappa} \int_{t_0}^{t} \mathrm{Tr}\left(\rho_s \mathcal{H}[L]\rho_s\right) dW_s.
\end{aligned} \tag{18}
$$

Moreover, by Theorem 1.51 in [44], for $0 < \epsilon < (1/2)$, one has

$$
\begin{aligned}
&\mathrm{Tr}\left(\rho_t^2\right) \\
&< \mathrm{Tr}\left(\rho_{t_0}^2\right) + \Big\{ -2\kappa(1-\eta)\alpha + 2\kappa\varphi_1(L,\rho_d)\sqrt{2(\Delta_d - \epsilon)} \\
&\qquad + \eta\kappa\varphi_2(L,\rho_d)\sqrt{2(\Delta_d - \epsilon)} \Big\} (t - t_0) \\
&+ o\left(t^{\frac{1}{2}+\epsilon}\right) + O(1) \\
&\to -\infty, \quad \text{as } t \to \infty, \quad \text{on } A
\end{aligned}
$$

where the limit is derived by the definition of Δ_d. This is a contradiction. Thus the theorem is proved. □

Theorem 3 points out that under some conditions, there is at least one eigenstate such that no matter how to choose the control channel and how to design the control law, one cannot prepare this target state with arbitrary high fidelity for all sufficient large t.

Note that if $[H_0, L] = 0$, then all the off-diagonal elements of L must be zero, so by (13), we have $\Delta_d = 0$. Hence, the above limit to the MFC is due to the non-perfect detection and the noncommutative relationship between the effective Hamiltonian H_0 and the measurement channel L. Theorem 3 is actually inspired by the Heisenberg uncertainty principle.

Now we want to compare the effects of the quantum OLC and MFC in dealing with the initial state uncertainties. This is not a theoretically obvious problem, because during the MFC process the quantum measurement itself will introduce additional uncertainty (the stochastic state collapse). Some initial analysis for a special model has been given in [22].

From Propositions 1, 2 and Theorem 1, we know that for the OLC model, one cannot achieve an arbitrarily desired target state from an arbitrary given initial state no matter how to select the measurement channel L, the control channel H_b and the control law u_t. Hence, in comparison with the MFC result as established in Theorem 2, we conclude that the measurement-based quantum feedback control is superior to the quantum OLC in dealing with the initial state uncertainties.

Note that in contrast to the classical control theory, the measurement-based quantum feedback control model (2) is different from the OLC models (5) and (6). This is due to the inherent quantum measurement back action effects which consist of the deterministic drift part and the uncertainty part. Specifically, the change from H_0' in (5) to H_0 in (2) and $\kappa\mathcal{D}[L]\rho_t dt$ describe the deterministic back action effects, while $\sqrt{\kappa\eta}\mathcal{H}[L]\rho_t dW_t$ depicts the stochastic effect of the measurement collapse.[5] Note that for pure states, the diffusion term $\sqrt{\kappa\eta}\mathcal{H}[L]\rho_t dW_t$ becomes zero almost surely only when ρ_t is one of the eigenstates of L, and hence this property will help to prepare the target state if L can be chosen appropriately. Moreover, for the open loop control, even if L is non-degenerate, $L = L^*$ and $[L, H_0] = 0$, by Propostion 2 we know that one cannot prepare an arbitrarily desired eigenstate of H_0 from any mixed initial state. Thus, it is this stochastic part of the back action effect that helps to design a feedback control law to achieve the control target.

[5]This is because $dW_t = dY_t - \sqrt{\eta\kappa}\mathcal{E}[j_t(L + L^*)|\mathcal{B}_t]dt$ with $\mathcal{E}$ being the quantum expectation, and $\mathcal{B}_t$ being the σ-algebra generated by $Y_{s \le t}$ [16]. Moreover, remember that Y_t is the observation process which is stochastic.

V. Conclusion

In this note, by investigating the stabilizing control of quantum systems with OLC and MFC respectively, we have compared their effects in dealing with the system initial state uncertainties, and have shown that the MFC is still superior to the OLC for the quantum systems in spite of the additional uncertainty induced by the quantum measurement itself. For the MFC, we have demonstrated how to select a measurement channel properly to achieve the control target. Moreover, if not, we gave a limitation theorem of the MFC. For future investigations, it may be necessary to further study the limitations of the quantum MFC, e.g., in dealing with structural uncertainties and/or in achieving other control objectives. Also, it would be interesting to study how the measurement channel and/or the control channel can be adjusted adaptively according to the quantum state in question, as discussed in e.g., [45]. The study of the above problems with the non-Markovian model would also be interesting. There is no doubt that these investigations will help us understand more about the MFC.

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On Functional Observers for Linear Time-Varying Systems

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***Abstract*—The technical note deals with existence conditions of a functional observer for linear time-varying systems in the case where the order of the observer is equal to the number of observed variables. Constructive procedures for the design of such a linear functional observer are deduced from the existence conditions. As a specific feature, the proposed procedures do not require the solution of a differential Sylvester equation. Some examples illustrate the presented results.**

***Index Terms*—Functional observer, linear time-varying system, Luenberger observer.**

I. Introduction

The interest to consider linear time-varying systems is twofold [5], [8], [17]: on the one hand as general models of linear behaviour for a plant, on the other hand as linearized models of non linear systems about a given trajectory. For state feedback control or fault diagnosis purposes the need of asymptotic observers of a given linear functional is of primary importance. Therefore, we consider the problem of observing a linear functional

$$v(t) = L(t)x(t) \tag{1}$$

where, for every time t in $\mathbb{R}^+$, $L(t)$ is a constant full row rank $(l \times n)$ differentiable matrix, and $x(t)$ is the n-dimensional state vector of the state space system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \\ y(t) &= C(t)x(t) \end{aligned} \tag{2}$$

where $u(t)$ is the p-dimensional control, and $y(t)$ is the m-dimensional output. For every t in $\mathbb{R}^+$, $A(t)$, $B(t)$, and $C(t)$ are known matrices of appropriate dimensions. To avoid tedious counts and distracting lists of differentiability requirements, we assume every time-varying matrices are such that all derivatives that appear are continuous for all t. Without loss of generality and in order to avoid useless dynamic parts in the observer, we suppose $\begin{bmatrix} C(t) \\ L(t) \end{bmatrix}$ has full row rank for all t. Indeed, the rows of an arbitrary given $L(t)$ which are linearly dependant of the rows of $C(t)$ induce obvious estimation of the corresponding components of $v(t)$ from the available informations.

The observability matrix of (2) is defined by

$$\Omega(t) = \begin{bmatrix} \Omega_0(t) \\ \Omega_1(t) \\ \vdots \\ \Omega_{n-1}(t) \end{bmatrix}$$

where $\Omega_0(t) = C(t)$, and $\Omega_j(t) = \Omega_{j-1}(t)A(t) + \dot{\Omega}_{j-1}(t)$ for $j = 1, 2, \ldots n-1$. System (2), or shortly $(A(t), C(t))$, is completely observable if $\mathrm{rank}\ (\Omega(t)) = n$ for some t in $\mathbb{R}^+$. It is uniformly observable if $\mathrm{rank}\ (\Omega(t)) = n$ for every t in $\mathbb{R}^+$ [21], [29].

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