FURTHER RESULTS ON LIMITATIONS OF SAMPLED-DATA FEEDBACK*

REN Jingli · CHENG Zhibo · GUO Lei

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Abstract To control continuous-time uncertain dynamical systems with sampled data-feedback is prevalent today, but the sampling rate is usually not allowed to be arbitrarily fast due to various physical and/or computational constrains. In this paper, the authors examine the limitations of sampled-data feedback control for a class of uncertain systems in continuous-time, with sampling rate not necessary fast enough and with the unknown system structure confined to a set of functions with both linear and nonlinear growth. The limitations of the sampled-data feedback control for the uncertain systems are established quantitatively, which extends the existing related results in the literature.

Key words Sampled-data feedback, stabilization, uncertain nonlinear systems.

1 Introduction

Feedback, a basic concept in automatic control, is used primarily for reducing the effects of various uncertainties on the desired performance of dynamical control systems, see [1–4] and references therein. The following fundamental questions arise naturally: How much uncertainty can be dealt with by feedback? and what are the limitations of feedback?

There are at least two areas in control theory which address similar problems, namely adaptive control and robust control. However, in spite of the extensive study over the past several decades, there are very few results on what the feedback mechanism cannot do (i.e.,

REN Jingli
School of Mathematics and Statistics, Zhengzhou University, Zhengzhou 454001, *China.*
Email: renjl@zzu.edu.cn.
CHENG Zhibo
School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo 454000, *China.*
GUO Lei
Key Laboratory of Systems and Control, Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, *China.* Email: lguo@amss.ac.cn.
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the limitations of the whole class of feedback laws, not only of a special class of feedbacks) for uncertain nonlinear dynamical systems. The study on the problem was originated in 1997 by Guo[5] who showed that for a typical first order discrete-time stochastic nonlinear control system with a scalar unknown parameter, the maximum nonlinear growth rate which can be dealt with by the feedback mechanism is $b = 4$, where b denotes the growth rate of the nonlinear function of the system. Afterwards, Xie and Guo[6] studied a basic class of nonparametric discrete-time uncertain nonlinear systems, and found that the maximum uncertainty that can be dealt with by the feedback mechanism is described by a ball with radius $3/2 + \sqrt{2}$ in a suitably defined normed space. In 2011, Li and Guo[7] investigated the maximum capability and limitations of the feedback mechanism in globally stabilizing a basic class of discrete-time nonlinearly parameterized dynamical systems with multiple unknown parameters. Both "possibility" and "impossibility" theorems together with a fairly complete characterization on the capability of feedback would be presented. It would be seen that to characterize the feedback capability, the growth rates of the sensitivity functions of the nonlinear dynamics with respect to the uncertain parameters play a crucial role, and a suitable decomposition of the family of the nonlinearly parameterized functions in question turns out to be necessary. Recently, Li and Lam[8] concerned with the use of the least squares (LS) algorithm to design feedback control law to stabilize a basic class of discrete-time nonlinear uncertain systems. The result shows that if a certain polynomial criterion is satisfied, the system can be stabilized by feedback based on LS algorithm for Gaussian distributed noise and unknown parameters. This result thus provides an answer of the question of what are the fundamental limitations of the discrete-time adaptive nonlinear control. Inspired by these results, there are some continuing works on the limitation of feedback, see [9–12].

All the above mentioned results are obtained for discrete-time control systems. However, a more prevalent and practical case seems to be sampled-data control systems where continuous-time uncertain systems are controlled by feedback laws constructed based on measured sampled-data with a prescribed sampling rate, due to various physical and/or communication constraints. This will inevitably give rise to a hybrid dynamical systems where continuous- and discrete-time signals are nonlinearly coupled, leading to some mathematical difficulties in theoretical analyzes and investigations. In fact, up to now, there are only few related results available in the literature.

To the best of our knowledge, the first paper in this direction seems to be Xue and Guo[13], where the authors considered the limitation of sampled-data feedback control for the following first order nonlinear system

$$\dot{x}_t = f(x_t) + u_t, \quad t \geq 0, \quad x_0 \in R^1, \tag{1}$$

the system signals are assumed to be sampled at a constant period $h > 0$, and the input is assumed to be implemented via the familiar zero-order hold device (piecewise constant function):

$$u_t = u_{kh}, \quad kh \leq t < (k+1)h, \tag{2}$$

where u_{kh} is a sampled-data feedback control, that is to say, at each step k, u_{kh} is a causal function of the past and present sampled data $\{x_0, x_h, \cdots, x_{kh}\}$. The nonlinear function f in (1) is assumed to be unknown but belongs to the following class of functions $G_c^L = \{f|f \text{ is locally Lipschitz and satifies } |f(x)| \le L|x| + c, \forall\ x \in R^1\}$, where $c > 0$ and $L > 0$ are constants. For the system, Xue and Guo[13] rigorously established an "impossibility theorem" on the capability of sampled data feedback, by showing that the class of uncertain systems (1) cannot be stabilized globally by any sampled-data feedback law whenever the sampling period h exceeds the inverse of the "slope" of the uncertain nonlinear functions (say, $1/L$) multiplied by a constant (≈7.53). It was also shown by Xue and Guo[13] that for a class of dynamical systems where the system function has a nonlinear growth rate with random disturbances described by the Brownian motion, the corresponding dynamical system will not be globally stabilizable by sampled-data feedback, even if the nonlinear function is known and the sampling rate is sufficiently fast. Afterwards, Ren and Guo[14] improved the result for the linear growth case, showing that actually once h is larger than L^{-1} multiplied by a constant (≈4.757), then there is no sampled-data control which can globally stabilize the prescribed class of uncertain nonlinear systems. Recently, Jiang and Guo[15] studied a class of uncertain systems that contains additional uncertain parameter in the input channel, and showed that the sufficient condition to globally stabilize the system by the sampled-data feedback is $Lh < \log 4$, giving the same upper bound as in [13].

In this paper, we continue on examining the limitations of sampled-data control systems (1) with sampled-data feedback control (2), and the nonlinear term f is assumed to be unknown but belongs to the following class of functions

$$G_{Lpc} = \{f|f \text{ is locally Lipschitz and satifies } |f(x)| \le L|x|^p + c, \forall x \in R^1\}, \tag{3}$$

where $c > 0$ and $L > 0$ are constants, and p is an odd number. The limitations of sampled-data feedback control for the uncertain systems are obtained, which depends on the parameters $(L,\ p,\ c)$ and subsumes the related results reported in [14].

Theorem 1 *For any given positive constants $h,\ b,\ c,\ p$ with $b > 1, c \ge 1,\ p \ge 1$ and p is odd, there exists a constant $L^* > b$ such that whenever $Lh > L^*$, then the uncertain class of systems described by G_{Lpc} is not stabilizable by sampled-data feedback control. To be specific, for any sampled-data feedback $\{u_{kh}, k \ge 0\}$, if $Lh > L^*$, then there always exists a function $f^* \in G_{Lpc}$, such that the state signal of (1)–(2) corresponding to f^* with initial point $x_0 = 0$ satisfies $(k \ge 1)$*

$$|x_{kh}| \ge (ch)^p \cdot b^{k-1} \underset{k\to\infty}{\longrightarrow} \infty. \tag{4}$$

Remark 1 The value of L^* can be any positive solution of the following equation

$$\frac{1}{L-b} + \frac{1}{L}\ln\frac{b(L-b+1)}{(L-b)(b-1)} + \frac{1}{(L-b)b+b} + \frac{1}{L}\ln\Big(2+\frac{b}{L-b}\Big) + \frac{b-1}{2L-b} = 1 \tag{5}$$

if $p = 1$; and

$$\frac{1}{(L-b)c^{p-1}} + \frac{1}{(L-b)b^pc^{p^2}+bc^p-L} + \frac{1}{(L-b)b^pc^{p^2-1}+bc^{p-1}} + \frac{1}{L-b} + \frac{b}{2L-b} = 1 \tag{6}$$

if $p > 1$.

Furthermore, in the case $p = 1$, the minimum value of L^* is $L^* \approx 4.757$ just for $b \approx 1.578$ by (5), see Figure 1. In the case $p > 1$, we take $c = 1$ for (6) for the sake of convenience. Then the equation (6) can be simplified by

$$\frac{1}{(L-b)} + \frac{1}{(L-b)(b^p-1)} + \frac{1}{(L-b)b^p+b} + \frac{1}{L-b} + \frac{b}{2L-b} = 1. \tag{7}$$

From Figures 2 we know that the minimum of L^* tends to 3.4 along with the increase of p. It shows that in this case if h is larger than L^{-1} multiplied by a constant (≈ 3.4), then there exists no sampled-data control which can globally stabilize the prescribed class of uncertain nonlinear systems.

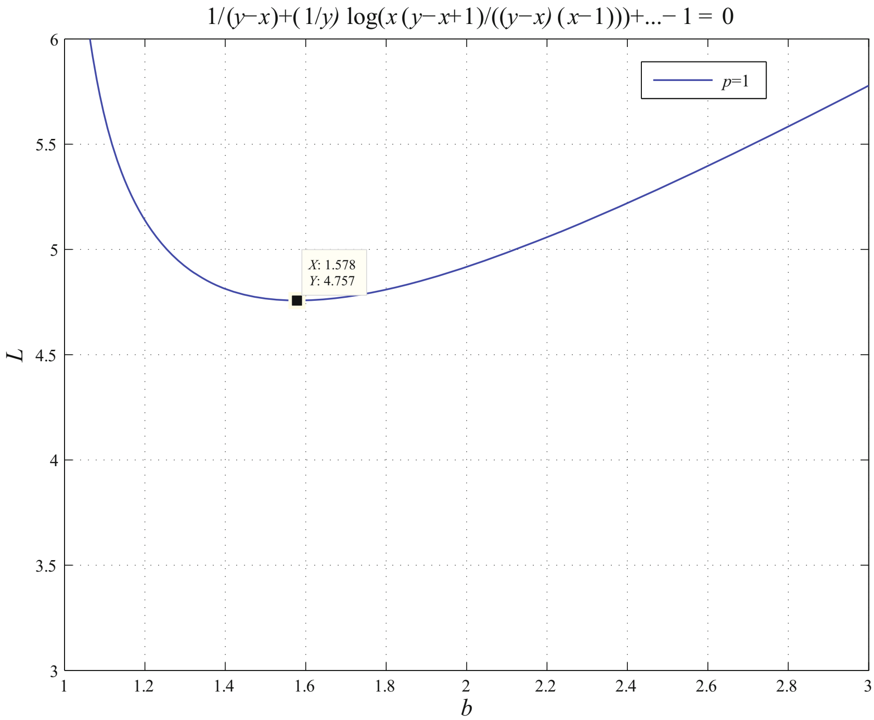

Figure 1 $p = 1$

2 Some Lemmas

In order to prove our result, we give a definition and some lemmas which are similar to those in [13].

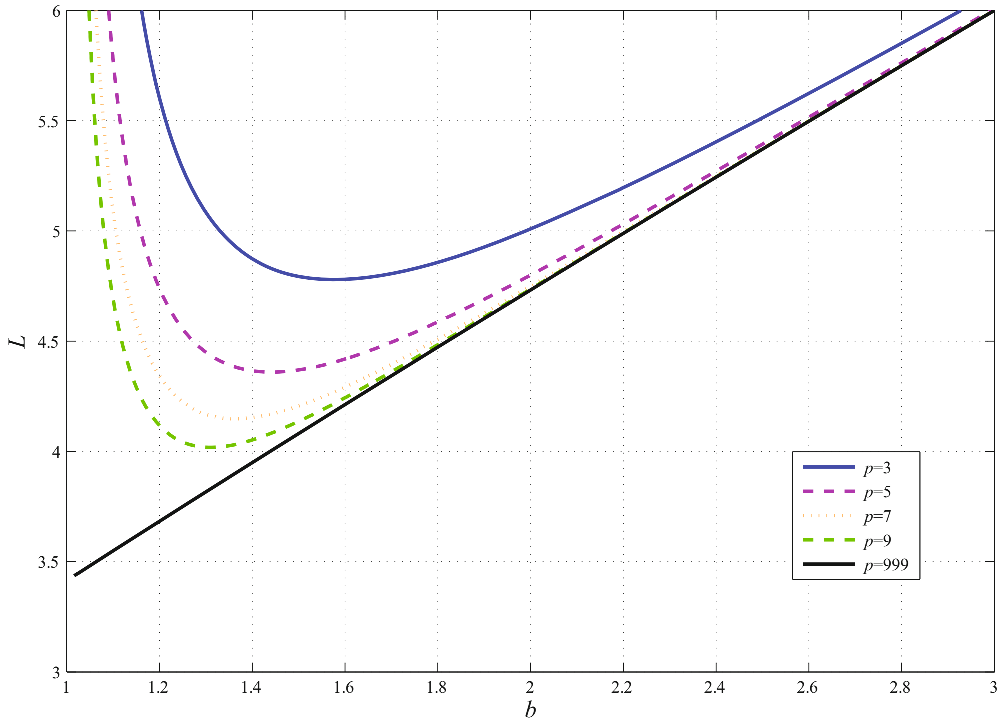

Figure 2 $p=3, p=5, p=7, p=9, p=999$

Definition 2.1 Consider the following two sampled-data control systems:

$$\Sigma_f : \begin{cases} \dot{x} = f(x) + u_t, & t \geq 0, \quad x(t_0) = a, \\ u_t = u_{kh}, & kh \leq t < (k+1)h; \end{cases} \tag{8}$$

$$\Sigma_g : \begin{cases} \dot{z} = g(z) + u_t, & t \geq 0, \quad z(t_0) = a, \\ u_t = u_{kh}, & kh \leq t < (k+1)h. \end{cases} \tag{9}$$

Under the same sampled-data control sequence $\{u_t\}$, the above two systems Σ_f and Σ_g are called N-step equivalent starting from the same initial point $a \in R^1$ and denoted by $\Sigma_f \overset{a}{\underset{N}{\Longleftrightarrow}} \Sigma_g$, s.t. $\{u_t\}$, if the sampled signals or observations of the two systems are equal, i.e., $x_{t_0+kh} = z_{t_0+kh}$, $k = 0, 1, \cdots N$. If $N = 1$, we will simply use the notation $\Sigma_f \overset{a}{\Longleftrightarrow} \Sigma_g$ s.t. u.

Lemma 2.1 (see [13]) *Consider the one dimensional autonomous system:*

$$\begin{cases} \dot{x} = \phi(x), \quad t \geq 0, \\ x(0) = x_0, \end{cases} \tag{10}$$

where $\phi(\cdot)$ is local Lipschiz. Then

(i) *the trajectory $x(t)$ is a monotonous function of t;*

(ii) *for any $T > 0$, and $x_T \neq x_0$, the necessary and sufficient condition for $x(T) = x_T$ is $\int_{x_0}^{x_T} \frac{dx}{\phi(x)} = T$ together with $\phi(x) \neq 0$ on $[\min(x_T, x_0), \max(x_T, x_0)]$.*

Lemma 2.2 *Let the function $g \in G_{Lpc}$ satisfy $g(z) \equiv L|z_0|^p + c$ for $z \geq |z_0|$ (or $g(z) \equiv -L|z_0|^p - c$ for $z \leq -|z_0|$), such that the state signal of the system*

$$\Sigma_g : \begin{cases} \dot{z} = g(z) + u_0, & t \geq 0, \\ z(0) = z_0 \end{cases} \tag{11}$$

satisfies $z(1) = z_1 > |z_0| > 0$ (or $z(1) = z_1 < -|z_0| < 0$). Then there exists a function $g_1 \in G_{Lpc}$ satisfying $g_1(z_1) = Lz_1^p + c$, and $g_1[z_0, |z_0|] = g$, $g_1[|z_0|, z_1] \geq 0$ (or $g_1(z_1) = Lz_1^p - c$ and $g_1[-z_0, |z_0|] = g$, $g_1[z_1, -|z_0|] \leq 0$), such that the state signal of the following system:

$$\Sigma_{g_1} : \begin{cases} \dot{x} = g_1(x) + u_0, & t \geq 0, \\ x(0) = z_0 \end{cases} \tag{12}$$

satisfies $x(1) = z_1$, where by definition $f_1[\alpha, \beta] = f_2$ means $f_1(x) = f_2(x)$, $\forall x \in [\min(\alpha, \beta),$ $\max(\alpha, \beta)]$.

Proof For convenience of presentation, we denote $\alpha := L|z_0|^p + c + u_0$, $\beta := z_1 - |z_0|$, and $\gamma := \mathrm{z}_1^p - |z_0|^p$ in the sequel. Obviously, we have $\beta > 0, \gamma > 0$. Also, it is easy to see $\alpha > 0$ by Lemma 2.1 and $|z_1| > z_0$.

Denote

$$\widehat{g}(z) = L|z_0|^p + c,$$

then by Lemma 2.1 we know

$$t_{|z_0| \to z_1} = \int_{|z_0|}^{z_1} \frac{dz}{\widehat{g}(z) + u_0} = \int_{|z_0|}^{z_1} \frac{dz}{L|z_0|^p + c + u_0} = \frac{\beta}{\alpha},$$

where and hereafter $t_{|z_0| \to z_1}$ denotes the time needed for $z(t)$ to travel form $|z_0|$ to z_1.

By the assumption and Lemma 2.1, we see that if we can construct a locally Lipschitz function g^* on $[|z_0|, z_1]$ to satisfy

a) $|g^*(x)| \leq L|x|^p + c,\ x \in [|z_0|, z_1]$;

b) $g^*(|z_0|) = \widehat{g}(|z_0|),\ g^*(z_1) = Lz_1^p + c$;

c) $g^*[|z_0|, z_1] \geq 0$;

d) $\int_{|z_0|}^{z_1} \frac{dz}{\overline{g}(z) + u_0} = \frac{\beta}{\alpha}$,

then $\widehat{g}[z_0, |z_0|] \oplus g^*[|z_0|, z_1]$ is just the desired function g_1.

Let s and l be two small positive constants, and let $\eta > 0$ satisfy $\alpha - \eta > 0$ and $L|z_0|^p + c - \eta > 0$. We define a function $g_{s,l}$ on the interval $[|z_0|, z_1]$:

$$g_{s,l}(x) = \begin{cases} L|z_0|^p + c - \dfrac{\eta}{s}(x - |z_0|), & x \in [|z_o|, |z_0| + s]; \\ L|z_0|^p + c - \dfrac{\eta}{s}(x - |z_0| - s), & x \in [|z_0| + s, |z_0| + 2s]; \\ L|z_0|^p + c, & x \in [|z_0| + 2s, z_1 - l]; \\ \dfrac{L(z_1^p - |z_0|^p)}{l}(x - z_1 + l) + L|z_0|^p + c, & x \in [z_1 - l, z_1]. \end{cases}$$

It is easy to verify that $g_{s,l}$ is locally Lipschitz and satisfies a), b), and c) required above when s and l are small enough.

Next, it is easy to calculate that

$$\int_{|z_0|}^{z_1} \frac{dx}{g_{s,l}(x)+u_0} = \left(\int_{|z_0|}^{|z_0|+s} + \int_{|z_0|+s}^{|z_0|+2s} + \int_{|z_0|+2s}^{z_1-l} + \int_{z_1-l}^{z_1} \right) \frac{dx}{g_{s,l}(x)+u_0}$$
$$= 2\frac{s}{\eta} \ln \frac{\alpha}{\alpha-\eta} + \frac{\beta-2s-l}{\alpha} + \frac{l}{L\gamma} \ln \frac{L\gamma+\alpha}{\alpha}.$$

Now, to make $g_{s,l}$ satisfy d), let

$$2\frac{s}{\eta} \ln \frac{\alpha}{\alpha-\eta} + \frac{\beta-2s-l}{\alpha} + \frac{l}{L\gamma} \ln \frac{L\gamma+\alpha}{\alpha} = \frac{\beta}{\alpha}.$$

We have

$$0 = \frac{2s}{\eta} \ln \frac{\alpha}{\alpha-\eta} - \frac{2s+l}{\alpha} + \frac{l}{L\gamma} \ln \frac{L\gamma+\alpha}{\alpha}.$$

So,

$$-\left[1 + \frac{\alpha}{\eta} \ln \left(1 - \frac{\eta}{\alpha}\right)\right] 2s = \left[1 - \frac{\alpha}{L\gamma} \ln \left(1 + \frac{L\gamma}{\alpha}\right)\right] l.$$

Since $\ln(1-x) < -x,\ \forall\ 0 < x < 1$, and $\ln(1+x) < x,\ \forall\ x > 0$, we know that both sides are positive. So, we can select $s > 0$ and $l > 0$ small enough to make the requirement d) hold. Finally, the $g_{s,l}$ is just the desired function g^*.

Lemma 2.3 (see [13]) *If we explicitly denote the system* (1)–(2) *as* $Sys(f, x_0, h, \{u_{kh}\})$*, then for any positive constant* λ*, there is a "linear time-transforming" relationship between the state signal* $x(t)$ *of the system* (1)–(2) *and the state signal* $z(t)$ *of the system* $Sys(\lambda f, x_0, \frac{1}{\lambda}h, \{\lambda u_{kh}\})$*, i.e.,*

$$z(t) = x(\lambda t), \quad \forall\ t \geq 0. \tag{13}$$

Lemma 2.4 *For any given constants* $b,\ c,\ p$ *with* $b > 1, c \geq 1$ *and* p *is odd, Equation* (6) (*or* (5), (7)) *has a unique solution* L_b *such that* $L_b > b$.

Proof Set

$$F(L) = \frac{1}{(L-b)c^{p-1}} + \frac{1}{(L-b)b^p c^{p^2} + bc^p - L} + \frac{1}{(L-b)b^p c^{p^2-1} + bc^{p-1}} + \frac{1}{L-b} + \frac{b}{2L-b}.$$

For any fixed $b,\ c,\ p$ with $b > 1,\ c \geq 1$ and p is odd, it is obviously $F(L)$ is strictly decreasing and continuous as $L > b$. Moreover, we have $F(L) \to 0$ as $L \to +\infty$ and $F(L) \to +\infty$ as $L \to b^+$. So, by intermediate value theorem, $F(L)$ has a unique solution $L^* > b$ such that $F(L_b) = 1$. ∎

3 Proofs of Theorem 1

In this section, we will prove Theorem 1.

Proof We first consider the case where $h = 1$.

Since the left-hand-side of (5) decreases with respect to L for any given $b > 1, c \geq 1$, we can select a small constant $\delta \in (0, 1)$ such that

$$\frac{1}{L-b} + \frac{1}{L}\ln\frac{b(L-b+1)}{(L-b)(b-1)} + \frac{1+\delta}{(L-b+1)b} + \frac{1}{L}\ln\frac{2L-b}{L-b} + \frac{b-1}{2L-b} \leq 1 \tag{14}$$

and

$$\frac{1+\delta}{L-b} + \frac{1}{L}\ln\frac{2L-b}{L-b} + \frac{b-1}{2L-b} \leq 1 \tag{15}$$

for $L > L_b > b$, where L_b is the solution of (5).

Similarly, we can also select a small constant $\delta \in (0, 1)$ such that

$$\frac{1}{(L-b)c^{p-1}} + \frac{1}{(L-b)b^p c^{p^2} + bc^p - L} + \frac{1+\delta}{(L-b)b^p c^{p^2-1} + bc^{p-1}} + \frac{1}{L-b} + \frac{b}{2L-b} \leq 1 \tag{16}$$

and

$$\frac{1+\delta}{(L-b)c^{p-1}} + \frac{1}{(L-b)} + \frac{b}{2L-b} \leq 1 \tag{17}$$

for $L > L_b > b$, where L_b is the solution of (6).

Denote

$$f_1[\alpha_1, \beta_1] \oplus f_2[\alpha_2, \beta_2] = \begin{cases} f_1(x), & x \in [\alpha_1, \beta_1], \\ f_2(x), & x \in [\alpha_2, \beta_2]; \end{cases}$$

where we assume $[\alpha_1, \beta_1] \bigcap [\alpha_2, \beta_2] = \phi$.

We will construct the function f^* step by step in a spanning manner to deal with the possible control effects of any given feedback sequences $\{u_{kh}\}$. It is divided into three steps.

Step 1 $t = 0$.

Given the initial input u_0 and $x_0 = 0$. We consider two cases separately.

Case (i) $u_0 \geq 0$.

Denote $a_1^+ = u_0 + c \geq c$ and define f^* on $[-a_1^+, 0]$ to be

$$f^*[-a_1^+, 0] = \begin{cases} (L + 2\delta^{-p}c^{-p+1})x^p + c, & x \in [-\delta c, 0]; \\ Lx^p - c, & x \in [-a_1^+, -\delta c). \end{cases} \tag{18}$$

Define $g_0^+[0, a_1^+] \equiv c$, then the system $\Sigma_{g_0^+}: \ \dot{x} = g_0^+(x) + u_0, \ t \geq 0, \ x_0 = 0$ satisfies $x(1) = a_1^+$. By Lemma 2.2 with $z_0 = 0$ and $z_1 = a_1^+$, we know that there exists a $\phi_0^+ \in G_c^L$ satisfying: $\Sigma_{\phi_0^+} \overset{0}{\Longleftrightarrow} \Sigma_{g_0^+}$, s.t. u_0, $\phi_0^+(a_1^+) = L(a_1^+)^p + c$ and $\phi_0^+[0, a_1^+] \geq 0$.

Let G_0^+ be the set consisting of only two functions defined on $[-a_1, a_1]$, i.e.,

$$G_0^+ := \left\{ f^*[-a_1^+, 0] \oplus g_0^+(0, a_1^+], \quad f^*[-a_1^+, 0] \oplus \phi_0^+(0, a_1^+] \right\} \subseteq G_{Lpc},$$

where and hereafter $G_0^+ \subseteq G_{Lpc}$ simply signifies that any function f in G_0^+ is locally Lipschitz and satisfies $|f(x)| \leq L|x|^p + c$ on its defined interval. Then the state $x(t)$ of the uncertain system (1)–(2) may be produced by a system corresponding to any function in G_0^+. But it is easily seen that for whichever function in G_0^+, we will always get $x_1 = a_1^+$ under u_0.

Case (ii) $u_0 < 0$.

Denote $a_1^- = -u_0 + c \geq c$ and define f^* on $[0, a_1^-]$ to be

$$f^*[0, a_1^-] = \begin{cases} (L + 2\delta^{-p}c^{-p+1})x^p - c, & x \in [0, \delta c]; \\ Lx^p + c, & x \in [\delta c, a_1^-). \end{cases} \tag{19}$$

Define $g_0^-[-a_1^-, 0) \equiv -c$, then the system $\Sigma_{g_0^-} : \dot{x} = g_0^-(x) + u_0, t \geq 0, x_0 = 0$ satisfies $x(1) = -a_1^-$. By Lemma 2.2 with $z_0 = 0$ and $z_1 = -a_1^-$, there exists a $\phi_0^- \in G_c^L$ satisfying: $\Sigma_{\phi_0^-} \overset{0}{\Longleftrightarrow} \Sigma_{g_0^-}$, s.t. u_0, $\phi_0^-(-a_1^-) = -L(a_1^-)^p - c$, and $\phi_0^-[-a_1^-, 0] \leq 0$.

Similar to the previous case, let us denote

$$G_0^- := \left\{ g_0^-[-a_1^-, 0) \oplus f^*[0, a_1^-], \quad \phi_0^-[-a_1^-, 0) \oplus f^*[0, a_1^-] \right\} \subseteq G_{Lpc}.$$

Then the state $x(t)$ of the uncertain system (1)–(2) has the possibility to evolve as the state of a system corresponding to any function in G_0^-. But we can see that for whichever function in G_0^-, we are bound to have $x_1 = -a_1^-$ under u_0.

Step 2 $t = 1$

We are now given the control u_1 and obverse x_1. The following discussion is divided into four cases according to the values of (x_1, u_1).

Case (i)

$$x_1 > 0, \quad u_1 \geq -(Lx_1^p + c) + b(|x_1|^p - |x_0|^p). \tag{20}$$

In the case, define $f^*[0, x_1] = \phi_0^+$, where ϕ_0^+ is defined in Case (i) of Step 1. And consequently, we have $f^*(x_1) = Lx_1^p + c$.

Next, denote

$$a_2^{++} := x_1 + (u_1 + Lx_1^p + c), \tag{21}$$

and extend the function f^* already defined on $[-x_1, x_1]$ to $[-a_2^{++}, -x_1)$ as

$$f^*[-a_2^{++}, -x_1) = \begin{cases} \dfrac{L(x_1 + \delta c)^p + f^*(-x_1) + c}{(\delta c)^p}(x + x_1)^p + f^*(-x_1), \\ \qquad x \in [-x_1 - \delta c, -x_1); \\ Lx^p - c, \quad x \in [-a_2^{++}, -x_1 - \delta c). \end{cases} \tag{22}$$

On the interval $[x_1, a_2^{++}]$, we define a function $g_1^{++}[x_1, a_2^{++}] \equiv Lx_1^p + c$. Then it is easy to verify that the system $\Sigma_{g_1^{++}} : \dot{x} = g_1^{++}(x) + u_1, t \geq 1$ travels from $x(1) = x_1$ to $x(2) = a_2^{++}$.

By Lemma 2.2 with $z_0 = x_1$ and $z_1 = a_2^{++}$, there exists a $\phi_1^{++} \in G_{Lpc}$ satisfying $\Sigma_{\phi_1^{++}} \overset{x_1}{\Longleftrightarrow} \Sigma_{g_1^{++}}$, s.t. u_1, $\phi_1^{++}(a_2^{++}) = L(a_2^{++})^p + c$ and $\phi_1^{++}[x_1, a_2^{++}] \geq 0$.

Now, denote

$$G_1^{++} := \left\{ f^*[-a_2^{++}, x_1] \oplus g_1^{++}(x_1, a_2^{++}], \quad f^*[-a_2^{++}, x_1] \oplus \phi_1^{++}(x_1, a_2^{++}] \right\} \subseteq G_{Lpc}.$$

It is clear that the state $x(t)$ of the uncertain system (1)–(2) may be produced by a system corresponding to any function in G_1^{++}. Obviously, for $f_1, f_2 \in G_1^{++}$, we have $f_1 \overset{0}{\underset{2}{\Longleftrightarrow}} f_2$, s.t. $\{u_0, u_1\}$. In particular, for whichever function in G_1^{++}, we will always have from (20)–(21) that $x_2 = a_2^{++} \geq b|x_1|^p$ under u_1.

Case (ii)

$$x_1 > 0, \qquad u_1 < -(Lx_1^p + c) + b(|x_1|^p - |x_0|^p). \tag{23}$$

In the case, define $f^*[0, x_1] = g_0^+$, hence we have $f^*[0, x_1] \equiv c \equiv Lx_0^p + c$.

Let $g_1^{+-}(-\infty, -x_1) \equiv f^*(-x_1)$ and $f_1^+(-\infty, x_1] := g_1^{+-}(-\infty, -x_1) \oplus f^*[-x_1, x_1]$. By the construction of $f_1^+(-\infty, x_1]$, it follows that

$$f_1^+(x) = \begin{cases} c, & x \in [0, x_1]; \\ (L + 2\delta^{-p}c^{1-p})x^p + c, & x \in [-\delta c, 0]; \\ Lx^p - c, & x \in [-x_1, -\delta c]; \\ -Lx_1^p - c, & x \leq -x_1. \end{cases} \tag{24}$$

We claim that the system: $\dot{z} = f_1^+(z) + u_1, t \geq 1, z(1) = x_1$ satisfies

$$z(2) \leq -bx_1^p. \tag{25}$$

In fact, from (23) it is easy to see $u_1 \leq -[(L-b)x_1^p + c]$ and $f_1^+(x) + u_1 < 0, \forall x \leq x_1$, so we can apply Lemma 2.1 here. By Lemma 2.1, it is clear that to verify the desired result (25) we need only to show that

$$t := \int_{x_1}^{-bx_1^p} \frac{dx}{f_1^+(x) + u_1} \leq 1.$$

For simplicity, we will continually use $t_{\alpha \to \beta}$ to denote the time that the trajectory needs to travel from α to β in the sequel.

By (24), we know that $f_1^+(x) \leq c$ on $[-\delta c, x_1]$. Recalling that $x_1 \geq c$, if $p = 1$, we have

$$t_{x_1 \to -\delta c} \leq \int_{x_1}^{-\delta c} \frac{dx}{c - [(L-b)x_1 + c]} \leq \frac{1+\delta}{(L-b)}.$$

Also, we have

$$\begin{aligned} t_{-\delta c \to -x_1} &\leq \int_{-\delta c}^{-x_1} \frac{dx}{Lx - (L-b)x_1 - 2c} \\ &\leq \frac{1}{L} \ln \frac{(2L-b)x_1 + 2c}{(L-b)x_1 + (L\delta + 2)c} \leq \frac{1}{L} \ln \frac{2L-b}{L-b}, \end{aligned}$$

and

$$t_{-x_1 \to -bx_1} \leq \frac{bx_1 - x_1}{(2L-b)x_1 + 2c} \leq \frac{b-1}{2L-b}.$$

In view of (15), we have $t_{x_1 \to -bx_1} \leq \frac{1+\delta}{L-b} + \frac{1}{L} \ln \frac{2L-b}{L-b} + \frac{b-1}{2L-b} \leq 1$ and then (25) holds.

If $p > 1$,

$$t_{x_1 \to -\delta c} \le \int_{x_1}^{-\delta c} \frac{dx}{c - [(L-b)x_1^p + c]} \le \frac{1+\delta}{(L-b)c^{p-1}}.$$

Also, we have

$$\begin{aligned} t_{-\delta c \to -x_1} &\le \int_{-\delta c}^{-x_1} \frac{dx}{Lx^p - (L-b)x_1^p - 2c} \\ &\le \int_{-\delta c}^{-x_1} \frac{dx}{-(L-b)x_1^p} \le \frac{1}{(L-b)} \end{aligned}$$

and

$$t_{-x_1 \to -bx_1^p} \le \frac{bx_1^p - x_1}{(2L-b)x_1^p + 2c} \le \frac{b}{2L-b}.$$

In view of (17), we have

$$t_{x_1 \to -bx_1^p} \le \frac{1+\delta}{(L-b)c^{p-1}} + \frac{1}{L-b} + \frac{b}{2L-b} \le 1$$

and then (25) holds.

Next, denote $a_2^{+-} := -z(2) > x_1$. By Lemma 2.2 with $z_0 = x_1$ and $z_1 = -a_2^{+-}$, there exists a $\phi_1^{+-} \in G_{Lpc}$ satisfying: $\Sigma_{\phi_1^{+-}} \overset{x_1}{\Longleftrightarrow} \Sigma_{\hat{f}_1^+(-\infty,x_1]}$, s.t. u_1; $\phi_1^{+-}[-x_1, x_1] = f^*[-x_1, x_1]$, $\phi_1^{+-}(-a_2^{+-}) = -L(a_2^{+-})^p - c$, and $\phi_1^{+-}[-a_2^{+-}, -x_1] \le 0$.

Let

$$f^*(x_1, a_2^{+-}] = \begin{cases} \dfrac{L(x_1+\delta c)^p - f^*(x_1) + c}{(\delta c)^p}(x - x_1)^p + f^*(x_1), & x \in (x_1, x_1 + \delta c]; \\ Lx^p + c, & x \in (x_1 + \delta c, a_2^{+-}], \end{cases} \tag{26}$$

and denote

$$G_1^{+-} := \left\{ g_1^{+-}[-a_2^{+-}, -x_1) \oplus f^*[-x_1, a_2^{+-}], \quad \phi_1^{+-}[-a_2^{+-}, -x_1) \oplus f^*[-x_1, a_2^{+-}] \right\} \subseteq G_{Lpc}.$$

Then the state $x(t)$ of the uncertain system (1)–(2) may be produced by a system corresponding to any function in G_1^{+-}. But it can be easily seen that, for whichever function in G_1^{+-}, we will bound to get $x_2 = -a_2^{+-}$ under u_1. Obviously, we get from (25) that $|x_2| \ge b|x_1|^p$.

Case (iii)

$$x_1 < 0, \quad u_1 \ge -(Lx_1^p - c) - b(|x_1|^p - |x_0|^p). \tag{27}$$

The conditions in this case are "symmetric" to those in Case (ii), so the proof ideas are similar.

Define $f^*[x_1, 0] = g_0^-$. Let

$$g_1^{-+}(-x_1, \infty) \equiv f^*(-x_1), \quad f_1^-[x_1, , \infty) := f^*[x_1, -x_1] \oplus g_1^{-+}(-x_1, \infty).$$

Similarly, we can show that $\dot{z} = f_1^-(z) + u_1,\ t \ge 1,\ z(1) = x_1$ satisfies

$$z(2) \ge -bx_1^p. \tag{28}$$

Now denote $a_2^{-+} := z(2) > 0$. By Lemma 2.2, there exists a $\phi_1^{-+} \in G_{Lpc}$ satisfying: $\Sigma_{\phi_1^{-+}} \overset{x_1}{\Longleftrightarrow} \Sigma_{\hat{f}_1^{-}[x_1,\infty)}$, s.t. u_1; $\phi_1^{-+}[x_1,-x_1] = f^*[x_1,-x_1]$, $\phi_1^{-+}(a_2^{-+}) = L(a_2^{-+})^p + c$, and $\phi_1^{-+}[-x_1, a_2^{-+}] \geq 0$.

Let

$$f^*[-a_2^{-+}, x_1) = \begin{cases} \dfrac{-L(x_1-\delta c)^p + f^*(x_1) + c}{(\delta c)^p}(x-x_1)^p + f^*(x_1), & x \in [x_1-\delta c, x_1); \\ Lx^p - c, & x \in [-a_2^{-+}, x_1-\delta c), \end{cases} \tag{29}$$

and denote

$$G_1^{-+} := \left\{ f^*[-a_2^{-+}, -x_1] \oplus g_1^{-+}(-x_1, a_2^{-+}], \quad f^*[-a_2^{-+}, -x_1] \oplus \phi_1^{-+}(-x_1, a_2^{-+}] \right\} \subseteq G_{Lpc}.$$

Then the state $x(t)$ of the uncertain system (1)–(2) may be produced by a system corresponding to any function in G_1^{-+}. But it can be easily seen that, for whichever function in G_1^{-+}, we will get $x(2) = a_2^{-+}$ under u_1. Obviously, it follows from (28) that $|x_2| \geq b|x_1|^p$.

Case (iv)

$$x_1 < 0, \quad u_1 < -(Lx_1^p - c) - b(|x_1|^p - |x_0|^p). \tag{30}$$

This time we define $f^*[x_1, 0] = \phi_0^-$, and get $f^*(x_1) = Lx_1^p - c$.

Denote

$$a_2^{--} := -(x_1 + (u_1 + Lx_1^p - c)) > -x_1, \tag{31}$$

and extend the definition of f^* to $(-x_1, a_2^{--}]$ as

$$f^*(-x_1, a_2^{--}] = \begin{cases} \dfrac{L(-x_1+\delta c)^p - f^*(-x_1) + c}{(\delta c)^p}(x+x_1)^p + f^*(-x_1), \\ \qquad x \in (-x_1, -x_1+\delta c]; \\ Lx^p + c, \quad x \in (-x_1+\delta c, a_2^{--}]. \end{cases} \tag{32}$$

Define $g_1^{--}[-a_2^{--}, x_1] \equiv Lx_1^p - c$, then $\Sigma_{g_1^{--}} : \dot{x} = g_1^{--}(x) + u_1, t \geq 1$ travels from $x(1) = x_1$ to $x(2) = -a_2^{--}$.

By Lemma 2.2, there exists a $\phi_1^{--} \in G_{Lpc}$ satisfying: $\Sigma_{\phi_1^{--}} \overset{x_1}{\Longleftrightarrow} \Sigma_{g_1^{--}}$, s.t. u_1, $\phi_1^{--}(-a_2^{--}) = -L(a_2^{--})^p - c$, and $\phi_1^{--}[-a_2^{--}, x_1] \leq 0$.

Now, denote

$$G_1^{--} := \left\{ g_1^{--}[-a_2^{--}, x_1) \oplus f^*[x_1, a_2^{--}], \phi_1^{--}[-a_2^{--}, x_1) \oplus f^*[x_1, a_2^{--}] \right\} \subseteq G_{Lpc}.$$

Then the state $x(t)$ of the uncertain system (1)–(2) may be produced by a system corresponding to any function in G_1^{--} and for whichever function in G_1^{--}, we will get $x_2 = -a_2^{--}$ under u_1. Moreover, we get from (30)–(31) that $|x_2| \geq b|x_1|^p$.

To proceed further, we denote

$$g_1^+ := \begin{cases} g_1^{++}, \text{ in Case (i)}; \\ g_1^{-+}, \text{ in Case (iii)}. \end{cases} \qquad g_1^- := \begin{cases} g_1^{+-}, \text{ in Case (ii)}; \\ g_1^{--}, \text{ in Case (iv)}. \end{cases}$$

$$\phi_1^+ := \begin{cases} \phi_1^{++}, & \text{in Case (i);} \\ \phi_1^{-+}, & \text{in Case (iii).} \end{cases} \qquad \phi_1^- := \begin{cases} \phi_1^{+-}, & \text{in Case (ii);} \\ \phi_1^{--}, & \text{in Case (iv).} \end{cases}$$

Step 3 $t = k$.

We now use the induction argument. Suppose that at some time k, for the given feedback sequence $\{u_0, u_1, \cdots, u_k\}$ we have found a trajectory $\{x_1, x_1, \cdots, x_{k+1}\}$ together with the corresponding nonlinear system or function f^*, which have the following properties:

a) $|x_{k+1}| \geq b|x_k|^p$, $|x_1| \geq c$.

b) If $x_{k+1} > 0$, then f^* is defined on the interval $[-|x_{k+1}|, |x_k|]$, together with g_k^+ and ϕ_k^+ defined on $(|x_k|, x_{k+1}]$, such that

$$\Sigma_{f^*[x_k,|x_k|]\oplus g_k^+(|x_k|,x_{k+1}]} \overset{0}{\underset{k+1}{\Longleftrightarrow}} \Sigma_{f^*[x_k,|x_k|]\oplus \phi_k^+(|x_k|,x_{k+1}]}$$

$$\text{s.t.} \quad \{u_t, t = 0, 1, \cdots, k\}.$$

c) If $x_{k+1} < 0$, then f^* is defined on the interval $[-|x_k|, |x_{k+1}|]$, together with g_k^- and ϕ_k^- defined on $[x_{k+1}, -|x_k|)$, such that

$$\Sigma_{g_k^+[x_{k+1},-|x_k|)\oplus f^*[-|x_k|,x_k]} \overset{0}{\underset{k+1}{\Longleftrightarrow}} \Sigma_{\phi_k^+[x_{k+1},-|x_k|)\oplus f^*[-|x_k|,x_k]}$$

$$\text{s.t.} \quad \{u_t, t = 0, 1, \cdots, k\}.$$

Given the control u_{k+1} and the observation x_{k+1}, we need to show that a)–c) still hold with k replaced by $k+1$. Similar to Step 2, we consider four cases separately.

Case (i)

$$x_{k+1} > 0, \qquad u_{k+1} \geq -(Lx_{k+1}^p + c) + b(|x_{k+1}|^p - |x_k|^p). \tag{33}$$

In the case define $f^*[|x_k|, x_{k+1}] = \phi_k^+$, and consequently we have $f^*(x_{k+1}) = Lx_{k+1}^p + c$.

Next, denote

$$a_{k+2}^{++} := x_{k+1} + (u_{k+1} + Lx_{k+1}^p + c) > x_{k+1}, \tag{34}$$

and extend the definition of f^* already defined on $[-x_{k+1}, x_{k+1}]$ to $[-a_{k+2}^{++}, -x_{k+1})$ by

$$f^*[-a_{k+2}^{++}, -x_{k+1}) = \begin{cases} \dfrac{L(x_{k+1}+\delta c)^p + f^*(-x_{k+1}) + c}{(\delta c)^p}(x + x_{k+1})^p + f^*(-x_{k+1}), \\ \qquad x \in [-x_{k+1} - \delta c, -x_{k+1}); \\ Lx^p - c, \qquad x \in [-a_{k+2}^{++}, -x_{k+1} - \delta c). \end{cases} \tag{35}$$

Define $g_{k+1}^{++}[x_{k+1}, a_{k+2}^{++}] \equiv Lx_{k+1}^p + c$, so the system $\Sigma_{g_{k+1}^{++}} : \dot{x} = g_{k+1}^{++}(x) + u_{k+1}, t \geq k+1$, travels from $x(k+1) = x_{k+1}$ to $x(k+2) = a_{k+2}^{++}$.

By Lemma 2.2 there exists a ϕ_{k+1}^{++} satisfying: $\Sigma_{\phi_{k+1}^{++}} \overset{x_{k+1}}{\Longleftrightarrow} \Sigma_{g_{k+1}^{++}}$, s.t. u_{k+1} and $\phi_{k+1}^{++}(a_{k+2}^{++}) = L(a_{k+2}^{++})^p + c$, $\phi_{k+1}^{++}[x_{k+1}, a_{k+2}^{++}] \geq 0$.

Now, denote

$$G_{k+1}^{++} := \{f^*[-a_{k+2}^{++}, x_{k+1}] \oplus g_{k+1}^{++}(x_{k+1}, a_{k+2}^{++}], f^*[-a_{k+2}^{++}, x_{k+1}] \oplus \phi_{k+1}^{++}(x_{k+1}, a_{k+2}^{++}]\}$$
$$\subseteq G_{Lpc}.$$

Then the state $x(t)$ of the uncertain system (1)–(2) may be produced by a system corresponding to any function in G_{k+1}^{++} (obviously $\forall f_1, f_2 \in G_{k+1}^{++}$,we have $\Sigma_{f_1} \overset{0}{\underset{k+2}{\Longleftrightarrow}} \Sigma_{f_2}$, s.t. $\{u_t, t = 0, 1, \cdots, k+1\}$). But it is easily seen that, for whichever function in G_{k+1}^{++}, we will get $x_{k+2} = a_{k+2}^{++}$ under u_{k+1}.

In view of (33), (34), and above induction, we have

$$|x_{k+2}| = a_{k+2}^{++} \geq x_{k+1} + b|x_{k+1}|^p - b|x_k|^p \geq b|x_{k+1}|^p.$$

Case (ii)

$$x_{k+1} > 0, \quad u_{k+1} < -(Lx_{k+1}^p + c) + b(|x_{k+1}|^p - |x_k|^p). \tag{36}$$

In the case, we define $f^*[|x_k|, x_{k+1}] = g_k^+$, i.e., $f^*[|x_k|, x_{k+1}] \equiv L|x_k|^p + c$.

Let $g_{k+1}^{+-}(-\infty, -x_{k+1}) \equiv f^*(-x_{k+1})(=-Lx_{k+1}^p - c)$, and let

$$f_{k+1}^+(-\infty, x_{k+1}] := g_{k+1}^{+-}(-\infty, -x_{k+1}) \oplus f^*[-x_{k+1}, x_{k+1}].$$

Now we prove that the system: $\dot{z} = f_{k+1}^+(z) + u_{k+1},\ t \geq k+1,\ z(k+1) = x_{k+1}$ satisfies

$$z(k+2) \leq -bx_{k+1}^p. \tag{37}$$

By the construction of f_{k+1}^+, we see that

$$f_{k+1}^+(z) \leq M(z), \quad \forall z \leq x_{k+1},$$

where $M(\cdot)$ is defined by

$$M(z) = \begin{cases} L|x_k|^p + c, & z \in [|x_k|, x_{k+1}]; \\ Lz^p + c, & z \in [0, |x_k|]; \\ c, & z \in [-\delta c, 0]; \\ 0, & z \in [-\delta c - |x_k|, -\delta c]; \\ Lz^p - c, & z \in [-x_{k+1}, -\delta c - |x_k|]; \\ -(Lx_{k+1}^p + c), & z \leq -x_{k+1}. \end{cases}$$

Also, by induction we have

$$|x_k| \geq c, \quad x_{k+1} \geq b|x_k|^p > 0, \tag{38}$$

and by (36)

$$u_{k+1} < -[(L-b)x_{k+1}^p + b|x_k|^p + c]. \tag{39}$$

Now, we define $y(t)$ to satisfy

$$\begin{cases} \dot{y} = M(y) - [(L-b)x_{k+1}^p + b|x_k|^p + c]; \\ y(k+1) = x_{k+1}. \end{cases} \tag{40}$$

Since $M(y(k+1)) - [(L-b)x_{k+1}^p + b|x_k|^p + c] < 0$, by Lemma 2.1, we know that $y(t)$ is monotonically decreasing. By the comparison principle for differential equations, we have $z(t) \le y(t)$, $t \ge k+1$. So, to prove the desired result we need only to show that $y(k+2) \le -bx_{k+1}^p$.

Now, by the definition of $M(z)$ and Lemma 2.1, and with the help of (38) and (39), it is clear that the time needed for the system (40) to travel from x_{k+1} to $-bx_{k+1}^p$ via $|x_k|$, 0, $-\delta c$, $-|x_k| - \delta c$, and $-x_{k+1}$ can be calculated as follows.

If $p = 1$, we have

$$\begin{aligned}
t_{x_{k+1}\to|x_k|} &\le \int_{x_{k+1}}^{|x_k|} \frac{dz}{(L-b)(|x_k| - x_{k+1})} = \frac{1}{L-b}, \\
t_{|x_k|\to 0} &\le \int_{|x_k|}^{0} \frac{dz}{Lz + c - [(L-b)x_{k+1} + b|x_k| + c]} \le \frac{1}{L} \ln \frac{b(L-b+1)}{(L-b)(b-1)}, \\
t_{0\to-\delta c} &\le \int_{0}^{-\delta c} \frac{dz}{-[(L-b)x_{k+1} + b|x_k|]} \le \frac{\delta c}{(L-b)b^p|x_k| + b|x_k|} \le \frac{\delta}{(L-b)b + b}, \\
t_{-\delta c\to-\delta c-|x_k|} &\le \int_{-\delta c}^{-\delta c-|x_k|} \frac{dz}{-[(L-b)x_{k+1} + b|x_k + c]} \le \frac{|x_k|}{(L-b)b|x_k| + b|x_k| + c} \\
&\le \frac{1}{(L-b)b + b}, \\
t_{-|x_k|-\delta c\to-x_{k+1}} &\le \int_{-|x_k|-\delta c}^{-x_{k+1}} \frac{dz}{Lz - c - [(L-b)x_{k+1} + b|x_k| + c]} \\
&= \frac{1}{L} \ln \frac{(2L-b)x_{k+1} + b|x_k| + 2c}{(L-b)x_{k+1} + (L+b)|x_k| + (L\delta + 2)c} \\
&\le \frac{1}{L} \ln \frac{2L-b}{L-b}, \\
t_{-x_{k+1}\to-bx_{k+1}^p} &\le \int_{-x_{k+1}}^{-bx_{k+1}} \frac{dz}{-(Lx_{k+1} + c) - [(L-b)x_{k+1} + b|x_k| + c]} \\
&= \frac{bx_{k+1} - x_{k+1}}{(2L-b)x_{k+1} + b|x_k| + 2c} \\
&\le \frac{b-1}{2L-b}.
\end{aligned}$$

Therefore, by (14), we know $t_{x_{k+1}\to-bx_{k+1}} \le 1$ and $y(k+2) \le -bx_{k+1}$.

If $p > 1$,

$$\begin{aligned}
t_{x_{k+1}\to|x_k|} &\le \int_{x_{k+1}}^{|x_k|} \frac{dz}{(L-b)(|x_k|^p - x_{k+1}^p)}, \\
t_{|x_k|\to 0} &= \int_{|x_k|}^{0} \frac{dz}{Lz^p + c - [(L-b)x_{k+1}^p + b|x_k|^p + c]}
\end{aligned}$$

$$\leq \int_{|x_k|}^{1} \frac{dz}{Lz^p - [(L-b)x_{k+1}^p + b|x_k|^p]} + \int_{1}^{0} \frac{dz}{Lz^p - [(L-b)x_{k+1}^p + b|x_k|^p]}.$$

Let

$$t_{|x_k|\to 1} = \int_{|x_k|}^{1} \frac{dz}{Lz^p - [(L-b)x_{k+1}^p + b|x_k|^p]};$$

$$t_{1\to 0} = \int_{1}^{0} \frac{dz}{Lz^p - [(L-b)x_{k+1}^p + b|x_k|^p]}.$$

So, we have

$$\begin{aligned}
t_{x_{k+1}\to 1} &= t_{x_{k+1}\to |x_k|} + t_{|x_k|\to 1} \\
&\leq \int_{x_{k+1}}^{|x_k|} \frac{dz}{(L-b)(|x_k|^p - x_{k+1}^p)} + \int_{|x_k|}^{1} \frac{dz}{Lz^p - [(L-b)x_{k+1}^p + b|x_k|^p]} \\
&\leq \int_{x_{k+1}}^{|x_k|} \frac{dz}{(L-b)(|x_k|^p - x_{k+1}^p)} + \int_{|x_k|}^{1} \frac{dz}{L|x_k|^p - [(L-b)x_{k+1}^p + b|x_k|^p]} \\
&= \int_{x_{k+1}}^{1} \frac{dz}{(L-b)(|x_k|^p - x_{k+1}^p)} \\
&= \frac{x_{k+1} - 1}{(L-b)(x_{k+1}^p - |x_k|^p)} \\
&\leq \frac{x_{k+1} - 1}{(L-b)(x_{k+1}^p - x_{k+1})} \\
&\leq \frac{1}{(L-b)c^{p-1}};
\end{aligned}$$

$$\begin{aligned}
t_{1\to 0} &= \int_{1}^{0} \frac{dz}{Lz^p - [(L-b)x_{k+1}^p + b|x_k|^p]} \\
&\leq \frac{1}{(L-b)x_{k+1}^p + b|x_k|^p - L} \\
&\leq \frac{1}{(L-b)b^p c^{p^2} + bc^p - L};
\end{aligned}$$

$$\begin{aligned}
t_{0\to -\delta c} &\leq \int_{0}^{-\delta c} \frac{dz}{-[(L-b)x_{k+1}^p + b|x_k|^p]} \\
&\leq \frac{\delta c}{(L-b)b^p|x_k|^{p^2} + b|x_k|^p} \\
&\leq \frac{\delta}{(L-b)b^p c^{p^2-1} + bc^{p-1}},
\end{aligned}$$

$$\begin{aligned}
t_{-\delta c\to -\delta c-|x_k|} &\leq \int_{-\delta c}^{-\delta c-|x_k|} \frac{dz}{-[(L-b)x_{k+1}^p + b|x_k|^p + c]} \\
&\leq \frac{|x_k|}{(L-b)b^p|x_k|^{p^2} + b|x_k|^p + c} \\
&\leq \frac{1}{(L-b)b^p c^{p^2-1} + bc^{p-1}},
\end{aligned}$$

$$
\begin{aligned}
t_{-|x_k|-\delta c\to -x_{k+1}} &\le \int_{-|x_k|-\delta c}^{-x_{k+1}} \frac{dz}{Lz^p - c - [(L-b)x_{k+1}^p + b|x_k|^p + c]}\\
&\le \int_{-|x_k|-\delta c}^{-x_{k+1}} \frac{dz}{-(L-b)x_{k+1}^p}\\
&\le \frac{x_{k+1} - (|x_k| + \delta c)}{(L-b)x_{k+1}^p}\\
&\le \frac{1}{L-b},\\
t_{-x_{k+1}\to -bx_{k+1}^p} &\le \int_{-x_{k+1}}^{-bx_{k+1}^p} \frac{dz}{-(Lx_{k+1}^p + c) - [(L-b)x_{k+1}^p + b|x_k|^p + c]}\\
&= \frac{bx_{k+1}^p - x_{k+1}}{(2L-b)x_{k+1}^p + b|x_k|^p + 2c}\\
&\le \frac{bx_{k+1}^p}{(2L-b)x_{k+1}^p} = \frac{b}{2L-b}.
\end{aligned}
$$

Therefore, by (15), we know $t_{x_{k+1}\to -bx_{k+1}^p} \le 1$ and $y(k+2) \le -bx_{k+1}^p$.

Next, denote $a_{k+2}^{+-} := -z(k+2) > 0$, and there exists a $\phi_{k+1}^{+-} \in G_{Lpc}$ satisfying: $\Sigma_{\phi_{k+1}^{+-}} \overset{x_{k+1}}{\Longleftrightarrow} \Sigma_{\hat{f}_{k+1}^{+}(-\infty, x_{k+1}]}$, s.t. u_{k+1}; $\phi_{k+1}^{+-}[-x_{k+1}, x_{k+1}] = f^*[-x_{k+1}, x_{k+1}]$, $\phi_{k+1}^{+-}(-a_{k+2}^{+-}) = -L(a_{k+2}^{+-})^p - c$ and $\phi_{k+1}^{+-}[-a_{k+2}^{+-}, -x_{k+1}] \le 0$.

Now, let

$$
f^*(x_{k+1}, a_{k+2}^{+-}] = \begin{cases} \dfrac{L(x_{k+1}+\delta c)^p - f^*(x_{k+1}) + c}{(\delta c)^p}(x - x_{k+1})^p + f^*(x_{k+1}), \\ \qquad\qquad x \in (x_{k+1}, x_{k+1} + \delta c]; \\ Lx^p + c, \quad x \in (x_{k+1} + \delta c, a_{k+2}^{+-}], \end{cases} \tag{41}
$$

and denote

$$
\begin{aligned}
G_{k+1}^{+-} := \big\{ & g_{k+1}^{+-}[-a_{k+2}^{+-}, -x_{k+1}) \oplus f^*[-x_{k+1}, a_{k+2}^{+-}],\\
& \phi_{k+1}^{+-}[-a_{k+2}^{+-}, -x_{k+1}) \oplus f^*[-x_{k+1}, a_{k+2}^{+-}] \big\} \subseteq G_{Lpc}.
\end{aligned}
$$

Then the state $x(t)$ of the uncertain system (1)–(2) may be produced by a system corresponding to any function in G_{k+1}^{+-} and for whichever function in G_{k+2}^{+-}, we have $x_{k+2} = -a_{k+2}^{+-} \ge b|x_{k+1}|^p$.

Case (iii)

$$
x_{k+1} < 0, \quad u_{k+1} > -(Lx_{k+1}^p - c) - b(|x_{k+1}|^p - |x_k|^p). \tag{42}
$$

Case (iv)

$$
x_{k+1} < 0, \qquad u_{k+1} \le -(Lx_{k+1}^p - c) - b(|x_{k+1}|^p - |x_k|^p). \tag{43}
$$

The case (iii) and (iv) are "symmetric" to the Case (ii) and (i), respectively, so similarly we can also get that a)–c) still hold in the cases with k replaced by $k+1$.

Therefore, according to the induction principle, for any given feedback sequence $\{u_i, i \geq 0\}$ we can define a nonlinear function $f^* \in G_c^L$ such that the corresponding closed-loop system with initial point $x_0 = 0$ is unstable in the sense that $|x_k| \geq c^p \cdot b^{k-1}$.

Moreover, by Lemma 2.3, the stabilization of $Sys(f, x_0 = 0, h, \{\widehat{u}_{kh}\})$ is equivalent to that of $Sys(hf, x_0 = 0, 1, \{h\widehat{u}_{kh}\})$. If $hL > l$ where l is defined by (10), then according to the results established above, there exists a function hf^* in G_{hc}^{hL} which makes the state of $Sys(hf^*, 0, 1, \{h\widehat{u}_{kh}\})$ to satisfy

$$|z(k)| \geq (ch)^p \cdot b^{k-1}, \quad k = 1, 2, \cdots.$$

Thus by Lemma 2.3, f^* is the desired function such that the state of $Sys(f, x_0 = 0, h, \{\widehat{u}_{kh}\})$ satisfies

$$|x(kh)| \geq (ch)^p \cdot b^{k-1}, \quad k = 1, 2, \cdots,$$

and hence the proof is completed. ∎

4 Concluding Remarks

We have in this paper investigated the limitations of sampled-data feedback in globally stabilizing a class of (unstable) dynamical systems with structural uncertainty described by a set of functions with both linear and nonlinear growth. Some impossibility results are established which show that as long as the sampling period is larger than a certain value, the corresponding uncertain class of systems cannot be globally stabilized by any sampled-data feedback. Of course, due to the hybrid and nonlinear nature of the closed-loop control systems, where both continuous-time and discrete-time signals are mixed, it is quite challenging to obtain a critical value for the sampling period. This belongs to further investigation.

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