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The Smallest Possible Interaction Radius for Synchronization of Self-Propelled Particles*

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Abstract. A central and fundamental issue in the theory of complex systems is to understand how local rules lead to collective behavior of the whole system. This paper will investigate a typical collective behavior (synchronization) of a self-propelled particle system modeled by the nearest neighbor rules. While connectivity of the dynamic neighbor graphs associated with the underlying systems is crucial for synchronization, it is widely known that the verification of such dynamical connectivity is at the core of theoretical analysis. Ideally, conditions used for synchronization should be imposed on the model parameters and the initial states of the particles. One crucial model parameter is the interaction radius, and we are interested in the following natural and basic question: What is the smallest interaction radius for synchronization? In this paper, we will show that, in a certain sense, the smallest possible interaction radius approximately equals $\sqrt{\log n/(\pi n)}$, with n being the population size, which coincides with the critical radius for connectivity of static random geometric graphs known in the literature.

Key words. multiagent system, Vicsek's model, consensus, random geometric graphs, the second largest eigenvalue

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1. Introduction.

A complex system is composed of multiple interacting elements, which produces some global behaviors at the macro level called collective behavior or emergent behavior [1]. Complex systems exist almost everywhere in nature and in human social and economic systems, and so have generated great interest in researchers from various fields. A central issue of complex system study is to understand how local interactions among the elements lead to collective behavior of the whole group.

Self-propelled particle (SPP) systems are a typical kind of complex system. To investigate clustering, transport, and phase transition in nonequilibrium systems, a well-known SPP model was proposed by Vicsek et al. [2]. This model is constructed by the following simple rule: at each time step each particle is driven by a constant

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absolute velocity, where its heading is updated by the average value in its neighborhood with some random perturbation added. In computer simulations, this model exhibits a kinetic phase transition through spontaneous symmetry breaking of the rotational symmetry [2]. Despite the simplicity of its rule, this SPP model captures some common features of a number of physical, biological, and social systems. As pointed out by the authors of [2], such a model is a nonequilibrium analogue of the ferromagnetic type of model and may be applied to investigate the collective motion of a wide range of biological systems such as schools of fish, herds of quadrupeds, and flocks of birds. Moreover, because the model reveals that a simple local interacting rule may result in interesting global behaviors, it can be considered as a starting point for the theoretic research of complex systems. Due to such fundamental importance, the model has attracted much attention from various fields such as biology, physics, chemistry, engineering, and mathematics. For example, the phase transition of the model was investigated in [3, 4, 5, 6]; a one-dimensional version [7] was used to study the collective motion of desert locusts [8, 9]; an initial step toward mathematically analyzing a simplified model was made by [10]; and more studies related to the model were reviewed in [11, 12, 13].

A basic goal of the analysis of models with local interacting particles is to establish conditions under which the system reaches ordered states. Intuitively, the larger the interacting range is, the easier the system becomes ordered; so, a natural and basic question is: How small must the local interaction range be in order to guarantee that the system is ordered? This paper will investigate this question on an SPP model under a random framework. We will show that to guarantee all particles move in the same direction, in a certain sense, the smallest possible interaction radius approximately equals $\sqrt{\log n/(\pi n)}$, with n the population size, which coincides with the critical radius for connectivity of random geometric graphs given by Gupta and Kumar [14]. This investigation should offer insights into the collective behaviors of more general nonequilibrium dynamic systems with local interactions. Also, our analyzes and results have possible applications in several other fields, as indicated below.

Biological systems: It has been demonstrated that if the population density is large, a group of locusts will become ordered after a short time [8]. Similar phenomena have been observed in some bacterial colonies [15] and fish keratocytes [16]. It is interesting to note that for the SPP model studied in the current paper, all particles will move in almost the same direction after a short time, provided that the velocity and noise are small and that the population density is large enough; see Remark 3.3.

Random geometric graphs: This paper will estimate the isoperimetric constant (Cheeger's constant) and essential spectrum radius of random geometric graphs; see Theorem 4.3 and its proof. These results and methods may be used to investigate the properties of random walk on random geometric graphs, such as convergence rate, cover time, and mixing time [17, 18].

Wireless sensor networks: A large-scale wireless sensor network can be modeled by a random geometric graph [14], so our results about random geometric graphs have potential applications in the investigation of wireless sensor networks. For example, the essential spectrum radius may be used to further study consensus algorithms [19, 20, 21, 22] and distributed optimization [23, 24, 25].

The rest of the paper is organized as follows. Section 2 introduces the model we will study and some related works. In section 3, we present the main results whose proofs are given in section 4. More detailed analysis of the auxiliary results is given in section 5. A simulation example is given in section 6. Section 7 concludes the paper with remarks.

2. Model and Related Work. To be consistent with [26] we will use the term *agent* rather than *particle* in the rest of this paper. The model in [2] consists of n autonomous agents moving in the plane with the same speed $v_n (v_n > 0)$, but with different headings. Two agents are called neighbors if and only if the distance between them is less than a predefined radius $r_n\,(r_n > 0)$. Let us assume that the n agents are labeled $1, 2, \dots, n$. Two agents i and j are neighbors at time t if and only if $\|X_i(t) - X_j(t)\|_2 \le r_n$, where $\|\cdot\|_2$ denotes the Euclidean norm. For any agent $i\,(1 \le i \le n)$, the set of its neighbors at time $t\,(t = 0, 1, \dots)$ is denoted by $\mathcal{N}_i(t)$. By the definition of neighbors, we see that each agent is a neighbor of itself, i.e., $i \in \mathcal{N}_i(t)$ for all $t \ge 0$ and $1 \le i \le n$. The position and heading of the agent i at time t are denoted by $X_i(t) (\in \mathbb{R}^2)$ and $\theta_i(t) (\in (-\pi, \pi])$, respectively, which are updated by

$$X_i(t+1) = X_i(t) + v_n(\cos\theta_i(t+1), \sin\theta_i(t+1)), \tag{2.1}$$

$$\theta_i(t+1) = \arctan \frac{\sum_{j\in\mathcal{N}_i(t)} \sin\theta_j(t)}{\sum_{j\in\mathcal{N}_i(t)} \cos\theta_j(t)} + \delta_i(t), \tag{2.2}$$

where $\delta_i(t)$ denotes a random noise [2].

As mentioned previously, this model has been of interest to researchers from many fields. However, the theoretical analysis of system (2.1)–(2.2) is difficult because of the nonlinearity and randomness of (2.2). An important step forward in analyzing the above model was given by Jadbabaie, Lin, and Morse in [10], where they omitted the noise effect and linearized the heading updating rule (2.2) as

$$\theta_i(t+1) = \frac{1}{|\mathcal{N}_i(t)|} \sum_{j\in\mathcal{N}_i(t)} \theta_j(t), \tag{2.3}$$

where $|\cdot|$ denotes the cardinality of the corresponding set. They proved that if the associated dynamical neighbor graphs are contiguously jointly connected, the above model will reach *synchronization* (or *consensus*) in the sense that there exists a common $\bar{\theta}$ such that for all $i\,(1 \le i \le n)$,

$$\lim_{t\to\infty} \theta_i(t) = \bar{\theta}. \tag{2.4}$$

Subsequently, Savkin [27] investigated the model with discrete headings and showed that if the limit of the neighbor graphs is connected, then synchronization can also be achieved. In [28], Ren and Beard studied the case where the neighbor graphs are directed and showed that synchronization can be achieved if the union of the interaction graphs has a spanning tree frequently enough.

In fact, most existing studies resort to certain connectivity conditions on the dynamical neighbor graphs, and these conditions are hard to verify. Therefore the corresponding analysis is not theoretically complete. One notable exception in the study of flocks is the interesting paper by Cucker and Smale [29], where global interactions are considered with weights of interactions decaying with the distances among agents. However, an unresolved central issue is how to guarantee the connectivity of the dynamical neighbor graphs resulting from local interactions using conditions imposed on only the initial states, the moving speed v_n, and the interaction radius r_n.

To give a complete analysis of the synchronization behavior of the system, Tang and Guo [30] introduced a random framework, assuming that the initial positions and headings of all agents are uniformly and independently distributed, as those in [2]. They showed that for any given positive model parameters, the system based on (2.1) and (2.3) will synchronize with large probability, giving the first complete theoretical result in this direction. Furthermore, in [32] they proved that if $\sqrt[6]{\log n/n} = o(r_n)$ and $v_n = O\left(r_n^5/\log n\right)$, then the model will synchronize.[1,2] Based on their results, Liu and Guo [31] investigated the system (2.1)–(2.2) without noise and provided a similar condition for synchronization. However, a theoretical analysis of the (linearized) Vicsek model with the radius $r_n = O(\sqrt[6]{\log n/n})$ is still lacking, and the question concerning the smallest possible radius for synchronization has never been investigated in this context.

We will carry out our analysis under the assumption that all agents are independently and uniformly distributed in $[0,1]^2$ with arbitrary headings in $(-\pi,\pi]$ at the initial time. As pointed out by Jadbabaie, Lin, and Morse in [10], the connectivity of the neighbor graphs is important for synchronization. Gupta and Kumar in [14] proved that the initial neighbor graph with radius $\sqrt{(c_n + \log n)/\pi n}$ is connected with high probability (w.h.p.)[3] if and only if $c(n) \to \infty$. We refer to $\sqrt{(c_n + \log n)/\pi n}$ with $c(n) \to \infty$ as the *supercritical radius* for connectivity. In this paper, we will show that if the interaction radius is taken as the supercritical radius, then the system can reach synchronization w.h.p. under some restriction on the speed; otherwise, if the radius satisfies (3.4) given in the next section, then the system may not synchronize w.h.p. for any nonnegative speed. From the analysis in [30], the spectral gap of the initial neighbor graph plays an important role for the synchronization rate of the model. However, the methods used in [30] are not suitable for the case of $r_n = O(\sqrt{\log n/n})$ since the radius is too small to meet the prerequisite of the method. In this paper, we will provide a novel approach to estimate the spectral gap of the random geometric graph with radius $O(\sqrt{\log n/n})$. Furthermore, by analyzing the system dynamics, we will prove the synchronization condition without resorting to any assumption on the dynamical behaviors of the self-propelled agents themselves.

3. Main Results.

The objective of this paper is to study the synchronization behavior of the dynamical system (2.1) and (2.3). From the description of the model, we know that the initial states of all agents and the model parameters will determine the trajectories of all agents. Throughout this paper, we assume that the initial positions of all agents are independently and uniformly distributed in $[0,1]^2$ with arbitrary initial headings in $(-\pi,\pi]$. All analysis proceeds under the above assumption without further explanation.

Similar to [31], we use a graph sequence $\{G(t), t = 0,1,\ldots\}$ to describe the relationship among neighbors. For $t \geq 0$, define

$$G(t) = G(\{X_1(t),\ldots,X_n(t)\}, E(t))$$

to be the *position graph* of the model at time t, where $E(t) = \{(i,j) : \|X_i(t) - X_j(t)\| \leq r_n\}$. Obviously, the graphs formed in this way are undirected, and for all $1 \leq i \leq n$ and $t \geq 0$, $(i,i) \in E(t)$. Denote by $P(t)$ the average matrix of the graph $G(t)$, i.e.,

$$\forall\ i,j = 1,2,\ldots,n, \quad (P(t))_{ij} = \begin{cases} \frac{1}{|\mathcal{N}_i(t)|} & \text{if } (i,j) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

[1]For two positive sequences $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$, $a_n = o(b_n)$ means that $\lim_{n\to\infty}(a_n/b_n) = 0$.

[2]For two positive sequences $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$, $a_n = O(b_n)$ means that there exists a positive constant c independent of n such that $a_n \leq cb_n$ for large enough n.

[3]We say that a sequence of events E_n occurs w.h.p. if $\lim_{n\to\infty} P[E_n] = 1$.

Let $\theta(t) := (\theta_1(t), \theta_2(t), \ldots, \theta_n(t))^T$; then the iteration rule of the headings and positions of the model based on (2.1) and (2.3) can be rewritten as

$$\left\{ \begin{array}{l} \theta(t+1) = P(t)\theta(t) \\ X_i(t+1) = X_i(t) + v_n(\cos\theta_i(t+1), \sin\theta_i(t+1)) \end{array} \right. \quad \forall t \geq 0, 1 \leq i \leq n. \tag{3.1}$$

Note that under the assumption on the initial positions, the graph $G(0)$ is a *random geometric graph* that has been studied in detail in, e.g., [34]. One of the classical results concerning the connectivity of the random geometric graph can be stated as follows.

LEMMA 3.1 (see [14]). *The initial random geometric graph* $G(0)$ *is connected w.h.p. if and only if* r_n *satisfies*

$$\lim_{n\to\infty} \left(\pi n r_n^2 - \log n\right) = \infty. \tag{3.2}$$

Based on this lemma, Gupta and Kumar in [33] called $\sqrt{\log n/(\pi n)}$ the *critical radius* for connectivity of $G(0)$. In this paper, we will show that in a probability sense, this critical radius can be regarded as the smallest possible radius for synchronization of our SPP model. The main results are formulated as the following theorem.

THEOREM 3.2. *Suppose that the n agents are independently and uniformly distributed in* $[0,1]^2$ *at the initial time* $t = 0$. *If* r_n *satisfies* (3.2) *and* v_n *satisfies*

$$v_n = o\left(r_n (\log n)^{-1} n^{-2}\right), \tag{3.3}$$

then the system (3.1) *will synchronize w.h.p. for arbitrary initial headings. Moreover, if* r_n *satisfies*

$$\lim_{n\to\infty} \left(\pi n r_n^2 + 3\log\log n - \log n\right) = -\infty, \tag{3.4}$$

then w.h.p. there exist some initial headings such that the system (3.1) *cannot reach synchronization for any speed* $v_n \geq 0$.

The proof of this theorem is in section 4.

Remark 3.3. *Consider the system* (3.1) *with noise added to* (2.3). *Suppose the* n *agents are independently and uniformly distributed in a square with density* ρ. *For arbitrary initial headings, by a similar method to Theorem* 3.2 *we find that if the noise and* v_n *are small, and* r_n *satisfies* $\pi\rho r_n^2 \geq (1+\varepsilon)\log n$ *with* ε *an arbitrary positive constant, then all agents will move in almost the same direction with large probability after a short time. However, this does not mean such ordered states will still hold after a very long time.*

Before closing this section, we propose a conjecture (which is intuitively correct) on the system (3.1), in terms of the values of the speed and the radius for synchronization.

CONJECTURE 3.4. *Suppose* n *agents are distributed in a plane and the initial positions are given. If the system* (3.1) *can synchronize with speed* v *and radius* r, *then it will also uniformly synchronize with speed* $v_1 \in (0, v)$ *and radius* r, *or with speed* v *and radius* $r_1 > r$.

4. Proof of Theorem 3.2.

To prove Theorem 3.2, we need to estimate the maximum degree, the minimum degree, and the eigenvalues of the average matrix of the random geometric graph $G(0)$. For this purpose, we need to introduce some notation.

Define the large deviation rate function $H : [0,\infty) \to \mathbb{R}$ by $H(0) = 1$ and

$$H(a) = 1 - a + a\log a, \qquad a > 0.$$

Note that $H(1) = 0$ and that the unique turning point of H is the minimum at 1. Also, $H(a)/a$ is increasing on $(1, \infty)$. Let $H_-^{-1} : [0, 1] \to [0, 1]$ be the unique inverse of the restriction of H to $[0, 1]$, and let $H_+^{-1} : [0, \infty) \to [1, \infty)$ be the inverse of the restriction of H to $[1, \infty)$; see [34] for the properties of H. Denote by d_i the degree of the vertex i in $G(0)$, i.e., the number of neighbors of the agent i at the initial time instant. Set

$$d_{\max} := \max_{1 \le i \le n} d_i \quad \text{and} \quad d_{\min} := \min_{1 \le i \le n} d_i.$$

The estimations of the maximum and minimum degrees of the initial random geometric graph $G(0)$ were given by Penrose [34], as is described by the following lemma.

LEMMA 4.1. *Suppose that* $\pi n r_n^2 / \log n \to w \in (1, \infty]$ *and* $r_n \to 0$ *as* $n \to \infty$. *Then with probability* 1,

$$\lim_{n\to\infty} \left(\frac{d_{\max}}{n\pi r_n^2} \right) = H_+^{-1} \left(\frac{1}{w} \right) \tag{4.1}$$

and

$$\lim_{n\to\infty} \left(\frac{d_{\min}}{n\pi r_n^2} \right) = \min \left(H_-^{-1} \left(\frac{1}{w} \right), \frac{1}{4} \right). \tag{4.2}$$

Proof. The assertions (4.1) and (4.2) are indicated by Theorems 6.14 and 7.14 of [34]. □

COROLLARY 4.2. *If* r_n *satisfies* (3.2), *then* $d_{\max} < 3 d_{\min} \log n$ *w.h.p.*

Proof. For the case where $\pi n r_n^2 \ge 3 \log n / e$, by Lemma 4.1 we see that $d_{\max} < d_{\min} \log n$ holds almost surely (a.s.) for large n. Next, we will discuss the case where $\pi n r_n^2 < 3 \log n / e$. Note that $d_{\max}$ increases with r_n; by Lemma 4.1, the following inequality holds a.s. for large n:

$$d_{\max} \le \frac{3 \log n}{e} H_+^{-1} \left(\frac{e}{3} \right) (1 + o(1)) < \frac{3 \log n}{e} H_+^{-1} (1) = 3 \log n.$$

Also, by Lemma 3.1, $d_{\min} \ge 1$ w.h.p., and thus our result is obtained. □

Next, we will estimate the eigenvalues of $G(0)$. Let $D = (d_{ij})_{n \times n}$ denote the degree matrix of $G(0)$, which is a diagonal matrix with diagonal entries $d_{ii} = d_i$. Obviously, the matrix $D^{1/2} P(0) D^{-1/2}$ is symmetric, so all eigenvalues of $P(0)$ are real numbers. On the other hand, all entries of $P(0)$ are nonnegative and $\sum_{j=1}^n (P(0))_{ij} = 1, i = 1, 2, \ldots, n$, so the average matrix $P(0)$ is stochastic. The i-largest eigenvalues of $P(0)$, denoted by $\lambda_i, 1 \le i \le n$, satisfy the inequalities

$$|\lambda_i| \le 1, \quad 1 \le i \le n,$$

which means that

$$1 = \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge -1.$$

Define the *essential spectral radius* $\bar{\lambda}$ of $G(0)$ as

$$\bar{\lambda} = \bar{\lambda}(P(0)) := \max\{|\lambda_2|, |\lambda_n|\}.$$

We remark that for the case where $\lim_{n\to\infty}(nr_n^2/\log n) = \infty$, Tang and Guo [30] proved that the essential spectral radius of $G(0)$ satisfies the following inequality w.h.p. for large n:

$$\bar{\lambda} \le 1 - \frac{\pi r_n^2}{512(r_n + \sqrt{6})^4}(1 + o(1)). \tag{4.3}$$

However, the methods used in [30] cannot be applied to estimate the spectral gap of $G(0)$ for the case of $r_n = O(\sqrt{\log n/n})$, since the interaction radius is too small to satisfy the condition of Lemma 4 in [30], which plays a key role in the estimation of $\bar{\lambda}$. In this paper, we will use some methods from percolation theory to study the essential spectral radius of $G(0)$ for the case where r_n satisfies (3.2).

THEOREM 4.3. *Assume that $r_n \le 1$. Then there exists a constant $c > 0$ such that the inequality $\bar{\lambda} \le 1 - cr_n^2$ holds w.h.p. if and only if r_n satisfies* (3.2).

The proof of Theorem 4.3 is given in section 5.

For $\eta > 0$, we write $Po(\eta)$ for any Poisson random variable with parameter η. Define a Poisson point process $\mathcal{P}_\eta$ by $\mathcal{P}_\eta := \{Y_1, Y_2, \ldots, Y_{Po(\eta)}\}$, where $\{Y_1, Y_2, \ldots\}$ is the set of vertices independently and uniformly distributed in $[0,1]^2$ and $Po(\eta)$ is independent of $\{Y_1, Y_2, \ldots\}$; see section 1.7 in [34]. For a Borel set $A \subseteq [0,1]^2$, $|\mathcal{P}_\eta \cap A|$, the number of vertices lying in A, is a Poisson random variable with parameter $\eta\mathcal{L}(A)$, where $\mathcal{L}(\cdot)$ denotes the Lebesgue measure in this paper. For any Borel set $A_1, A_2 \subseteq [0,1]^2$, if $\mathcal{L}(A_1 \cap A_2) = 0$, then the random variables $|\mathcal{P}_\eta \cap A_1|$ and $|\mathcal{P}_\eta \cap A_2|$ are mutually independent. This property is called *spatial independence* of a Poisson point process.

Proof of Theorem 3.2. We will first prove the sufficient part of Theorem 3.2.

For $r_n > 1$, under the condition (3.3), we can directly deduce that the system (3.1) can reach synchronization by Theorem 1 of [30]. Thus, we just need to consider the case where $r_n \le 1$. By Theorem 4.3 and (3.2), we see that there exists a constant $c > 0$ such that

$$\lim_{n\to\infty} P\left(\bar{\lambda} \le 1 - cr_n^2\right) = 1. \tag{4.4}$$

Let E_n denote the event $\bar{\lambda} \le 1 - cr_n^2$, and let $\widetilde{E}_n$ denote the event $d_{\max} < 3d_{\min}\log n$. Define F_n to be the event

$$\bigcap_{1\le i,j\le n}\left\{\|X_i(0) - X_j(0)\|_2 \notin \left(r_n - o\Big(\frac{1}{n^2 r_n}\Big), r_n + o\Big(\frac{1}{n^2 r_n}\Big)\right)\right\}.$$

Using Boole's inequality, we have

$$\begin{aligned} P(F_n^c) &\le \sum_{i\ne j} P\left(\|X_i(0) - X_j(0)\|_2 \in \left(r_n - o\Big(\frac{1}{n^2 r_n}\Big), r_n + o\Big(\frac{1}{n^2 r_n}\Big)\right)\right) \\ &< n^2 \int_{x\in[0,1]^2} P\left(\|x - X_j(0)\|_2 \in \left(r_n - o\Big(\frac{1}{n^2 r_n}\Big), r_n + o\Big(\frac{1}{n^2 r_n}\Big)\right)\right) dx \\ &< n^2 5\pi r_n \cdot o\Big(\frac{1}{n^2 r_n}\Big) \to 0 \quad \text{as} \quad n \to \infty, \end{aligned}$$

where the property that the initial positions are independently and uniformly distributed in $[0,1]^2$ is used in the last inequality. Combining (4.5) with (4.4) and Corollary 4.2, we can deduce that

$$P(E_n \cap \widetilde{E}_n \cap F_n) \to 1 \quad \text{as} \quad n \to \infty. \tag{4.5}$$

We assert that if the speed v_n satisfies (3.3), then for all $t \geq 0$, the topology of $G(t)$ remains unchanged given $E_n \cap \widetilde{E}_n \cap F_n$. We will prove this assertion by induction.

For $t = 0$, the assertion is obviously true. Assume that the assertion holds for all $s \leq t$, that is, $P(s) = P(0)$ for all $s \leq t$. Thus, by (3.1) we have

$$\theta(s+1) = P^s(0)\theta(0) \quad \forall\, 0 \leq s \leq t.$$

Combining this with Proposition 3 in [35], for all integers $s \in [0,t]$ and $i,j \in [1,n]$ we have

$$\begin{aligned} |\theta_i(s+1) - \theta_j(s+1)| &= \left| \sum_{k=1}^{n} \left[(P^s(0))_{ik} - (P^s(0))_{jk} \right] \theta_k(0) \right| \\ &\leq \pi \sum_{k=1}^{n} \left| (P^s(0))_{ik} - (P^s(0))_{jk} \right| \leq \pi\sqrt{n} \left(\sqrt{\frac{d_{\max}}{d_i}} + \sqrt{\frac{d_{\max}}{d_j}} \right) \bar{\lambda}^s \\ &\leq 2\pi\sqrt{3n\log n} \cdot \bar{\lambda}^s, \end{aligned} \tag{4.6}$$

where the assertion conditions E_n and $\widetilde{E}_n$ are used in the last inequality. Set

$$d_{ij}(t+1) := \|X_i(t+1) - X_j(t+1) - X_i(0) + X_j(0)\|_2.$$

Subsequently, using (3.1), the triangle inequality, and standard goniometric formulae, we have

$$\begin{aligned} d_{ij}(t+1) &= \left\| v_n \sum_{s=1}^{t+1} (\cos\theta_i(s), \sin\theta_i(s)) - v_n \sum_{s=1}^{t+1} (\cos\theta_j(s), \sin\theta_j(s)) \right\|_2 \\ &\leq v_n \sum_{s=1}^{t+1} \|\, (\cos\theta_i(s) - \cos\theta_j(s), \sin\theta_i(s) - \sin\theta_j(s))\, \|_2 \\ &= v_n \sum_{s=1}^{t+1} \sqrt{2 - 2\cos[\theta_i(s) - \theta_j(s)]} \leq v_n \sum_{s=1}^{t+1} |\theta_i(s) - \theta_j(s)|, \end{aligned} \tag{4.7}$$

where the inequality $\cos x \geq 1 - x^2/2$ is also used. Set $t_0 := \min\{t : 2\pi\sqrt{3n\log n} \cdot \bar{\lambda}^t \leq 2\pi\}$. Then

$$t_0 = \left\lceil \frac{\log \frac{1}{\sqrt{3n\log n}}}{\log \bar{\lambda}} \right\rceil \leq \frac{-\log(3n\log n)}{2\log \bar{\lambda}} + 1,$$

where $\lceil x \rceil$ denotes the smallest integer not less than x. Hence, by (4.6) and the inequality $1 - x < -\log x$ for $x \in (0,1)$, we have

$$\begin{aligned} &\max_{i,j} \sum_{s=1}^{t+1} |\theta_i(s) - \theta_j(s)| \leq 2\pi t_0 + \sum_{s=t_0+1}^{t+1} 2\pi\sqrt{3n\log n} \cdot \bar{\lambda}^s \\ &< 2\pi \left(\frac{-\log(3n\log n)}{2\log\bar{\lambda}} + 1 \right) + \left(\frac{2\pi\sqrt{3n\log n}}{1-\bar{\lambda}} \right) \bar{\lambda}^{\frac{-\log(3n\log n)}{2\log\bar{\lambda}}} \\ &= O\left(\frac{\log n}{1-\bar{\lambda}} \right) = O\left(r_n^{-2}\log n \right). \end{aligned}$$

Substituting this inequality and (3.3) into (4.7), we can obtain that

$$\max_{i,j} d_{ij}(t+1) \leq v_n \max_{i,j} \sum_{s=1}^{t+1} |\theta_i(s) - \theta_j(s)| \tag{4.8}$$
$$= o\left(\frac{r_n}{n^2 \log n} \cdot \frac{\log n}{r_n^2}\right) = o\left(\frac{1}{n^2 r_n}\right),$$

which means that the position between any two agents changed at time t is bounded by $o(\frac{1}{n^2 r_n})$, in comparison with that at the initial time. Combining (4.8) with the condition F_n, we know that, compared with $G(0)$, the topology of the graph $G(t+1)$ is unchanged w.h.p.

By induction, our assertion holds for all $t \geq 0$, which means that the inequality (4.6) holds for all $t \geq 0$. Thus, system (3.1) can reach synchronization.

It remains to prove the necessary part of the theorem. Set

$$M_n := \left\lfloor \sqrt{\pi n/(4 \log n)} \right\rfloor - 1,$$

where $\lfloor x \rfloor$ denotes the largest integer no bigger than x. Define the point

$$x_k := \left((2k+1)\sqrt{\log n/(\pi n)}, 0\right) \in [0,1]^2, \quad k = 0, \ldots, K_n.$$

Let $b_n := \log n - 3 \log\log n - \pi n r_n^2$; then by (3.4)

$$b_n < \log n - 3 \log\log n \quad \text{and} \quad \lim_{n\to\infty} b_n = \infty.$$

Take $\varepsilon_n = \sqrt{1/(\pi n \log n)}$. Let

$$\mathcal{X}_n := \{X_1(0), X_2(0), \ldots, X_n(0)\} \tag{4.9}$$

denote the n vertices independently and uniformly distributed in $[0,1]^2$. For any integer $k \in [0, M_n]$, define the event

$$A_k := \{\mathcal{X}_n \cap B(x_k, \varepsilon_n) \neq \emptyset, \mathcal{X}_n \cap [B(x_k, r_n + \varepsilon_n) \setminus B(x_k, \varepsilon_n)] = \emptyset\},$$

where $B(x,r) := \{y \in \mathbb{R}^2 : \|x-y\|_2 \leq r\}$ denotes the ball centered at x with radius r. If the event $A_k (k \in [0, M_n])$ occurs, then the agents lying in $B(x_k, \varepsilon_n)$ do not have any neighbor at the initial time. For such a case, the system (3.1) will not synchronize by setting the initial headings of the agents lying in $B(x_k, \varepsilon_n)$ to be $-\pi/2$ and the others to be $\pi/2$; see Figure 1. Thus, to prove the necessary part we just need to verify the following equation:

$$\lim_{n\to\infty} P\left(\bigcup_{0 \leq k \leq M_n} A_k\right) = 1. \tag{4.10}$$

Set $\eta(n) := n + n^{3/4}$ and $\lambda(n) := n - n^{3/4}$. Let $\mathcal{P}_{\eta(n)}$ and $\mathcal{P}_{\lambda(n)}$ denote a Poisson point process in $[0,1]^2$ with parameters $\eta(n)$ and $\lambda(n)$, respectively. Using Lemma 1.4 in [34], for large n we find

$$P\left(\mathcal{X}_n \subseteq \mathcal{P}_{\eta(n)}\right) = P\left(Po(\eta(n)) \geq n\right) > 1 - e^{-n^{1/4}} \tag{4.11}$$

$$\text{and} \quad P\left(\mathcal{P}_{\lambda(n)} \subseteq \mathcal{X}_n\right) = P(Po(\lambda(n)) \leq n) > 1 - e^{-n^{1/4}}. \tag{4.12}$$

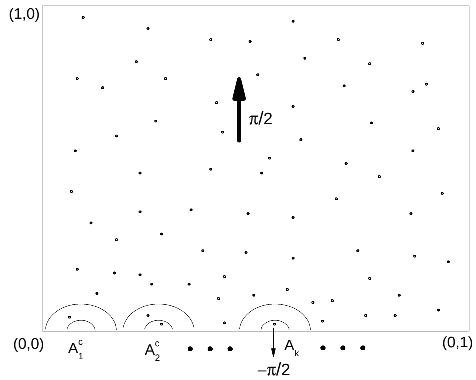

Fig. 1 *If A_k occurs, then the system will not synchronize by setting the initial headings of the agents lying in $B(x_k, \varepsilon_n)$ to be $-\pi/2$ and the others to be $\pi/2$.*

Define the event

$$\widetilde{A}_k := \left\{\mathcal{P}_{\lambda(n)} \cap B(x_k, \varepsilon_n) \neq \emptyset, \mathcal{P}_{\eta(n)} \cap [B(x_k, r_n + \varepsilon_n) \setminus B(x_k, \varepsilon_n)] = \emptyset\right\};$$

then by (4.11) and (4.12),

$$\begin{aligned} P\left(\bigcup_{0\le k\le M_n} A_k\right) &\ge P\left(\bigcup_{0\le k\le M_n} \widetilde{A}_k, \mathcal{P}_{\lambda(n)} \subseteq \mathcal{X}_n, \mathcal{X}_n \subseteq \mathcal{P}_{\eta(n)}\right) \\ &\ge P\left(\bigcup_{0\le k\le M_n} \widetilde{A}_k\right) + P\left(\mathcal{P}_{\lambda(n)} \subseteq \mathcal{X}_n\right) + P\left(\mathcal{X}_n \subseteq \mathcal{P}_{\eta(n)}\right) - 2 \qquad (4.13) \\ &> P\left(\bigcup_{0\le k\le M_n} \widetilde{A}_k\right) - 2e^{-n^{1/4}}. \end{aligned}$$

Also, using the spatial independence of the Poisson point process and Taylor's expansion,

$$\begin{aligned} &P\left(\bigcup_{0\le k\le M_n} \widetilde{A}_k\right) = 1 - P\left(\bigcap_{0\le k\le M_n} \widetilde{A}_k^c\right) = 1 - \prod_{0\le k\le M_n}\left[1 - P\left(\widetilde{A}_k\right)\right] \\ &= 1 - \left[1 - \left(1 - e^{-\lambda(n)\pi\varepsilon_n^2/2}\right) e^{-\eta(n)\pi(r_n^2 + 2r_n\varepsilon_n)/2}\right]^{M_n+1} \\ &= 1 - \left[1 - \frac{1}{2\log n} \cdot n^{-\frac{1}{2}} (\log n)^{\frac{3}{2}} e^{\frac{b_n}{2} - \sqrt{1 - b_n/\log n}} (1 + o(1))\right]^{M_n+1} \\ &= 1 - \exp\left(-\frac{1}{2}(M_n + 1) n^{-\frac{1}{2}} (\log n)^{\frac{1}{2}} e^{\frac{b_n}{2} - \sqrt{1 - \frac{b_n}{\log n}}}\right)(1 + o(1)) \\ &\to 1 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Combining this with (4.13) yields (4.10). □

Remark 4.4. From the proof of Theorem 3.2, we see that the speed v_n is so small that the topology of the neighbor graph remains unchanged during the evolution of the system. However, the relaxation of the restriction on the speed is very hard, since the estimation of the essential spectral radius of $P(t)P(t-1)\cdots P(0)$ is still an open question in inhomogeneous Markov chain theory, even if only one edge is changed in the neighbor graph; see Problem 1.1 in [36]. The restriction on the speed could be relaxed if the above open problem was resolved.

5. Proof of Theorem 4.3. First, we will provide the proof of the sufficient part of Theorem 4.3. For the case where $\pi n r_n^2 \geq (\log n)^2$, the inequality $\bar{\lambda} \leq 1 - c r_n^2$ holds w.h.p. by Theorem 3 in [30]. Therefore, we just need to consider the case where

$$\pi n r_n^2 \leq (\log n)^2 \quad \text{and} \quad \lim_{n\to\infty} \left(\pi n r_n^2 - \log n\right) = \infty. \tag{5.1}$$

In this section we use $G(\mathcal{X}_n; r_n)$ to denote the initial random geometric graph $G(0)$. Divide the unit square $[0,1]^2$ into K_n^2 small squares with the length of each side equal to $1/K_n$, where $K_n := \lceil \sqrt{5}/r_n \rceil$. Denote these small squares by $S_1, S_2, \ldots, S_{K_n^2}$. Set

$$\alpha_n := E\left[|\mathcal{X}_n \cap S_1|\right] = \frac{n}{\lceil \frac{\sqrt{5}}{r_n} \rceil^2},$$

where $\mathcal{X}_n$ is defined by (4.9). Define

$$\Delta_n := \max_{1 \leq i \leq K_n^2} |\mathcal{X}_n \cap S_i|.$$

We will consider the upper bound of Δ_n first.

LEMMA 5.1. *Assume that r_n satisfies* (5.1). *Then, with probability* 1, $\Delta_n < 21\alpha_n$ *for large enough n.*

Proof. Since the initial positions $X_j(0), j = 1, 2, \ldots, n$, are independently and uniformly distributed in $[0,1]^2$, $P\left(X_j(0) \in S_i\right) = 1/K_n^2, 1 \leq j \leq n, i \in [1, K_n^2]$, and $|\mathcal{X}_n \cap S_i|$ is a binomial random variable. By (1.7) in [34], for large enough n,

$$\begin{aligned} P\left(|\mathcal{X}_n \cap S_i| \geq 21\alpha_n\right) &\leq \exp\left(\frac{-21\alpha_n}{2} \log\left(\frac{21\alpha_n}{E\left[|\mathcal{X}_n \cap S_i|\right]}\right)\right) \\ &\leq \exp\left(\frac{-21n}{2(\frac{\sqrt{5}}{r_n} + 1)^2} \log(21)\right) \\ &< \exp(-2.03 \cdot \log n) = n^{-2.03}. \end{aligned}$$

Thus, by the definition of Δ_n, for large enough n we have

$$\begin{aligned} P\left(\Delta_n \geq 21\alpha_n\right) &= P\left(\bigcup_{i=1}^{K_n^2} \{|\mathcal{X}_n \cap S_i| \geq 21\alpha_n\}\right) \\ &\leq \sum_{i=1}^{K_n^2} P\left(|\mathcal{X}_n \cap S_i| \geq 21\alpha_n\right) \\ &< n \cdot n^{-2.03} = n^{-1.03}. \end{aligned}$$

Hence, using the Borel–Cantelli lemma yields our result. ◻

Remark 5.2. Using a method similar to that of Theorem 6.14 in [34], we find that, with probability 1, the inequality

$$\alpha_n H_+^{-1}\left(\frac{\log n}{\alpha_n}\right)(1 - o(1)) \leq \Delta_n \leq \alpha_n H_+^{-1}\left(\frac{\log n}{\alpha_n}\right)(1 + o(1))$$

holds for large n. However, the proof of this result is complicated, so we do not include it in this paper.

For what follows we need to introduce some definitions. Let $\|\cdot\|_1$ denote the l_1-norm, and let $\|\cdot\|_\infty$ denote the infinity norm. For any $x, y \in \mathbb{Z}^2$, if $\|x-y\|_1 = 1$, then we say that x and y are *adjacent* and we write $x \sim y$. Also, given $A \subseteq \mathbb{Z}^2$, if, for any $x, y \in A$, there exists a vertex sequence $x_1, x_2, \dots, x_n$ in A such that $x \sim x_1, x_1 \sim x_2, x_2 \sim x_3, \dots, x_n \sim y$, then we say A is connected. Similarly, if $\|x-y\|_\infty \leq k$, $k \geq 1$, we say that x and y are *k-adjacent* and we write $x \sim_k y$. Given $A \subseteq \mathbb{Z}^2$, if, for any $x, y \in A$, there exists a vertex sequence $x_1, x_2, \dots, x_n$ in A such that $x \sim_k x_1, x_1 \sim_k x_2, x_2 \sim_k x_3, \dots, x_n \sim_k y$, then we say A is *k-connected.* It can be seen that if A is connected, then A must be k-connected for all $k \geq 1$. In particular, a single vertex set $\{x\} \subset \mathbb{Z}^2$ is both connected and k-connected.

We define the lattice box $B_{\mathbb{Z}}(K_n)$ by $B_{\mathbb{Z}}(K_n) := \prod_{i=1}^2([1, K_n] \cap \mathbb{Z})$. If $A \subset B_{\mathbb{Z}}(K_n)$, set $A^c := B_{\mathbb{Z}}(K_n) \backslash A$ and let ∂A denote the *internal vertex-boundary* of A, that is, the set of vertices $z \in A$ such that $\{y \in A^c : \|z-y\|_1 = 1\}$ is nonempty. To prove Theorem 4.3, several lemmas are needed.

Lemma 5.3. *Let $\beta \in (0,1)$. If A is a subset of $B_{\mathbb{Z}}(K_n)$ (not necessarily connected), with $|A| \leq \beta K_n^2$, then*

$$|\partial A| \geq \frac{1}{4}(1-\sqrt{\beta})\sqrt{|A|}.$$

Proof. Replacing 2/3 with β in the proof of Lemma 9.9 of [34], the result can be deduced. □

Lemma 5.4 (Lemma 9.6 in [34]). *Suppose $A \subset B_{\mathbb{Z}}(K_n)$ is such that both A and A^c are connected. Then ∂A is 1-connected.*

Remark 5.5. If both A and A^c are connected, by Lemma 5.4 both ∂A and $\partial(A^c)$ are 1-connected since $(A^c)^c = A$.

Lemma 5.6. *Suppose $A \subset B_{\mathbb{Z}}(K_n)$. If A is 3-connected and A^c is connected, then ∂A is 3-connected and $\partial(A^c)$ is 2-connected.*

Proof. Let $A_1, A_2, \dots, A_m$ denote the connected components of A, which indicates that $A_1, \dots, A_m$ are connected, but $A_i \cup A_j$, $1 \leq i \neq j \leq m$, is not connected. By the fact that $B_{\mathbb{Z}}(K_n)$ is connected, $A_i, i \in [1, m]$, are all connected with A^c. Note that A^c is connected, so for any $i \in [1, m]$, $A_i^c = A^c \cup A_1 \cup \dots \cup A_{i-1} \cup A_{i+1} \cup A_m$ is also connected. By Lemma 5.4, we know that both ∂A_i and $\partial(A_i^c)$ are 1-connected.

Moreover, if $A_i \cup A_j (i \neq j)$ is 3-connected, then there exists a pair $(z_i, z_j) \in (\partial A_i, \partial A_j)$ such that z_i and z_j are 3-connected, and there exists another pair $(\widetilde{z}_i, \widetilde{z}_j) \in (\partial(A_i^c), \partial(A_j^c))$ such that $\widetilde{z}_i$ and $\widetilde{z}_j$ are 2-connected; see Figure 2. Thus, $\partial A_i \cup \partial A_j$ is 3-connected, and $\partial(A_i^c) \cup \partial(A_j^c)$ is 2-connected since ∂A_i, ∂A_j, $\partial(A_i^c)$, and $\partial(A_j^c)$ are 1-connected. Combining this with the fact that $A = \cup_{i=1}^m A_i$ is 3-connected, we have that $\partial A = \cup_{i=1}^m \partial A_i$ is 3-connected and $\partial(A^c) = \cup_{i=1}^m \partial(A_i^c)$ is 2-connected. □

Lemma 5.7 (Corollary 9.4 in [34]). *Given integer $k \geq 1$, the number of k-connected subsets of the lattice box $B_{\mathbb{Z}}(K_n)$ of cardinality m is at most $K_n^2 2^{4k(k+1)m}$.*

For each small square S_i, $1 \leq i \leq K_n^2$, let x_i denote its center point. Set $z_i := K_n x_i + \frac{1}{2} \in \mathbb{Z}^2$; see Figure 3. By the definition of $B_{\mathbb{Z}}(K_n)$, we find that the set $\{z_i : 1 \leq i \leq K_n^2\}$ is equal to $B_{\mathbb{Z}}(K_n)$.

Recall that $\lambda(n) = n - n^{3/4}$ and $\mathcal{P}_{\lambda(n)}$ denotes a Poisson point process in $[0,1]^2$ with parameter $\lambda(n)$. Define the function

$$f_1(A) := \sum_{z_i \in A, z_j \in A^c, z_i \sim z_j} |\mathcal{P}_{\lambda(n)} \cap S_i| \cdot |\mathcal{P}_{\lambda(n)} \cap S_j|.$$

We then find the following lemmas.

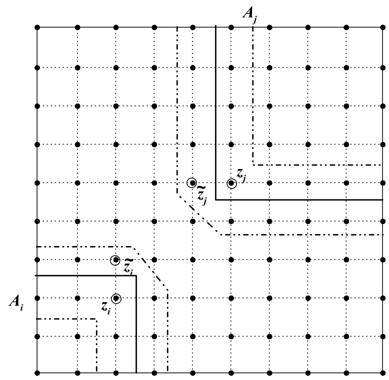

Fig. 2 *A_i and A_j are the connected components surrounded by the solid lines. If $A_i \cup A_j$ is 3-connected, then there exist $z_i \in \partial A_i$ and $z_j \in \partial A_j$ such that $z_i \sim_3 z_j$, and there exist $\widetilde{z}_i \in \partial(A_i^c)$ and $\widetilde{z}_j \in \partial(A_j^c)$ such that $\widetilde{z}_i \sim_2 \widetilde{z}_j$.*

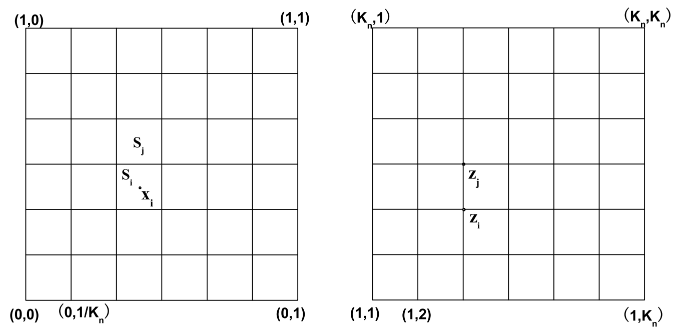

Fig. 3 *The relationships among S_i, x_i, and z_i are shown. If $z_i \sim z_j$, then any two vertices x, y in $S_i \cup S_j$ will satisfy $\|x - y\|_2 \le r_n$.*

LEMMA 5.8. *Assume that r_n satisfies* (5.1). *Suppose $A \subset B_{\mathbb{Z}}(K_n)$ and integer $k \ge 1$. Then, for any constant $\beta \in (0,1)$, there exists a constant $\eta = \eta(k,\beta) > 0$ such that, for large enough n,*

$$P\left[\inf_{\substack{\beta\alpha_n^{-2}K_n \le |A| \le (1-\beta)K_n^2 \\ \partial A \text{ is } k\text{-connected}}} \frac{f_1(A)}{|A|} \le \frac{\eta\alpha_n^2}{K_n}\right] < e^{-n^{1/5}}.$$

Proof. This proof partly uses the ideas appearing in [37]. Let

$$c_1 := \frac{1-\sqrt{1-\beta}}{4\sqrt{1-\beta}} \quad \text{and} \quad c_2 := \frac{1}{4}(1-\sqrt{1-\beta})\sqrt{\beta}.$$

If $\beta\alpha_n^{-2}K_n \le |A| \le (1-\beta)K_n^2$, then by Lemma 5.3,

$$|\partial A| \ge \frac{1}{4}(1-\sqrt{1-\beta})\sqrt{|A|} \ge \frac{c_1|A|}{K_n}, \tag{5.2}$$

and also

$$|\partial A| \geq \frac{1}{4}(1-\sqrt{1-\beta})\sqrt{|A|} \geq \frac{c_2\sqrt{K_n}}{\alpha_n}. \tag{5.3}$$

For any $\varepsilon > 0$, by the definition of f_1 we find

$$f_1(A) \geq (\varepsilon\alpha_n)^2 \sum_{z_i\in\partial A, z_j\in\partial(A^c), z_i\sim z_j} I_{\{|\mathcal{P}_{\lambda(n)}\cap S_i|\geq\varepsilon\alpha_n, |\mathcal{P}_{\lambda(n)}\cap S_j|\geq\varepsilon\alpha_n\}}. \tag{5.4}$$

For any set $\Lambda, \Gamma \subset B_{\mathbb{Z}}(K_n)$, let

$$\xi(\Lambda,\Gamma) := \sum_{z_i\in\Lambda, z_j\in\Gamma, z_i\sim z_j} I_{\{|\mathcal{P}_{\lambda(n)}\cap S_i|\geq\varepsilon\alpha_n, |\mathcal{P}_{\lambda(n)}\cap S_j|\geq\varepsilon\alpha_n\}}.$$

Therefore, by (5.2) and (5.4) we have

$$\frac{f_1(A)}{|A|} \geq \frac{c_1(\varepsilon\alpha_n)^2\xi(\partial A, \partial(A^c))}{K_n|\partial A|}.$$

Combining the above inequality with (5.3) yields

$$\begin{aligned}
&\inf_{\substack{\beta\alpha_n^{-2}K_n\leq|A|\leq(1-\beta)K_n^2\\ \partial A \text{ is } k\text{-connected}}} \frac{f_1(A)}{|A|}\\
&\geq \frac{c_1(\varepsilon\alpha_n)^2}{K_n} \inf_{\substack{|\partial A|\geq c_2\alpha_n^{-1}\sqrt{K_n}\\ \partial A \text{ is } k\text{-connected}}} \frac{\xi(\partial A,\partial(A^c))}{|\partial A|}.
\end{aligned} \tag{5.5}$$

Note that, for any $z \in \partial(A^c)$, there exists at least one vertex $\widetilde{z} \in \partial A$ such that $z \sim \widetilde{z}$, so if ∂A is k-connected, then $\partial A \cup \partial(A^c)$ is also k-connected. Let $(\Lambda_1^{M,m}, \Gamma_1^{M,m}), (\Lambda_2^{M,m}, \Gamma_2^{M,m}), \ldots, (\Lambda_{i_{M,m}}^{M,m}, \Gamma_{i_{M,m}}^{M,m})$ denote all possible pairs of $(\partial A, \partial(A^c))$ satisfying (i) ∂A is k-connected, (ii) $|\partial A \cup \partial(A^c)| = M$, and (iii) $|\partial A| = m$. Then, by Lemma 5.7,

$$\begin{aligned}
\sum_{m=1}^{M} i_{M,m} &\leq K_n^2 2^{4k(k+1)M} \sum_{m=1}^{M} \binom{M}{m}\\
&= K_n^2 2^{4k(k+1)M}\cdot 2^M = K_n^2 2^{(2k+1)^2 M}.
\end{aligned} \tag{5.6}$$

Thus, for any constant $c_3 > 0$, using Boole's inequality we find

$$\begin{aligned}
&P\left(\inf_{\substack{|\partial A|\geq c_2\alpha_n^{-1}\sqrt{K_n}\\ \partial A \text{ is } k\text{-connected}}} \frac{\xi(\partial A,\partial(A^c))}{|\partial A|} \leq c_3\right)\\
&= P\left(\bigcup_{m\geq c_2\alpha_n^{-1}\sqrt{K_n}} \bigcup_{M\geq m} \bigcup_{l=1}^{i_{M,m}} \left\{\frac{\xi(\Lambda_l^{M,m},\Gamma_l^{M,m})}{m} \leq c_3\right\}\right)\\
&\leq P\left(\bigcup_{M\geq c_2\alpha_n^{-1}\sqrt{K_n}} \bigcup_{m=1}^{M} \bigcup_{l=1}^{i_{M,m}} \left\{\frac{\xi(\Lambda_l^{M,m},\Gamma_l^{M,m})}{M} \leq c_3\right\}\right)\\
&\leq \sum_{M\geq c_2\alpha_n^{-1}\sqrt{K_n}} \sum_{m=1}^{M} \sum_{l=1}^{i_{M,m}} P\left(\xi(\Lambda_l^{M,m},\Gamma_l^{M,m}) \leq c_3 M\right).
\end{aligned} \tag{5.7}$$

For any $z \in \Lambda_l^{M,m}$ (or $\Gamma_l^{M,m}$), $1 \le l \le i_{M,m}$, there exist at least one and at most four vertices in $\Gamma_l^{M,m}$ (or $\Lambda_l^{M,m}$) which are connected with z, and thus we can choose the vertex pairs $(z_{i_1}, z_{\widetilde{i_1}}), (z_{i_2}, z_{\widetilde{i_2}}), \ldots, (z_{i_{j(l)}}, z_{\widetilde{i_{j(l)}}}) \in (\Lambda_l^{M,m}, \Gamma_l^{M,m})$, $j(l) \ge M/8$, such that $z_{i_1} \sim z_{\widetilde{i_1}}, z_{i_2} \sim z_{\widetilde{i_2}}, \ldots, z_{i_{j(l)}} \sim z_{\widetilde{i_{j(l)}}}$ and $z_{i_1}, z_{\widetilde{i_1}}, z_{i_2}, z_{\widetilde{i_2}}, \ldots, z_{i_{j(l)}}, z_{\widetilde{i_{j(l)}}}$ are mutually different. Thus, by the spatial independence of the Poisson point process, for any $1 \le k_1 \ne k_2 \le j(l)$, the corresponding events $I_{\{|\mathcal{P}_{\lambda(n)} \cap S_{i_{k_1}}| \ge \varepsilon\alpha_n, |\mathcal{P}_{\lambda(n)} \cap S_{\widetilde{i_{k_1}}}| \ge \varepsilon\alpha_n\}}$ and $I_{\{|\mathcal{P}_{\lambda(n)} \cap S_{i_{k_2}}| \ge \varepsilon\alpha_n, |\mathcal{P}_{\lambda(n)} \cap S_{\widetilde{i_{k_2}}}| \ge \varepsilon\alpha_n\}}$ are mutually independent. Let $E_k = I_{\{|\mathcal{P}_{\lambda(n)} \cap S_{i_k}| \ge \varepsilon\alpha_n, |\mathcal{P}_{\lambda(n)} \cap S_{\widetilde{i_k}}| \ge \varepsilon\alpha_n\}}$; then

$$P\left(\xi(\Lambda_l^{M,m}, \Gamma_l^{M,m}) \le c_3 M\right) \le P\left(\sum_{k=1}^{j(l)} E_k \le c_3 M\right), \tag{5.8}$$

where $j(l) \ge M/8$ and the events E_k, $1 \le k \le j(l)$, are mutually independent.

Choose $\varepsilon = 1/2$; then for all large n and $1 \le k \le j(l)$,

$$\begin{aligned}
P(E_k) &= P\left(|\mathcal{P}_{\lambda(n)} \cap S_{i_k}| \ge \frac{\alpha_n}{2}\right) P\left(|\mathcal{P}_{\lambda(n)} \cap S_{\widetilde{i_k}}| \ge \frac{\alpha_n}{2}\right) \\
&= P^2\left(Po\left(\frac{n - n^{3/4}}{\lceil \frac{\sqrt{5}}{r_n} \rceil^2}\right) \ge \frac{\alpha_n}{2}\right) \\
&\ge \left(1 - \exp\left\{-\frac{n - n^{3/4}}{\lceil \frac{\sqrt{5}}{r_n} \rceil^2} H\left(\frac{\alpha_n \lceil \frac{\sqrt{5}}{r_n} \rceil^2}{2(n - n^{3/4})}\right)\right\}\right)^2 \\
&\ge \left(1 - \exp\left\{-\frac{\log n}{5} H\left(\frac{2}{3}\right)\right\}\right)^2 = \left(1 - n^{-H(\frac{2}{3})/5}\right)^2,
\end{aligned}$$

where the last inequality follows from Lemma 1.2 in [34]. Therefore, for any $\rho > 0$ and large enough n, by Markov's inequality we have

$$\begin{aligned}
P\left(\sum_{k=1}^{j(l)} E_k \le c_3 M\right) &= P\left(\exp\left(-\rho \sum_{k=1}^{j(l)} E_k\right) \ge e^{-\rho c_3 M}\right) \\
&\le e^{\rho c_3 M} \prod_{k=1}^{j(l)} E\left[e^{-\rho E_k}\right] \\
&\le e^{\rho c_3 M} \left(\left(1 - n^{-H(\frac{2}{3})/5}\right)^2 e^{-\rho} + 1 - \left(1 - n^{-H(\frac{2}{3})/5}\right)^2\right)^{M/8}.
\end{aligned} \tag{5.9}$$

Choose $c_3 > 0$ small enough; then there exist constants $\rho > 0$ and $c_4 > 0$ such that, for large enough n,

$$\begin{aligned}
&(2k+1)^2 \log 2 + \rho c_3 \\
&\quad + \frac{1}{8} \log\left(\left(1 - n^{-H(\frac{2}{3})/5}\right)^2 e^{-\rho} + 1 - \left(1 - n^{-H(\frac{2}{3})/5}\right)^2\right) \le -c_4.
\end{aligned} \tag{5.10}$$

Combining (5.6)–(5.9) with (5.10), for large enough n we have

$$\begin{aligned}
&P\left(\inf_{\substack{|\partial A|\geq c_2\alpha_n^{-1}\sqrt{K_n}\\ \partial A \text{ is } k\text{-connected}}} \frac{\xi(\partial A,\partial(A^c))}{|\partial A|}\leq c_3\right)\\
&\leq \sum_{M\geq c_2\alpha_n^{-1}\sqrt{K_n}}\sum_{m=1}^{M}\sum_{l=1}^{i_{M,m}} P\left(\sum_{k=1}^{j(l)} E_k\leq c_3 M\right)\\
&\leq \sum_{M\geq c_2\alpha_n^{-1}\sqrt{K_n}} K_n^2 2^{(2k+1)^2M} e^{\rho c_3 M}\\
&\qquad\cdot\left(\left(1-n^{-H(\frac{2}{3})/5}\right)^2 e^{-\rho}+1-\left(1-n^{-H(\frac{2}{3})/5}\right)^2\right)^{M/8}\\
&\leq \sum_{M\geq c_2\alpha_n^{-1}\sqrt{K_n}} K_n^2 e^{-c_4 M}\leq \exp\left(\frac{-c_4c_2\alpha_n^{-1}\sqrt{K_n}}{2}\right)< e^{-n^{1/5}}.
\end{aligned}$$

The above inequality and (5.5) yield our result. ∎

For any $z_i\in B_{\mathbb{Z}}(K_n)$, we call z_i *open* if $S_i\cap\mathcal{P}_{\lambda(n)}\neq\emptyset$ and call z_i *closed* otherwise. Let $\mathcal{O}_n$ denote the set of open vertices in $B_{\mathbb{Z}}(K_n)$, and let $\mathcal{C}_n$ denote the largest open clusters of $\mathcal{O}_n$.

LEMMA 5.9. *Assume that r_n satisfies* (5.1). *Then, with probability* 1, $|\mathcal{C}_n| = (1-o(1))K_n^2$ *for all large enough n.*

Proof. For any $z\in B_{\mathbb{Z}}(K_n)$,

$$P(\{z \text{ is closed}\})=\exp\left(-\frac{\lambda(n)}{K_n^2}\right)\to 0\quad\text{as } n\to\infty.$$

By Theorem 8.65 in [38] and Theorem 1 in [39] our result can be deduced. ∎

LEMMA 5.10. *Assume that r_n satisfies* (5.1). *Suppose $A\subset B_{\mathbb{Z}}(K_n)$. Then, for any constant $\beta\in(0,1)$, there exists a constant $\eta=\eta(\beta)>0$ such that, for large enough n,*

$$\inf_{\substack{\beta\alpha_n^{-2}K_n\leq|A|\leq(1-\beta)K_n^2\\ A \text{ is 3-connected}}}\frac{f_1(A)}{|A|}\geq\frac{\eta\alpha_n^2}{K_n}\qquad a.s.$$

Proof. For any $A\subset B_{\mathbb{Z}}(K_n)$ with $\beta\alpha_n^{-2}K_n\leq|A|\leq(1-\beta)K_n^2$, let $\Lambda_1,\ldots,\Lambda_{m_A}$ denote the connected components of A^c, taken in decreasing order. In other words, $\Lambda_1,\ldots,\Lambda_{m_A}$ are connected, but $\Lambda_i\cup\Lambda_j$, $1\leq i\neq j\leq m_A$, is not connected and $|\Lambda_1|\geq|\Lambda_2|\geq\cdots\geq|\Lambda_{m_A}|$. Since $\Lambda_1,\ldots,\Lambda_{m_A}$ are all connected with A and A is 3-connected, Λ_i^c, $1\leq i\leq m_A$, are all 3-connected. By Lemma 5.6, for $1\leq i\leq m_A$, $\partial(\Lambda_i^c)$ is 3-connected and $\partial\Lambda_i$ is 2-connected. By the definition of f_1 we find

$$f_1(A)=\sum_{i=1}^{m_A}f_1(\Lambda_i)=\sum_{i=1}^{m_A}f_1(\Lambda_i^c). \tag{5.11}$$

If $|\Lambda_1| > K_n^2/2$, then $|\Lambda_1^c| \le K_n^2/2$. Note that $A \subseteq \Lambda_1^c$, and by (5.11) and Lemma 5.8 we have

$$\begin{aligned}\inf_{\substack{|A|\ge\beta\alpha_n^{-2}K_n,|\Lambda_1|>\frac{1}{2}K_n^2\\ A\text{ is 3-connected}}}\frac{f_1(A)}{|A|}&\ge \inf_{\substack{\beta\alpha_n^{-2}K_n\le|\Lambda_1^c|\le\frac{1}{2}K_n^2\\ \partial(\Lambda_1^c)\text{ is 3-connected}}}\frac{f_1(\Lambda_1^c)}{|\Lambda_1^c|}\\ &> \frac{\eta\alpha_n^2}{K_n}\qquad\text{a.s.}\end{aligned}\tag{5.12}$$

Next we consider the case of $|\Lambda_1| \le K_n^2/2$. Without loss of generality, we assume that $|\Lambda_i| \ge \frac{1}{2}\alpha_n^{-2}K_n$ for $1 \le i \le i_A$, and $|\Lambda_i| < \frac{1}{2}\alpha_n^{-2}K_n$ for $i_A+1 \le i \le m_A$, where $i_A \in [1, m_A]$. Since $\partial\Lambda_i$ is 2-connected, by Lemma 5.8 and the Borel–Cantelli lemma, with probability 1,

$$\frac{f_1(\Lambda_i)}{|\Lambda_i|} > \frac{\eta\alpha_n^2}{K_n}\qquad\forall 1\le i\le i_A$$

for all large n. Thus,

$$\begin{aligned}&\inf_{A\text{ is 3-connected},|\Lambda_1|\le\frac{1}{2}K_n^2}\frac{\sum_{i=1}^{i_A}f_1(\Lambda_i)}{\sum_{i=1}^{i_A}|\Lambda_i|}\\ &\ge\min_{1\le i\le i_A}\left\{\inf_{\substack{\frac{1}{2}\alpha_n^{-2}K_n\le|\Lambda_i|\le\frac{1}{2}K_n^2,\\ \partial\Lambda_i\text{ is 2-connected}}}\frac{f_1(\Lambda_i)}{|\Lambda_i|}\right\}>\frac{\eta\alpha_n^2}{K_n}\qquad\text{a.s.}\end{aligned}\tag{5.13}$$

holds for large enough n.

For Λ_i, $i_A+1 \le i \le m_A$, if $\Lambda_i \cap \mathcal{C}_n \ne \emptyset$, then $f_1(\Lambda_i) \ge 1$, which indicates that

$$\frac{f_1(\Lambda_i)}{|\Lambda_i|} > \frac{1}{\frac{1}{2}\alpha_n^{-2}K_n} = \frac{2\alpha_n^2}{K_n}.\tag{5.14}$$

Let $\eta' := \min\{\eta, 2\}$. By (5.13) and (5.14) we find, with probability 1,

$$\begin{aligned}&\inf_{A\text{ is 3-connected},|\Lambda_1|\le\frac{1}{2}K_n^2}\frac{\sum_{i=1}^{i_A}f_1(\Lambda_i)+\sum_{i_A+1\le i\le m_A,\Lambda_i\cap\mathcal{C}_n\ne\emptyset}f_1(\Lambda_i)}{\left(\sum_{i=1}^{i_A}+\sum_{i_A+1\le i\le m_A,\Lambda_i\cap\mathcal{C}_n\ne\emptyset}\right)|\Lambda_i|}\\ &\ge\frac{\eta'\alpha_n^2}{K_n}.\end{aligned}\tag{5.15}$$

For the case of $\Lambda_i \cap \mathcal{C}_n = \emptyset$, by Lemma 5.9, for large enough n,

$$\sum_{i_A+1\le i\le m_A,\Lambda_i\cap\mathcal{C}_n=\emptyset}|\Lambda_i| \le K_n^2 - |\mathcal{C}_n| = o(K_n^2)\quad\text{a.s.}$$

Moreover, note that $\sum_{i=1}^{m_A}|\Lambda_i| = |A^c| \ge \beta K_n^2$, so we have

$$\begin{aligned}&\left(\sum_{i=1}^{i_A}+\sum_{i_A+1\le i\le m_A,\Lambda_i\cap\mathcal{C}_n\ne\emptyset}\right)|\Lambda_i|\\ &\quad=|A^c|-\sum_{i_A+1\le i\le m_A,\Lambda_i\cap\mathcal{C}_n=\emptyset}|\Lambda_i|\ge\frac{\beta}{2}K_n^2\quad\text{a.s.}\end{aligned}$$

Combining the above inequality with (5.11) and (5.15), for large enough n, we have

$$\inf_{\substack{|A|\le(1-\beta)K_n^2,|\Lambda_1|\le\frac{1}{2}K_n^2\\ A \text{ is 3-connected}}} f_1(A) \ge \frac{\eta'\alpha_n^2}{K_n}\cdot\frac{\beta K_n^2}{2} = \frac{\eta'\beta\alpha_n^2 K_n}{2} \qquad \text{a.s.}$$

By the above inequality, we can deduce that, with probability 1,

$$\inf_{\substack{|A|\le(1-\beta)K_n^2,|\Lambda_1|\le\frac{1}{2}K_n^2\\ A \text{ is 3-connected}}} \frac{f_1(A)}{|A|} \ge \frac{\eta'\beta\alpha_n^2 K_n}{2}\cdot\frac{1}{(1-\beta)K_n^2} = \frac{\eta'\beta\alpha_n^2}{2(1-\beta)K_n}$$

for large n. Combining this with (5.12) yields our result. ∎

Recall that d_i denotes the degree of vertex i in $G(0)$, and set $d^* := \sum_{i=1}^n d_i$. We obtain the following lemma.

LEMMA 5.11. *Assume that r_n satisfies* (5.1). *Then, for any constant $s > 1/\pi$, with probability* 1, *$d^* > n^2 r_n^2/s$ for large enough n.*

Proof. Given a constant $s' \in (1/\pi, s)$, let $Z_n(s')$ denote the number of vertices of $G(0)$ of degree at least nr_n^2/s'. By Theorem 4.2 in [34], $n^{-1}Z_n(s')$ converges completely to 1 as $n\to\infty$. Since $d^* \ge Z_n(s')nr_n^2/s'$, this yields our result. ∎

Proof of Theorem 4.3. If $G(0)$ is not connected, then $P(0)$ is reducible and therefore $\lambda_2 = 1$, so our necessary condition can be deduced directly using Lemma 3.1. Also, for the case of $\pi n r_n^2 \ge (\log n)^2$, our sufficient condition has been indicated by (4.3). Thus, we just need to consider the sufficient condition for the case that r_n satisfies (5.1).

Given $\lambda\in\mathbb{R}$, if $\lambda < \frac{1}{d_{max}} - 1$, then $P(0)-\lambda I_n$ is a strictly diagonally dominant matrix and $\det(P(0)-\lambda I_n)\ne 0$. Therefore, λ is not an eigenvalue of $P(0)$. By Lemma 4.1 we find that for all large enough n, with probability 1,

$$\lambda_n \ge \frac{1}{e\pi n r_n^2(1+o(1))} - 1.$$

Note that $1\ge\lambda_2\ge\lambda_n\ge -1$, so we just need to estimate λ_2 to get our result.

Let $F\subseteq\{1,2,\ldots,n\}$ denote a subset of agents and define $\widetilde{F} := \{X_i(0) : i\in F\}\subseteq \mathcal{X}_n$ to be the initial positions of agents in F. Let $F^c = \{1,2,\ldots,n\}\backslash F$. For any area $D_1, D_2\subset[0,1]^2$, set

$$f_{D_1,D_2}(F) := \sum_{x\in D_1\cap\widetilde{F},\, y\in D_2\cap\widetilde{F^c}} I_{\{\|x-y\|_2\le r_n\}}$$

and take $f(F) = f_{[0,1]^2,[0,1]^2}(F)$. Define Cheeger's constant Φ of $P(0)$ by

$$\Phi = \inf_{\sum_{i\in F} d_i\le\frac{1}{2}d^*} \frac{f(F)}{\sum_{i\in F} d_i}.$$

We assert that there exists a constant $\eta>0$ such that w.h.p., $\Phi\ge\eta r_n$ for large enough n. Next we will prove this assertion.

For any $F\subseteq\{1,2,\ldots,n\}$, set

$$A_F := \left\{z_i : |S_i\cap\widetilde{F}| > \frac{1}{2}|S_i\cap\mathcal{X}_n|\right\}\subseteq B_{\mathbb{Z}}(K_n)$$

and define

$$\widetilde{A_F} := \bigcup_{z_i\in A_F} S_i\cap\mathcal{X}_n.$$

By the condition $\sum_{i\in F} d_i \le \frac{1}{2}d^*$, we have

$$|F^c|d_{\max} \ge \sum_{i\in F^c} d_i \ge \frac{1}{2}d^*;$$

then by Corollary 4.2 and Lemma 5.11, for all large enough n,

$$|F^c| \ge \frac{d^*}{2d_{\max}} > \frac{2e\pi n^2 r_n^2}{6}\cdot\frac{1}{2e\pi n r_n^2} = \frac{n}{6} \quad \text{a.s.}$$

Set $\beta := \frac{1}{252}$. If $|A_F| > (1-\beta)K_n^2$, then $|A_F^c| \le \beta K_n^2$. By Lemma 5.1,

$$\sum_{z_i\in A_F^c} |S_i| \le |A_F^c|\Delta_n \le \frac{K_n^2}{252}\cdot 21\alpha_n = \frac{n}{12}$$

holds a.s. for large enough n. Note that $|\widetilde{F^c}| = |F^c| > \frac{n}{6}$; then there exist at least $\frac{n}{12}$ vertices of $\widetilde{F^c}$ contained in $\widetilde{A_F}$. For $x \in \widetilde{F^c}\cap\widetilde{A_F}$, without loss of generality we assume that $x \in S_i$ with $z_i \in A_F$; then by the definition of A_F we find $|\widetilde{F}\cap S_i| \ge |\widetilde{F^c}\cap S_i| \ge 1$, which indicates that there exists at least one vertex $y \in \widetilde{F}\cap S_i$ such that $\|x-y\|_2 \le r_n$. Thus, by Lemma 4.1,

$$\inf_{\sum_{i\in F} d_i \le \frac{1}{2}d^*, |A_F|>(1-\beta)K_n^2} \frac{f(F)}{\sum_{i\in F} d_i} \ge \frac{\frac{n}{12}}{\frac{1}{2}n d_{\max}} \ge \frac{1}{6e\pi n r_n^2(1+o(1))} > r_n \tag{5.16}$$

holds a.s. for large enough n.

Now we consider the case of $|A_F| \le (1-\beta)K_n^2$. Let $A_1, A_2, \dots, A_{m_F}$ be the 3-connected components of A_F, taken in decreasing order of size. In other words, $A_1, \dots, A_{m_F}$ are all 3-connected, but $A_i \cup A_j$, $1 \le i \ne j \le m_F$, is not 3-connected, and $|A_1| \ge |A_2| \ge \cdots \ge A_{m_F}$. Without loss of generality, we assume that $|A_i| \ge \beta\alpha_n^{-2}K_n$ for $1 \le i \le i_F$, and $|A_i| < \beta\alpha_n^{-2}K_n$ for $i_F+1 \le i \le m_F$, where $i_F \in [1, m_F]$. Then, by Lemma 5.10, there exists a constant $\eta' > 0$ such that

$$\inf_{|A_F|\le(1-\beta)K_n^2, 1\le i\le i_F} \frac{f_1(A_i)}{|A_i|} \ge \frac{\eta'\alpha_n^2}{K_n} \quad \text{a.s.} \tag{5.17}$$

For $i \in [1, i_F]$, it is easy to see that if $z_k \in A_i$ and $z_j \in A_i^c$ with $z_k \sim z_j$, then $z_j \in A_F^c$ and all pairs of vertices in $S_k \cup S_j$ are neighbors. So, by the definition of A_F, we have

$$f_{S_k,S_j}(F) = \sum_{x\in S_k\cap\widetilde{F}, y\in S_j\cap\widetilde{F^c}} I_{\{\|x-y\|_2\le r_n\}} \ge \frac{1}{4}|\mathcal{X}_n \cap S_k|\cdot|\mathcal{X}_n\cap S_j|.$$

Therefore, if $\mathcal{P}_{\lambda(n)} \subseteq \mathcal{X}_n$, then

$$\sum_{z_k\in A_i, z_j\in A_F^c, z_k\sim z_j} f_{S_k,S_j}(F) = \sum_{z_k\in A_i, z_j\in A_i^c, z_k\sim z_j} f_{S_k,S_j}(F) \ge \frac{1}{4}f_1(A_i). \tag{5.18}$$

Moreover, by (4.12) and the Borel–Cantelli lemma, we know that $\mathcal{P}_{\lambda(n)} \subseteq \mathcal{X}_n$ holds a.s. for large enough n. Set

$$S_F^1 := \bigcup_{i=1}^{i_F}\bigcup_{z_k\in A_i} S_k.$$

By (5.18), for large enough n, we have

$$f_{S_F^1,[0,1]^2\setminus S_F^1}(F) \geq \sum_{i=1}^{i_F} \sum_{z_k\in A_i, z_j\in A_F^c, z_k\sim z_j} f_{S_k,S_j}(F) \geq \sum_{i=1}^{i_F} \frac{1}{4} f_1(A_i) \quad \text{a.s.} \tag{5.19}$$

For $i \in [i_F+1, m_F]$, if $\bigcup_{z_j\in A_i} S_j \cap \widetilde{F^c} \neq \emptyset$, let $D_i = \bigcup_{z_j\in A_i} S_j$. Then we have $f_{D_i,D_i}(F) \geq 1$; otherwise, by Lemma 3.1, w.h.p. there exists at least one vertex $x^* \in (\bigcup_{z_j\in A_i} S_j)^c \cap \mathcal{X}_n$ such that the set

$$\left\{ y : y \in \bigcup_{z_j\in A_i} S_j \cap \widetilde{F}, \|x^* - y\| \leq r_n \right\}$$

is not empty. Assume that $x^* \in S_k (1 \leq k \leq K_n^2)$ and z_k is the corresponding integer point of S_k. Then z_k must be 3-connected with A_i, and $z_k \in A_F^c$. Set $D_i = \bigcup_{z_j\in A_i} S_j \cup S_k$. If $x^* \in \widetilde{F^c}$, then $f_{D_i,D_i}(F) \geq 1$; otherwise, by the definition of A_F we have $S_k \cap \widetilde{F^c} \neq \emptyset$, so

$$f_{D_i,D_i}(F) \geq f_{S_k,S_k}(F) \geq 1.$$

Let $S_F^2 = \bigcup_{i=i_F+1}^{m_F} D_i$. For $z \in \mathbb{Z}^2$, it is easy to see that the number of 3-connected components with which z is 3-connected is not more than 8. By the above argument we have w.h.p.

$$f_{S_F^2,S_F^2}(F) \geq \frac{1}{8}(m_F - i_F). \tag{5.20}$$

Let $S_F^3 = [0,1]^2 \setminus (S_F^1 \cup S_F^2)$. For $x \in S_F^3 \cap \widetilde{F}$, assume that $x \in S_k (1 \leq k \leq K_n^2)$ and $z_k \in B_{\mathbb{Z}}(K_n)$ is the corresponding integer point of S_k. Obviously $z_k \in A_F^c$, so the set $S_k \cap \widetilde{F^c}$ is not empty. Thus,

$$f_{S_F^3,S_F^3}(F) \geq \sum_{x\in S_F^3\cap\widetilde{F}} 1 \geq |S_F^3 \cap \widetilde{F}|. \tag{5.21}$$

Recall that $\mathcal{L}(\cdot)$ denotes the Lebesgue measure. By the definitions of S_F^1 and S_F^2 we have $\mathcal{L}(S_F^1 \cap S_F^2) = 0$. So by (5.19), (5.20), and (5.21), we have

$$\begin{aligned} f(F) &\geq f_{S_F^1,[0,1]^2\setminus S_F^1}(F) + f_{S_F^2,S_F^2}(F) + f_{S_F^3,S_F^3}(F) \\ &\geq \sum_{i=1}^{i_F} \frac{1}{4} f_1(A_i) + \frac{1}{8}(m_F - i_F) + |S_F^3 \cap \widetilde{F}| \quad \text{w.h.p.} \end{aligned}$$

Thus, w.h.p.,

$$\inf_{|A_F|\le(1-\beta)K_n^2} \frac{f(F)}{\sum_{i\in F} d_i}$$
$$\ge \frac{\sum_{i=1}^{i_F}\frac{1}{4}f_1(A_i)+\frac{1}{8}(m_F-i_F)+|S_F^3\cap\widetilde{F}|}{d_{\max}(|S_F^1\cap\widetilde{F}|+|S_F^2\cap\widetilde{F}|+|S_F^3\cap\widetilde{F}|)}$$
$$\ge \frac{\sum_{i=1}^{i_F}\frac{1}{4}f_1(A_i)+\frac{1}{8}(m_F-i_F)+|S_F^3\cap\widetilde{F}|}{d_{\max}(\Delta_n\sum_{i=1}^{i_F}|A_i|+(m_F-i_F)\Delta_n\beta\alpha_n^{-2}K_n+|S_F^3\cap\widetilde{F}|)} \tag{5.22}$$
$$\ge \min\left\{\frac{\frac{1}{4}\sum_{i=1}^{i_F}f_1(A_i)}{d_{\max}\Delta_n\sum_{i=1}^{i_F}|A_i|}, \frac{\frac{1}{8}(m_F-i_F)}{d_{\max}(m_F-i_F)\Delta_n\beta\alpha_n^{-2}K_n}, \frac{|S_F^3\cap\widetilde{F}|}{d_{\max}|S_F^3\cap\widetilde{F}|}\right\}$$
$$\ge \min\left\{\frac{\eta'\alpha_n^2}{4d_{\max}\Delta_n K_n}, \frac{\alpha_n^2}{8d_{\max}\Delta_n\beta K_n}, \frac{1}{d_{\max}}\right\},$$

where the last inequality can be deduced by (5.17).

Combining (5.1), (5.16), (5.22) with Lemmas 4.1 and 5.1, our assertion holds.

By Cheeger's inequality (Proposition 6 in [35]), we have $\lambda_2 \le 1-\Phi^2$. Hence, with probability 1, $\lambda_2 \le 1-\eta^2 r_n^2$ holds for large enough n. This completes the proof of our result. ∎

6. Simulation Example. In this section, we provide a simulation example. Here, the number of agents is taken as $n = 1000$, and the interaction radius is $r_n = \sqrt{1.1\log n/(\pi n)}$. The initial positions and headings of the n agents are independent, with positions uniformly and independently distributed in $[0,1]^2$ and with headings uniformly and independently distributed in $(-\pi,\pi]$. Figure 4 shows how the probability of synchronization changes with moving speed. From this simulation, we see that if the speed is small, the system can synchronize w.h.p., and the probability of synchronization will tend to zero as the speed increases.

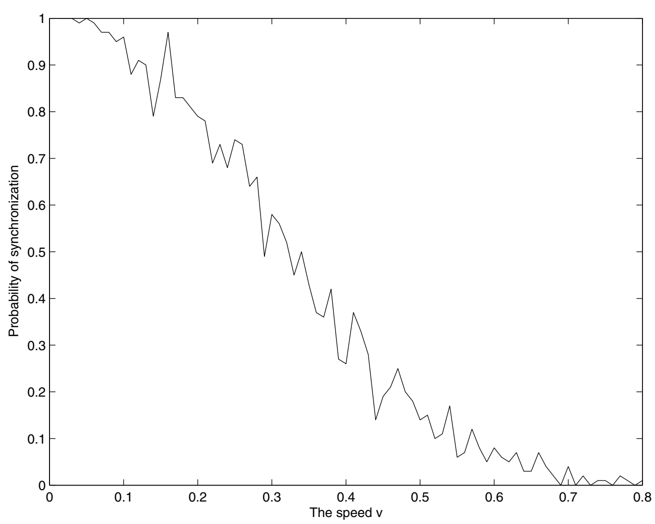

Fig. 4 *Simulation results for the system with* $n = 1000$, $r_n = \sqrt{1.1\log n/(\pi n)}$, *and the random initial states.*

7. Concluding Remarks. For the SPP system studied in this paper, it is intuitively obvious that the smaller the interaction radius is, the harder it is for the synchronization to happen. Thus, an important and interesting problem is how small the interaction radius must be in order to guarantee synchronization. This paper shows that, in a certain sense, the smallest possible interaction radius for synchronization can be considered to be the same as the critical radius for connectivity of the initial random geometric graph. We remark that an important step of this paper is to provide an estimation of the spectral gap of the average matrix of the random geometric graph. Our results have possible applications to other problems such as collective motion of biological systems, random walk on random geometric graphs, and consensus algorithms of wireless sensor networks.

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