



# Control of uncertain nonlinear systems based on observers and estimators<sup>☆</sup>



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## ABSTRACT

In this paper, we consider a class of nonlinear dynamical systems with zero dynamics, which is subject to both unknown nonparametric dynamics and external disturbances, and is required to track a given reference signal by using the output feedback. Our controller is designed based on both the extended state observer (ESO) and the projected gradient estimator. While the ESO is used to estimate the total uncertainties, the projected gradient algorithm is used to estimate the nonparametric uncertainties treated as “time-varying parameters”. This method overcomes the difficulties that the traditional active disturbance rejection control (ADRC) technique needs to have a “good” prior estimate for the uncertainties in the input channel. The closed-loop system is shown to be semi-globally stable, and at the same time, the tracking error can be made arbitrarily small.

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## 1. Introduction

Uncertainties always exist in the modeling of practical dynamical systems due to, e.g., complexity in understanding complex systems, unavoidable changes of systems structures, difficulty in predicting changes of the environment, etc. As a fundamental issue in automatic control, dealing with uncertainties has been the focus of many developments in control theory. Plenty of control methods have been developed for dealing with uncertainties over the past half a century, among which adaptive control (see, e.g., Åström & Wittenmark, 1995, Chen & Guo, 1991 and Krstić, Kanelakopoulos, & Kokotović, 1995) and robust control (see, e.g., Qu, 1998 and Zames, 1981) are two typical approaches. Traditional adaptive control design usually requires that the uncertainties can be expressed linearly in terms of unknown parameters. On

the other hand, robust control design usually requires that the uncertainties be bounded in some norm and have certain structural property. What is more, various disturbance estimation techniques have been proposed for rejecting disturbances, such as the unknown input observer (UIO) (Hostetter & Meditch, 1973), the disturbance observer (DOB) (Schrijver & Van Dijk, 2002), the perturbation observer (POB) (Kwon & Chung, 2003), etc. Brief surveys of disturbance observers can be found in Guo, Feng, and Chen (2006) and Radke and Gao (2006). Most estimators, like UIO, DOB and POB, are designed to handle small perturbations, and usually require the model of the plant to reconstruct the disturbances.

Owing to its less dependence on plant information, its capabilities to deal with a wide range of uncertainties, and its simplicity in the control structure, the active disturbance rejection control (ADRC) technique has received much attention in the control community (see, e.g. Gao, 2006, Gao, Huang, & Han, 2001 and Han, 1995, 1998, 2008, 2009). The key of ADRC is to online estimate the total uncertainties that lump unmodeled dynamics and external disturbances by an extended state observer (ESO) (Han, 1995, 2008, 2009). Thus, the uncertainties may then be compensated in real time. Up to now, the idea of ADRC technique has been applied in solving various kinds of engineering problems, e.g., motor control (Feng, Liu, & Huang, 2004; Li & Liu, 2009), flight control (Huang, Xu, Han, & Lam, 2001; Xia, Zhu, Fu, & Wang, 2011), robot control (Su, Ma, Qiu, & Xi, 2004; Talole, Kolhe, & Phadke, 2010), etc. Meanwhile, some progress has also been made in the

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theoretical analysis of the ADRC (Guo & Zhao, 2011; Xue & Huang, 2011a,b; Yang & Huang, 2009; Zheng, Gao, & Gao, 2007). In Zheng et al. (2007), the stability of the closed-loop system based on ADRC is discussed under the condition of bounded uncertainties. The case where there is additional uncertainty in the input channel (denoted by  $b(x, t)$ ) has been considered in Freidovich and Khalil (2008) and Xue and Huang (2011b), with two different control methods proposed to stabilize such uncertain systems. Based on the extended high-gain observer (EHGO), Freidovich and Khalil (2008) proposed an output feedback controller and proved that the closed-loop system was able to recover the performance of the nominal linear model. Xue and Huang (2011b), applying the ADRC method, also demonstrated the stability of the closed-loop system. Nevertheless, a “good” prior estimate of  $b(x, t)$ , satisfying some algebraic condition, is required for both the above control methods.

So a natural problem is: if it is possible to relax or remove the priori information on  $b(x, t)$ ? Actually, for many practical control systems, there do exist uncertainties in the input channel (e.g. the flight control system Xiao, 1987) and such “good” priori information is usually difficult to obtain. Therefore, it is of great significance, both theoretically and practically, to investigate this problem. To solve it, Huang and Guo (2012) proposed to estimate the uncertainties by combining observers and estimators for the output feedback control of a class of nonlinear systems with the integrators in series structure. Furthermore, in Scheinker and Krstić (2013), a state feedback based on the extremum seeking (ES) design has been developed for semi-global stabilization of unstable and time-varying systems, where the control direction is unknown and is allowed to persistently change signs.

In this paper, we will consider the tracking problem for a class of nonlinear uncertain systems with zero dynamics, which is an extension of the nonlinear systems considered in an earlier paper of Huang and Guo (2012) where no zero dynamics was considered. We remark that many practical plants may be described by nonlinear models with stable zero dynamics (see, e.g. Chen, Yan, & Sun, 2014), and the coupling between internal and external states makes the analyses more complicated, since the stability of both states have to be established simultaneously. By using the method of combining the ESO technique and the projected gradient estimator, we are able to design an output feedback controller, and to show that such controller can ensure the closed-loop stability and make the tracking error arbitrarily small.

In the rest of the paper, we will present the main results in Section 2, and give the detailed proof in Section 3. A numerical example will be given in Section 4. Finally, Section 5 will conclude the paper with some remarks.

## 2. Main results

### 2.1. Problem formulation

We consider the following single-input-single-output (SISO) nonlinear system

$$\begin{cases} \dot{x} = Ax + B[a(x, z, t) + b(x, z, t)u], \\ \dot{z} = f_0(x, z, t), \\ y = Cx, \end{cases} \quad t \geq t_0 \quad (1)$$

where  $x = [x_1 \ x_2 \ \dots \ x_n]^T \in \mathbb{R}^n$  and  $z = [z_1 \ z_2 \ \dots \ z_m]^T \in \mathbb{R}^m$  are the state variables,  $u \in \mathbb{R}$  is the control input,  $y \in \mathbb{R}$  is the measured output,  $t_0$  is the initial time, and  $a(x, z, t)$ ,  $b(x, z, t)$ ,  $f_0(x, z, t)$  are nonlinear time-varying functions which may contain unknown dynamics and external disturbances. In addition, the triple  $(A, B, C)$  represents a chain of  $n$  integrators,

i.e.,

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & \dots & 0 \end{bmatrix} \in \mathbb{R}^{n \times n}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^n, \\ C = [1 \ 0 \ \dots \ 0] \in \mathbb{R}^{1 \times n}.$$

Our control objective is to develop an output feedback controller to make sure that for all initial states in any given compact set, the state signals  $(x(t), z(t))$  are bounded, and  $x(t)$  tracks the reference trajectory which is generated from the target system

$$\dot{x}^*(t) = A_m x^*(t) + Br(t), \quad t \geq t_0 \quad (2)$$

where  $x^*(t) \in \mathbb{R}^n$ , the input signal  $r(t) \in \mathbb{R}$  satisfying

$$|r(t)| \leq \bar{r}, \quad |\dot{r}(t)| \leq \bar{r} \quad (3)$$

with  $\bar{r} > 0$  a known constant, and

$$A_m = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ -k_1 & -k_2 & \dots & \dots & -k_n \end{bmatrix} \in \mathbb{R}^{n \times n} \quad (4)$$

is a Hurwitz matrix (i.e., the polynomial  $s^n + k_n s^{n-1} + \dots + k_1$  is Hurwitz), so that there exists a positive definite matrix  $P_0 > 0$  such that

$$A_m^T P_0 + P_0 A_m = -I. \quad (5)$$

Throughout the paper, we need the following assumptions:

(A1)  $f_0(x, z, t)$  is locally Lipschitz,  $a(x, z, t)$  and  $b(x, z, t)$  are differentiable with locally Lipschitz derivatives. Moreover, for any constant  $\rho \geq 0$ , if  $\|(x, z)\| \leq \rho$ , then

$$\|f_0\| + \|a\| + \|b\| + \|\nabla a\| + \|\nabla b\| \leq \tau(\rho) \quad (6)$$

holds for all  $t \geq t_0$ , where  $\|\cdot\|$  is the Euclidean norm,  $\nabla f$  is the gradient of  $f$ , and  $\tau(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a known finite increasing function.

(A2) The nonlinear function  $b(x, z, t)$  is bounded away from zero for all  $(x, z, t) \in \mathbb{R}^n \times \mathbb{R}^m \times [t_0, \infty)$ , and the sign of  $b(x, z, t)$  is known. Without loss of generality, let  $b(x, z, t) \geq \underline{b}$  with a known positive number  $\underline{b}$ .

(A3) There exists a continuously differentiable function  $V_0(t, z) : [t_0, \infty) \times \mathbb{R}^m \rightarrow \mathbb{R}_+$ , such that for all  $(x, z, t) \in \mathbb{R}^n \times \mathbb{R}^m \times [t_0, \infty)$ ,

$$\alpha_1(\|z\|) \leq V_0(t, z) \leq \alpha_2(\|z\|), \quad (7)$$

$$\frac{\partial V_0}{\partial t} + \frac{\partial V_0}{\partial z} f_0(x, z, t) \leq 0, \quad \forall \|z\| \geq \alpha_0(\|x\|), \quad (8)$$

where  $\alpha_0(\cdot)$  is a known class  $\mathcal{K}$  function and  $\alpha_1(\cdot)$ ,  $\alpha_2(\cdot)$  are known class  $\mathcal{K}_\infty$  functions (Khalil, 2002).

We remark that Assumption (A3) ensures that the system  $\dot{z} = f_0(x, z, t)$ , with input  $x$ , is bounded-input-bounded-state stable (BIBS), which is less restrictive than the input-to-state stability (ISS) because it does not require the origin of  $\dot{z} = f_0(0, z, t)$  to be uniformly asymptotically stable (UAS). To our understanding, there is no conclusive assertion that the typical minimum phase condition is weaker than our Assumption (A3), and vice versa. The Assumption (A3) used in this paper is only for the convenience of proof. Of course, the main results in this paper are still true if the zero dynamics  $\dot{z} = f_0(0, z, t)$  is (locally uniformly) exponentially stable and Assumptions (A1)–(A2) hold.

We also remark that in our controller design to be given shortly, it is not necessary to know  $\tau(\rho)$  in (6) for all  $\rho$  as stated in Assumption (A1). Actually, from the proof of [Theorem 2.1](#) given later, it can be seen that we only need to know  $\tau(\rho_{\max})$  for some positive number  $\rho_{\max}$ , where  $\rho_{\max}$  can be determined by the range of the initial values, the parameters in the reference model, as well as the bounds in Assumption (A3), as will be shown shortly below. Furthermore, in practice, the upper bound  $\tau(\rho_{\max})$  can usually be obtained by analyzing the physical mechanism of the plant and experiment methods. Moreover, to the best of our knowledge, this assumption appears to be the weakest one under which the stabilization by output feedback for nonlinear uncertain plants can be established.

To give a precise value of  $\rho_{\max}$ , we suppose that the initial condition lies in a known bounded domain, say,  $\|(x(t_0), z(t_0))\| \leq \rho_0$  with  $\rho_0 > 0$ . Let us take  $\rho_s > \rho_0$  be any given constant, and define

$$\rho_r \triangleq 2 \left( \sqrt{\frac{c_{02}}{c_{01}}} \rho_s + \frac{2c_{02}\sqrt{2c_{02}}}{\sqrt{c_{01}}} \bar{r} \right), \quad (9)$$

where  $\bar{r}$  is the upper bound of the reference signal given by (3), and  $c_{01}$  and  $c_{02}$  are the minimum and maximum eigenvalues of the positive definite matrix  $P_0$  defined from (5), respectively, i.e.,

$$c_{01} = \lambda_{\min}(P_0), \quad c_{02} = \lambda_{\max}(P_0). \quad (10)$$

Furthermore, we take

$$\gamma_z(\rho_r) > \kappa(\rho_r), \quad (11)$$

where  $\kappa(\rho_r) = \alpha_1^{-1}(\alpha_2(\bar{\rho}))$  with  $\bar{\rho} = \max\{\alpha_0(\rho_r), \rho_s\}$ , in which  $\alpha_i(\cdot)$  ( $i = 0, 1, 2$ ) are given in Assumption (A3).

Consequently, we can take  $\rho_{\max} = \sqrt{\rho_r^2 + \gamma_z^2(\rho_r)}$  to replace  $\rho$  in Assumption (A1), and all the results still hold true.

## 2.2. Controller design

With the Assumption (A3), we concentrate on the design of an output feedback controller for the  $x$ -subsystem. Had  $(x, z)$  been available for feedback and the functions  $a(\cdot)$  and  $b(\cdot)$  been known, we could have used the following state feedback controller

$$u = \frac{-K^T x - a(x, z, t) + r(t)}{b(x, z, t)}, \quad K \triangleq [k_1 \quad k_2 \quad \cdots \quad k_n]^T$$

to achieve our control objective. Hence the key point to design the output feedback controller is to estimate the unmeasurable states  $\{x_2, \dots, x_n\}$  and the unknown functions  $a(\cdot)$  and  $b(\cdot)$ . To that end, we propose a novel estimation method by combining the ESO and the projected gradient estimator, which will be given below.

Firstly, by taking

$$x_{n+1} \triangleq a(x, z, t) + b(x, z, t)u \quad (12)$$

as the total uncertainty, we use the ESO to estimate both the states  $\{x_1, \dots, x_n\}$  and the total uncertainty  $x_{n+1}$ , which can be written as [Han \(1995\)](#)

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + \frac{\beta_1}{\epsilon}(x_1 - \hat{x}_1) \\ \vdots \\ \dot{\hat{x}}_n = \hat{x}_{n+1} + \frac{\beta_n}{\epsilon^n}(x_n - \hat{x}_n) \\ \dot{\hat{x}}_{n+1} = \frac{\beta_{n+1}}{\epsilon^{n+1}}(x_{n+1} - \hat{x}_{n+1}) \end{cases} \quad (13)$$

where  $\epsilon > 0$  is a gain parameter to be determined later, and  $\beta_i$  ( $i = 1, \dots, n+1$ ) are coefficients of the Hurwitz polynomial

$s^{n+1} + \beta_1 s^n + \cdots + \beta_{n+1}$ . Actually, if the parameter  $\epsilon$  is small enough, the ESO (13) is in essence a high gain observer, and  $\hat{x}_{n+1}$  can be a good estimate of the total uncertainty  $x_{n+1}$ . To make an intuitive explanation for this, we calculate the transfer function from  $x_{n+1}$  to its estimate  $\hat{x}_{n+1}$  as follows:

$$G(s)|_{x_{n+1} \rightarrow \hat{x}_{n+1}} = \frac{\beta_{n+1}}{(\epsilon s)^{n+1} + \beta_1 (\epsilon s)^n + \cdots + \beta_{n+1}},$$

which approaches 1 as  $\epsilon$  tends to zero. Hence,  $\hat{x}_{n+1}$  should be a good estimate of  $x_{n+1}$  intuitively.

Now having  $\hat{x}_{n+1}$  be the estimate of the total uncertainty  $a(x, z, t) + b(x, z, t)u$ , we can treat the nonlinear functions  $a(\cdot)$  and  $b(\cdot)$  as unknown time-varying parameters, and use the projected gradient method to estimate  $a(\cdot)$  and  $b(\cdot)$  respectively. Firstly, to calculate the projection boundaries, we need to define the following compact set

$$D \triangleq \left\{ (x, z) \in \mathbb{R}^{n+m} \mid \|x\| \leq \rho_r, \|z\| \leq \gamma_z(\rho_r) \right\}, \quad (14)$$

where  $\rho_r$  and  $\gamma_z(\rho_r)$  are given by (9) and (11).

We are now in a position to give the parameter estimation algorithms. Let  $\hat{a}(t)$  and  $\hat{b}(t)$  be estimates of  $a(x, z, t)$  and  $b(x, z, t)$  at time  $t$  respectively, and define the prediction error

$$w \triangleq \hat{x}_{n+1} - \hat{a}(t) - \hat{b}(t)u, \quad (15)$$

then for any  $t \geq t_0$ , the projected gradient estimators ([Ioannou & Sun, 1996](#)) for  $a(x, z, t)$  and  $b(x, z, t)$  are as follows:

$$\dot{\hat{a}} = \begin{cases} 0, & \text{if } (\hat{a} = \mu_1(\rho_r) \text{ and } w \geq 0) \\ & \text{or } (\hat{a} = -\mu_1(\rho_r) \text{ and } w \leq 0) \\ \gamma w, & \text{otherwise} \end{cases} \quad (16)$$

$$\dot{\hat{b}} = \begin{cases} 0, & \text{if } (\hat{b} = \mu_2(\rho_r) \text{ and } wu \geq 0) \\ & \text{or } (\hat{b} = \underline{b} \text{ and } wu \leq 0) \\ \gamma wu, & \text{otherwise} \end{cases} \quad (17)$$

where  $\gamma > 0$  is a parameter to be designed and we take  $\gamma = \epsilon^{-\frac{2}{3}}$  in order to minimize the ultimate upper bound of the tracking error (see the proof of [Theorem 2.1](#) for details). In addition, the projection boundaries are dependent on the set  $D$  defined by (14). To be specific,  $\mu_1(\rho_r) \triangleq \tau(\rho_r + \gamma_z(\rho_r))$  and  $\mu_2(\rho_r) \triangleq \max\{\tau(\rho_r + \gamma_z(\rho_r)), \underline{b}\}$  with the function  $\tau(\cdot)$  given in Assumption (A1), and  $\{\rho_r, \gamma_z(\rho_r)\}$  given by (9) and (11). Notice that the differential equations of the estimation law (16)–(17) are with discontinuous right-hand sides, so the solutions here are understood in the Filippov sense ([Filippov, 1988](#)). From previous works (e.g., [Polycarpou & Ioannou, 1993](#) and [Shevitz & Paden, 1994](#)), it can be verified that the solutions of Eqs. (16)–(17) exist for any  $t \geq t_0$  in the sense that the solutions are absolutely continuous and satisfy Eqs. (16)–(17) almost everywhere on  $[t_0, +\infty)$ . Furthermore, for the above algorithm (16)–(17), it is not difficult to see that if the initial conditions satisfy  $\hat{a}(t_0) \in [-\mu_1(\rho_r), \mu_1(\rho_r)]$  and  $\hat{b}(t_0) \in [\underline{b}, \mu_2(\rho_r)]$ , the estimates  $\{\hat{a}, \hat{b}\}$  will never escape the following compact set

$$S \triangleq \{(\hat{a}, \hat{b}) \in \mathbb{R}^2 \mid \hat{a} \in [-\mu_1(\rho_r), \mu_1(\rho_r)], \hat{b} \in [\underline{b}, \mu_2(\rho_r)]\},$$

which essentially ensures the boundedness of  $(\hat{a}, \hat{b})$ .

Now with the state estimates  $\{\hat{x}_1, \dots, \hat{x}_n\}$  from the ESO (13), the parameter estimates  $\{\hat{a}, \hat{b}\}$  from the estimator (16)–(17), and in order to track the reference signal  $x^*(t)$  given in (2), we can take the control  $u$  as

$$\psi(\hat{x}, \hat{a}, \hat{b}, t) \triangleq (-K^T \hat{x} - \hat{a} + r(t)) / \hat{b}, \quad (18)$$

where  $K = [k_1 \quad k_2 \quad \cdots \quad k_n]^T$  and  $\hat{x} \triangleq [\hat{x}_1 \quad \hat{x}_2 \quad \cdots \quad \hat{x}_n]^T$ . Moreover, to protect the closed-loop system from peaking in the

observer's transient response caused by the nonzero initial error  $\| [x_1(t_0) - \hat{x}_1(t_0), \dots, x_n(t_0) - \hat{x}_n(t_0)] \|$ , follow the idea proposed in Khalil (2002), we saturate the control outside a compact set of interest, and the control law is designed as follows:

$$u = M \operatorname{sat} \left( \psi(\hat{x}, \hat{a}, \hat{b}, t) / M \right), \quad (19)$$

where  $\psi(\cdot)$  is defined by (18),  $M$  is a constant depending on the compact set  $D$  defined by (14), which is satisfied

$$M > \max_{\{\|x\| \leq \rho_r, |\hat{a}| \leq \mu_1(\rho_r), \hat{b} \geq \underline{b}, |r| \leq \bar{r}\}} \left| \frac{-K^T x - \hat{a} + r(t)}{\hat{b}} \right|, \quad (20)$$

and  $\operatorname{sat}(\cdot)$  is the standard saturation function defined by  $\operatorname{sat}(s) = \min\{1, |s|\} \operatorname{sign}(s)$ . In general, the upper bound  $\rho_r$  might be calculated based on some Lyapunov function and be of conservative, so the calculation of  $M$  might end up with a conservative bound.

### 2.3. Main theorem

The performance of the closed-loop system under the above output feedback controller (19) can be presented by the following theorem:

**Theorem 2.1.** Consider the nonlinear uncertain plant (1). Let the output feedback controller be defined by (19), which is based on both the extended state observer (13) and the parameter estimator (16)–(17) with  $\rho_r$  given by (9). Suppose that Assumptions (A1)–(A3) are satisfied, then for any  $\rho_0 > 0$  and any initial state  $\|(x(t_0), z(t_0))\| \leq \rho_0$ , there exists  $\epsilon^* > 0$  such that for any  $\epsilon \in (0, \epsilon^*)$ , we have

- (1) all trajectories  $(x(t), z(t))$  are bounded for all  $t \geq t_0$ ;
- (2) The tracking performance satisfies

$$\limsup_{t \rightarrow \infty} \|x(t) - x^*(t)\| = O\left(\epsilon^{\frac{1}{3}}\right).$$

Furthermore, if  $x(t_0) = x^*(t_0)$ , then  $\|x(t) - x^*(t)\| = O(\epsilon^{\frac{1}{3}})$  holds for all  $t \geq t_0$ .

**Remark 2.1.** The tracking error  $\|x(t) - x^*(t)\|$  is determined by both parameters  $\epsilon$  and  $\gamma$ , which relate to the ESO and the projected gradient estimation algorithm. Moreover, the ultimate upper bound of  $\|x(t) - x^*(t)\|$  is of the order  $\frac{1}{\sqrt{\gamma}} + \gamma\epsilon$  for all large  $\gamma$  and small  $\epsilon$ . Hence we take  $\gamma = \epsilon^{-2/3}$  to minimize the bound for the tracking error.

**Remark 2.2.** Theoretically, the value of  $\epsilon$  can be taken as arbitrarily small to make the tracking error as small as possible. However, in practice, there will be some limitations on the values of  $\epsilon$  and  $\gamma$  due to both noises and sampling constraints when the algorithms (13), (16) and (17) are implemented on the computers. This issue will be addressed in detail elsewhere.

### 3. Proof of Theorem 2.1

In this section, we will prove Theorem 2.1. Before proceeding to explore the performance of the closed-loop system, we first derive the equation of the closed-loop system consisting of the plant (1) and the output feedback controller (19), then present several key lemmas upon which the stability analysis depends. At last, the detailed proof is given.

#### 3.1. Closed-loop system equation

We first derive the closed-loop system equation under the controller (19). For this, we first need to get the estimation error

equation of the ESO in (13). We now define the estimation errors as

$$e_i \triangleq x_i - \hat{x}_i, \quad \text{for } 1 \leq i \leq n \quad (21)$$

$$\begin{aligned} e_{n+1} &\triangleq a(x, z, t) + b(x, z, t) M g_\epsilon \left( \psi(x, \hat{a}, \hat{b}, t) / M \right) - \hat{x}_{n+1} \\ &\triangleq f_\epsilon(x, z, \hat{a}, \hat{b}, t) - \hat{x}_{n+1} \end{aligned} \quad (22)$$

where  $g_\epsilon(\cdot)$  is an odd function defined by Freidovich and Khalil (2008)

$$g_\epsilon(s) = \begin{cases} s, & \text{for } 0 \leq s \leq 1 \\ s + \frac{s-1}{\epsilon} - \frac{s^2-1}{2\epsilon}, & \text{for } 1 \leq s \leq 1+\epsilon \\ 1 + \frac{\epsilon}{2}, & \text{for } s \geq 1+\epsilon. \end{cases} \quad (23)$$

The function  $g_\epsilon(\cdot)$  is nondecreasing, continuously differentiable with a locally Lipschitz derivative, bounded uniformly in  $\epsilon$  on any bounded interval of  $\epsilon$ , and satisfies  $0 \leq g'_\epsilon(s) \leq 1$  and  $|g_\epsilon(s) - \operatorname{sat}(s)| \leq \epsilon/2$  for all  $s \in \mathbb{R}$ . To get a compact form of the closed-loop equation for the state estimation error, we introduce the following scaled variables

$$\xi_i \triangleq \frac{1}{\epsilon^{n+1-i}} e_i, \quad i = 1, \dots, n+1, \quad (24)$$

and set  $\xi_e \triangleq [\xi_1 \cdots \xi_{n+1}]^T \in \mathbb{R}^{n+1}$ . Then we have

$$\dot{\xi}_e = T^{-1}(\epsilon) E_e \xi_e,$$

where  $T(\epsilon) \triangleq \operatorname{diag}\{\epsilon^n, \dots, \epsilon, 1\}$  and  $E_e \triangleq [e_1 \cdots e_{n+1}]^T$ . Then by (21)–(22), (24) and combining the system (1), Eq. (12), the ESO (13) and the controller (19), it is not difficult to see that  $\xi_e$  satisfies:

$$\dot{\xi}_e = \frac{1}{\epsilon} A_e \xi_e + \frac{1}{\epsilon} B_1 \Delta_u(x, z, \xi_e, \hat{a}, \hat{b}, t) + B_2 \frac{df_\epsilon(x, z, \hat{a}, \hat{b}, t)}{dt}, \quad (25)$$

where  $A_e$  is a Hurwitz matrix defined by

$$A_e \triangleq \begin{bmatrix} -\beta_1 & 1 & 0 & \cdots & 0 \\ -\beta_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\beta_n & 0 & \cdots & 0 & 1 \\ -\beta_{n+1} & 0 & \cdots & \cdots & 0 \end{bmatrix}, \quad (26)$$

$$B_1 \triangleq [B^T \quad 0]^T, \quad B_2 \triangleq [0 \quad B^T]^T, \quad (27)$$

$$\begin{aligned} \Delta_u(x, z, \xi_e, \hat{a}, \hat{b}, t) &\triangleq a(x, z, t) + b(x, z, t) u - f_\epsilon(x, z, \hat{a}, \hat{b}, t) \\ &= b(x, z, t) M \left[ \operatorname{sat} \left( \frac{\psi(\hat{x}, \hat{a}, \hat{b}, t)}{M} \right) - g_\epsilon \left( \frac{\psi(x, \hat{a}, \hat{b}, t)}{M} \right) \right], \end{aligned} \quad (28)$$

and the function  $f_\epsilon(\cdot)$  is given by (22).

Consequently, by substituting (19) into (1) and combining (25), we see that the closed-loop system under the controller (19) can be written as:  $\forall t \geq t_0$ ,

$$\begin{cases} \dot{z} = f_0(x, z, t), \\ \dot{x} = Ax + B \left[ a(x, z, t) + b(x, z, t) M \operatorname{sat} \left( \frac{\psi(x, \xi_e, \hat{a}, \hat{b}, t)}{M} \right) \right], \\ \dot{\xi}_e = \frac{1}{\epsilon} A_e \xi_e + \frac{1}{\epsilon} B_1 \Delta_u(x, z, \xi_e, \hat{a}, \hat{b}, t) + B_2 \eta(x, z, \xi_e, \hat{a}, \hat{b}, t), \end{cases} \quad (29)$$

where  $A_e$  and  $B_i$  ( $i = 1, 2$ ) are given in (26) and (27),  $\psi(\cdot)$  and  $\Delta_u(\cdot)$  are functions defined by (18) and (28) respectively, and  $\eta(\cdot)$  is defined by

$$\begin{aligned} \eta(x, z, \xi_e, \hat{a}, \hat{b}, t) &\triangleq \frac{df_\epsilon(x, z, \hat{a}, \hat{b}, t)}{dt} \\ &= \frac{\partial a}{\partial t} + \frac{\partial b}{\partial t} Mg_\epsilon \left( \frac{\psi(x, \hat{a}, \hat{b}, t)}{M} \right) \\ &\quad + \left[ \frac{\partial a}{\partial z} + \frac{\partial b}{\partial z} Mg_\epsilon \left( \frac{\psi(x, \hat{a}, \hat{b}, t)}{M} \right) \right] f_0(x, z, t) \\ &\quad + \left[ \frac{\partial a}{\partial x} + \frac{\partial b}{\partial x} Mg_\epsilon \left( \frac{\psi(x, \hat{a}, \hat{b}, t)}{M} \right) \right] \\ &\quad \times \left( Ax + B \left[ a + bM \operatorname{sat} \left( \frac{\psi(x, \xi_e, \hat{a}, \hat{b}, t)}{M} \right) \right] \right) \\ &\quad + bg'_\epsilon \left( \frac{\psi(x, \hat{a}, \hat{b}, t)}{M} \right) \left[ -\frac{\dot{\hat{b}}\psi(x, \hat{a}, \hat{b}, t)}{\hat{b}} \right. \\ &\quad \left. + \frac{1}{\hat{b}} \left( -K^T Ax - k_n(a + bu) - \dot{\hat{a}} + \dot{r} \right) \right], \end{aligned} \quad (30)$$

in which  $\hat{a}, \hat{b}$  are given by the estimation law (16)–(17). Note that by (21) and (24) we have

$$\hat{x}_i = x_i - e_i = x_i - \epsilon^{n+1-i} \xi_i, \quad i = 1, \dots, n.$$

Substituting this into (18), we know that  $\psi(\hat{x}, \hat{a}, \hat{b}, t)$  can be expressed alternatively by

$$\psi(x, \xi_e, \hat{a}, \hat{b}, t) = \frac{-K^T x + \epsilon K_e^T \xi_e - \hat{a} + r(t)}{\hat{b}} \quad (31)$$

where  $K_e$  is defined by

$$K_e \triangleq [\epsilon^{n-1} k_1, \epsilon^{n-2} k_2, \dots, k_n, 0]^T. \quad (32)$$

Of course, the closed-loop equation (29) can be simplified in the case where  $|\psi(\hat{x}, \hat{a}, \hat{b}, t)| \leq M$ . In fact, in this case the controller (19) is

$$u = \frac{-K^T \hat{x} - \hat{a} + r(t)}{\hat{b}}. \quad (33)$$

Setting

$$\theta(x, z, t) \triangleq [a(x, z, t), b(x, z, t)]^T, \quad \varphi \triangleq [1, u]^T, \quad (34)$$

the estimation vector  $\hat{\theta}(t) \triangleq [\hat{a}(t), \hat{b}(t)]^T$ , and the estimation errors

$$\begin{aligned} \tilde{a}(t) &\triangleq a(x, z, t) - \hat{a}(t), & \tilde{b}(t) &\triangleq b(x, z, t) - \hat{b}(t), \\ \tilde{\theta}(t) &\triangleq \theta(x, z, t) - \hat{\theta}(t), \end{aligned} \quad (35)$$

and substituting the controller (33) into the plant (1), then from (21), (24) and (32), we can get

$$\begin{aligned} a + bu &= -K^T \hat{x} + \tilde{\theta}^T \varphi + r(t) \\ &= -K^T x + \tilde{\theta}^T \varphi + \epsilon K_e^T \xi_e + r(t). \end{aligned}$$

Hence, in the case where  $|\psi(\hat{x}, \hat{a}, \hat{b}, t)| \leq M$ , the closed-loop system (29) under the controller (33) can be simply expressed as

$$\begin{cases} \dot{z} = f_0(x, z, t), \\ \dot{x} = A_m x + B \left( \epsilon K_e^T \xi_e + \tilde{\theta}^T \varphi + r(t) \right), \\ \dot{\xi}_e = \frac{1}{\epsilon} A_e \xi_e + \frac{1}{\epsilon} B_1 \Delta_u(x, z, \xi_e, \hat{a}, \hat{b}, t) \\ \quad + B_2 \eta(x, z, \xi_e, \hat{a}, \hat{b}, t), \end{cases} \quad t \geq t_0 \quad (36)$$

where  $A_m$  and  $K_e$  are given in (4) and (32),  $A_e$  and  $B_i$  ( $i = 1, 2$ ) are given in (26) and (27), and  $\Delta_u(\cdot)$ ,  $\eta(\cdot)$  are functions defined by (28) and (30) respectively.

We remark that the solution of the closed-loop system (29), starting from any initial time  $t_0$  and any initial value  $(x(t_0), z(t_0), \xi_e(t_0))$ , exists in the sense that the solution is absolutely continuous and satisfies Eq. (29) almost everywhere on  $[t_0, +\infty)$ . Thus all derivatives in what follows can be carried out in a classical way. Moreover, the closed-loop system (29) is of the time-scale separation structure, where the  $\xi_e$ -dynamics has the fastest time scale and the adaptive law (16)–(17) has the second fastest one. From the following analysis, it can be seen that such structure plays a very important role in obtaining the effective estimates for uncertain functions  $a(x, z, t)$  and  $b(x, z, t)$ .

### 3.2. Key lemmas

In this subsection, we introduce several key lemmas upon which the stability analysis for the closed-loop system (29) depends. First of all, properties of the  $z$ -dynamics in plant (1) and the projected gradient algorithm (16)–(17) are given in the following lemmas.

**Lemma 3.1.** Consider the internal dynamics of the plant (1)

$$\dot{z} = f_0(x, z, t), \quad t \geq t_0.$$

Suppose  $f_0(\cdot)$  is locally Lipschitz and Assumption (A3) holds, then for any  $\rho_0 > 0$  and  $c > 0$ , if  $\|z(t_0)\| \leq \rho_0$  and  $\|x(t)\| \leq c$  ( $\forall t \geq t_0$ ), we have

$$\|z(t)\| \leq \alpha_1^{-1}(\alpha_2(\rho_*)), \quad \forall t \geq t_0,$$

where  $\rho_* \triangleq \max\{\rho_0, \alpha_0(c)\}$  and  $\alpha_i(\cdot)$  ( $i = 0, 1, 2$ ) are functions given in Assumption (A3).

**Proof.** See Appendix A. ■

**Lemma 3.2.** Consider the estimation law (16)–(17). Let the unknown functions  $a(x, z, t)$  and  $b(x, z, t)$  satisfy Assumptions (A1)–(A2), then for any  $(x, z) \in D = \{(x, z) : \|x\| \leq \rho_r, \|z\| \leq \gamma_z(\rho_r)\}$ , the Lyapunov function

$$V_1(\tilde{\theta}) = \frac{1}{2} \tilde{\theta}^T \tilde{\theta} \quad (37)$$

has the property that

$$\dot{V}_1(\tilde{\theta}) \leq \tilde{\theta}^T (\dot{\tilde{\theta}} - \gamma w \varphi), \quad \forall t \geq t_0. \quad (38)$$

**Proof.** See Appendix B. ■

Next, we show the properties of the trajectory of the closed-loop equation (29). More specifically, Lemma 3.3 demonstrates that the state estimation error  $\|\xi_e\|$  can enter into a small region of the origin, and the boundedness of the state  $(x, z, \xi_e)$  is shown in Lemma 3.4.

**Lemma 3.3.** Consider the dynamic system (25). Let Assumptions (A1)–(A2) hold and the state  $(x, z)$  satisfies

$$\|(x(t), z(t))\| \leq \rho_s, \quad \forall t \in [t_0, t_s] \quad (39)$$

for some constant  $\rho_s > \rho_0$  and finite time  $t_s > t_0$ , then there exists  $\epsilon_1 > 0$  such that for any  $\epsilon \in (0, \epsilon_1)$ , the trajectory of the system (25) has the following property: there exists  $t_c(\epsilon) = t_0 + O(\epsilon \ln \epsilon) < t_s$ , such that for any  $\rho_e \geq 0$  and any initial state estimation error  $\|[e_1(t_0), \dots, e_{n+1}(t_0)]\| \leq \rho_e$ , we have

$$\|\xi_e(t)\| \leq c_e(\rho_s, \rho_e, M) \epsilon^{\frac{1}{3}}, \quad \forall t \in [t_c(\epsilon), t_s], \quad (40)$$

where  $c_e$  is a constant depending on  $\{\rho_s, \rho_e, M\}$ , and  $M$  is defined by (20).

The detailed proof is given in [Appendix C](#). We remark that in the case where a reasonably good prior estimate for the unknown time-varying parameter ( $a(\cdot)$ ,  $b(\cdot)$ ) is available, similar result has been given by [Freidovich and Khalil \(2008\)](#) with a better estimation error upper bound  $O(\epsilon)$ . However, in this paper, since we have introduced an online parameter tracking algorithm and no prior estimate is needed in our controller design, our analyses have to involve the estimation algorithms and the upper bound turns out to be  $O(\epsilon^{\frac{1}{3}})$ . Furthermore, we remark that from the following analysis, it can be seen that the value of  $c_e$  only depends on the parameters  $\{\rho_s, \rho_e\}$ , the reference model parameters  $A_m$  and  $\bar{r}$ , the ESO parameters  $\beta_i$  ( $i = 1, \dots, n + 1$ ), and the bounds  $\tau(\cdot)$ ,  $\underline{b}$ ,  $\alpha_i(\cdot)$  ( $i = 0, 1, 2$ ) given in our assumptions.

**Lemma 3.4.** *Consider the closed-loop system (29) with Assumptions (A1)–(A3) hold. Then for any  $t_s > t_0$ , any  $\rho_s > \rho_0$  and  $c_e > 0$ , there exists  $\epsilon_* > 0$  such that whenever  $\epsilon \in (0, \epsilon_*)$  and the triple  $(x(t_s), z(t_s), \xi_e(t_s))$  belongs to the following compact set*

$$\Omega_0 \triangleq \left\{ (x, z, \xi_e) \mid \|x, z\| \leq \rho_s, \|\xi_e\| \leq c_e \epsilon^{\frac{1}{3}} \right\}, \quad (41)$$

the trajectory of the closed-loop system (29) starting from  $t_s$  can never escape from the following compact set

$$\Omega_r \triangleq \left\{ (x, z, \xi_e) \mid \|x\| \leq \rho_r, \|z\| \leq \gamma_z(\rho_r), \|\xi_e\| \leq \gamma_e(\rho_r) \epsilon^{\frac{1}{3}} \right\}, \quad (42)$$

where  $\rho_r > \rho_s$ ,  $\gamma_z(\rho_r) > \rho_s$  are constants given by (9) and (11), and  $\gamma_e(\rho_r) > c_e$  depends on  $\rho_r$  and  $c_e$ .

The proof is provided in [Appendix D](#). It shows that the trajectory of the closed-loop system (29) starting from the time  $t_s$  and the compact set  $\Omega_0$  is bounded if taking  $\epsilon$  properly. In fact, the value of  $\epsilon$  can be determined by the values  $\rho_s$  and  $c_e$ , the reference model parameters  $A_m$  and  $\bar{r}$ , the ESO parameters  $\beta_i$  ( $i = 1, \dots, n + 1$ ), and the known upper and lower bounds  $\tau(\cdot)$ ,  $\underline{b}$ ,  $\alpha_i(\cdot)$  ( $i = 0, 1, 2$ ) given in Assumptions (A1)–(A3).

### 3.3. Closed-loop performance analysis

Now we analyze the performance of the closed-loop system (29) and show the proof of [Theorem 2.1](#).

**Proof of Theorem 2.1.** We split the proof into two parts. First, we demonstrate the state  $(x, z, \xi_e)$  of the closed-loop system (29) is bounded and give the explicit boundary. Then the control performance is analyzed and the tracking error is given.

**Part I:** First, we assume the initial values<sup>2</sup> satisfy

$$\|(x(t_0), z(t_0))\| \leq \rho_0, \quad \|E_e(t_0)\| \leq \rho_e.$$

Since the trajectory of  $(x, z)$ -system is continuous, for any given  $\rho_s > \rho_0$ , there exists  $t_s > t_0$  such that

$$\|[x(t), z(t)]\| \leq \rho_s, \quad \forall t \in [t_0, t_s]. \quad (43)$$

Hence, by [Lemma 3.3](#) and (C.11), we have that there exists  $\epsilon$  small enough and a time  $t_c(\epsilon) = t_0 + O(\ln \epsilon) < t_s$  such that

$$\|\xi_e(t)\| \leq c_e \gamma \epsilon = c_e \epsilon^{\frac{1}{3}}, \quad \forall t \in [t_c(\epsilon), t_s], \quad (44)$$

where  $c_e$  is a constant depending on the initial values  $\{\rho_0, \rho_e\}$  and the upper bound  $M$  of the control (19). Furthermore, by (43)–(44) and according to [Lemma 3.4](#), we have that there exists  $\epsilon_* > 0$  such that for any  $\epsilon \in (0, \epsilon_*)$ , the trajectory of the closed-loop equation (29) starting from the following compact set

$$\left\{ \|[x(t_s), z(t_s)]\| \leq \rho_s, \|\xi_e(t_s)\| \leq c_e \gamma \epsilon = c_e \epsilon^{\frac{1}{3}} \right\}$$

satisfies that, for any  $t \geq t_s$ ,

$$\|x(t)\| \leq \rho_r, \quad \|z(t)\| \leq \gamma_z(\rho_r), \quad \|\xi_e(t)\| \leq \gamma_e(\rho_r) \epsilon^{\frac{1}{3}}, \quad (45)$$

where  $\rho_r > \rho_s$ ,  $\gamma_z(\rho_r) > \rho_s$  and  $\gamma_e(\rho_r) > c_e$  are constants given by (9), (11) and (D.26) respectively. Hence, the boundedness of the closed-loop system trajectories  $(x, z, \xi_e)$  is proved.

**Part II:** In this part, we calculate the bound of the tracking error  $e^*(t) \triangleq x(t) - x^*(t)$ .

Firstly, in the case where  $x(t_0) \neq x^*(t_0)$ , we only need to analyze the tracking error after the time  $t_c(\epsilon)$  since the time interval  $t_c(\epsilon) - t_0 = O(\ln \epsilon)$  is very short. Actually, from (43)–(45) and according to (20) and (31), it can be verified that there exists  $\epsilon^* \leq \epsilon_*$  such that for any  $\epsilon \in (0, \epsilon^*)$  and for any  $t \geq t_c(\epsilon)$ ,

$$|\psi(\hat{x}, \hat{a}, \hat{b}, t)| < M,$$

that means

$$u = (-K^T \hat{x} - \hat{a} + r(t))/\hat{b},$$

and the corresponding closed-loop system equation is given by (36). Hence by (2) and (36), the tracking error equation is as follows:

$$\dot{e}^* = A_m e^* + B \left( \epsilon K_e^T \xi_e + \tilde{\theta}^T \varphi \right), \quad \forall t \geq t_c(\epsilon), \quad (46)$$

where  $A_m$  is a Hurwitz matrix defined by (4) and  $K_e$  is given by (32). Next, to get an upper bound for  $\|e^*(t)\|$ , we consider the Lyapunov function

$$V_5(e^*) = e^{*T} P_0 e^*,$$

where  $P_0$  is a positive definite matrix given by (5) satisfying  $A_m^T P_0 + P_0 A_m = -I$ . Then the derivative of  $V_5$  along the trajectory of the system (46) is

$$\dot{V}_5(e^*) = -\|e^*\|^2 + 2B^T P_0 \left( \epsilon K_e^T \xi_e + \tilde{\theta}^T \varphi \right) e^*, \quad \forall t \geq t_c(\epsilon).$$

Moreover, from (44) to (45) and by simple calculations, we have  $\forall t \geq t_c(\epsilon)$ ,

$$\begin{aligned} \sqrt{V_5(e^*(t))} &\leq \sqrt{V_5(e^*(t_c))} e^{-\frac{1}{2c_{02}}(t-t_c)} \\ &\quad + \int_{t_c}^t \frac{1}{\sqrt{c_{01}}} \|B^T P_0 (\epsilon K_e^T \xi_e + \tilde{\theta}^T \varphi)\| e^{-\frac{1}{2c_{02}}(t-\tau)} d\tau \\ &\leq \sqrt{V_5(e^*(t_c))} e^{-\frac{1}{2c_{02}}(t-t_c)} \\ &\quad + \int_{t_c}^t \frac{c_{02}}{\sqrt{c_{01}}} |\tilde{\theta}^T \varphi| e^{-\frac{1}{2c_{02}}(t-\tau)} d\tau + O(\gamma \epsilon^2), \end{aligned} \quad (47)$$

where  $c_{01}$  and  $c_{02}$  are constants given by (10).

Now we analyze the property of the integral term in (47). First, from the inequality (D.9), we know

$$\begin{aligned} (\tilde{\theta}^T \varphi)^2 &\leq -\frac{1}{\gamma} \dot{V}_1 + (\tilde{\theta}^T \varphi)(\xi_{n+1} + \Delta_u) + \frac{1}{\gamma} \Delta_0 \\ &\leq -\frac{1}{\gamma} \dot{V}_1 + \frac{1}{2} (\tilde{\theta}^T \varphi)^2 + \xi_{n+1}^2 + \Delta_u^2 + \frac{1}{\gamma} \Delta_0, \end{aligned}$$

<sup>2</sup> The initial value we can design when implementing is  $\hat{x}(t_0)$  rather than  $E_e(t_0)$ . However, the upper bound of  $\|E_e(t_0)\|$  can be easily calculated based on  $x(t_0)$  and  $\hat{x}(t_0)$ , hence we directly use the initial condition  $\|E_e(t_0)\|$  for simplicity.

where  $V_1(\cdot)$ ,  $\Delta_u(\cdot)$  and  $\Delta_0(\cdot)$  are defined by (37), (D.3) and (D.10). Thus, by (43)–(45), (D.3), (D.10) and according to Assumptions (A1)–(A2), we have

$$\begin{aligned} (\tilde{\theta}^T \varphi)^2 &\leq -(2/\gamma)\dot{V}_1 + O(\xi_\epsilon^2) + O(1/\gamma) \\ &= -(2/\gamma)\dot{V}_1 + O((\gamma\epsilon)^2 + 1/\gamma). \end{aligned}$$

Furthermore, according to Hölder inequality, we have for any  $t \geq t_c(\epsilon)$ ,

$$\begin{aligned} &\left( \int_{t_c}^t |\tilde{\theta}^T \varphi| e^{-\frac{1}{2c_{02}}(t-\tau)} d\tau \right)^2 \\ &\leq \int_{t_c}^t \left( e^{-\frac{1}{4c_{02}}(t-\tau)} \right)^2 d\tau \cdot \int_{t_c}^t (|\tilde{\theta}^T \varphi| e^{-\frac{1}{4c_{02}}(t-\tau)})^2 d\tau \\ &\leq 2c_{02} \int_{t_c}^t \left( -\frac{2}{\gamma}\dot{V}_1 + O\left(\frac{1}{\gamma} + (\gamma\epsilon)^2\right) \right) e^{-\frac{1}{2c_{02}}(t-\tau)} d\tau \\ &\leq \left( \frac{4c_{02}}{\gamma} \right) (V_1(t_c) + V_1(t)) + \left( \frac{4c_{02}}{\gamma} \right) \cdot \max_{t_c \leq \tau \leq t} V_1(\tau) \\ &\quad + O\left(\frac{1}{\gamma} + (\gamma\epsilon)^2\right). \end{aligned}$$

Notice  $V_1(t) = \frac{1}{2}\tilde{\theta}^T(t)\tilde{\theta}(t)$ , then by (43), (45), Assumptions (A1)–(A2) and the boundedness of  $\{\hat{a}, \hat{b}\}$ , we have  $V_1(t)$  is bounded for any  $t \geq t_0$ . Hence we further get that for any  $t \geq t_c(\epsilon)$ ,

$$\int_{t_c}^t |\tilde{\theta}^T \varphi| e^{-\frac{1}{2c_{02}}(t-\tau)} d\tau = O(1/\sqrt{\gamma} + \gamma\epsilon). \quad (48)$$

By substituting (48) into (47), we have  $\forall t \geq t_c(\epsilon)$ ,

$$\begin{aligned} \sqrt{V_5(e^*(t))} &= \sqrt{V_5(e^*(t_c))} e^{-\frac{1}{2c_{02}}(t-t_c)} \\ &\quad + O(1/\sqrt{\gamma} + \gamma\epsilon) + O(\gamma\epsilon^2) \\ &= \sqrt{V_5(e^*(t_c))} e^{-\frac{1}{2c_{02}}(t-t_c)} + O(1/\sqrt{\gamma} + \gamma\epsilon). \end{aligned} \quad (49)$$

Moreover, it is obvious that

$$\lim_{t \rightarrow \infty} \sqrt{V_5(e^*(t_c))} e^{-\frac{1}{2c_{02}}(t-t_c)} = 0,$$

which together with (49) derives that

$$\limsup_{t \rightarrow \infty} \|e^*(t)\| = O(1/\sqrt{\gamma} + \gamma\epsilon).$$

Hence, by taking  $\gamma = \epsilon^{-\frac{2}{3}}$ , the order of the ultimate upper bound of  $\|e^*(t)\|$  is minimized, that is,

$$\limsup_{t \rightarrow \infty} \|e^*(t)\| = O\left(\epsilon^{\frac{1}{3}}\right).$$

Thus the tracking performance is proved.

In the following, we calculate the tracking error in the case where  $x(t_0) = x^*(t_0)$ . First of all, from the reference model (2) and  $t_c(\epsilon) = t_0 + O(\epsilon \ln \epsilon)$  given in Lemma 3.3, it is not difficult to see that

$$\|x^*(t) - x^*(t_0)\| = O(\epsilon \ln \epsilon), \quad \forall t \in [t_0, t_c(\epsilon)].$$

Similarly, from (29), (43), (19) and according to Assumption (A1), we get

$$\|x(t) - x(t_0)\| = O(\epsilon \ln \epsilon), \quad \forall t \in [t_0, t_c(\epsilon)].$$

Thus we get that  $\forall t \in [t_0, t_c(\epsilon)]$ ,

$$\|e^*(t)\| \leq \|x^*(t) - x^*(t_0)\| + \|x(t) - x(t_0)\| = O(\epsilon \ln \epsilon).$$

This together with (49) further derives that

$$\|e^*(t)\| = O\left(\epsilon \ln \epsilon + \epsilon^{\frac{1}{3}}\right) = O\left(\epsilon^{\frac{1}{3}}\right), \quad \forall t \geq t_c(\epsilon).$$

Hence we have  $\|e^*(t)\| = O(\epsilon^{\frac{1}{3}})$  holds for any  $t \geq t_0$ . The proof is completed. ■

#### 4. Simulation

We illustrate the performance of the proposed control method by a numerical example. Here we consider the following system presented in Freidovich and Khalil (2008)

$$\dot{\xi}_1 = \xi_2 + p\xi_1^2, \quad \dot{\xi}_2 = bu + \xi_3 + d, \quad \dot{\xi}_3 = -\xi_3 + \xi_1$$

where the unknown parameters  $b$  and  $p$  satisfy  $b \in [0.75, 1.75]$ ,  $p \in [-1, 1]$ , and  $d(t)$  is a bounded disturbance with bounded derivative. The control task is to have  $\xi_1$  asymptotically track a reference signal  $r(t)$ , where  $r(t)$  and its derivatives up to the third one are bounded. Also the transient response of the tracking error  $\xi_1 - r$  is required to follow the target system

$$\dot{x}_1^* = x_2^*, \quad \dot{x}_2^* = -\omega_0^2 x_1^* - 1.4\omega_0 x_2^*$$

where  $\omega_0 > 0$  is the desired natural frequency. By the change of variables  $x_1 = \xi_1 - r$ ,  $x_2 = \xi_2 + p\xi_1^2 - \dot{r}$ ,  $z = \xi_3$ , the system is transformed into the normal form

$$\begin{aligned} \dot{x}_1 &= x_2, & \dot{x}_2 &= a(x, z, t) + bu \\ \dot{z} &= -z + x_1 + r, & y &= x_1 \end{aligned}$$

where  $a(x, z, t) = z + d + 2p(x_1 + r)(x_2 + \dot{r}) - \ddot{r}$ . We assume:  $|r(t)| \leq 1$ ,  $|\dot{r}(t)| \leq 0.1$ ,  $|\ddot{r}(t)| \leq 0.1$ , and  $|d(t)| \leq 1$ . Taking  $\omega_0 = 1$ ,  $\rho_r = 1.2$  and  $\gamma_z(\rho_r) = 2.3 > \rho_r + 1$ , then we have

$$\begin{aligned} |a| &\leq |z| + |d| + 2|p|(|x_1 x_2| + |\dot{r} x_1| + |r x_2| + |r \dot{r}|) + |\ddot{r}| \\ &\leq \gamma_z(\rho_r) + \rho_r^2 + 2.2\rho_r + 1.3 < 8. \end{aligned}$$

Then it can be verified that

$$\max_{\{\|x\| \leq 1.2, |\hat{a}| \leq 8, \hat{b} \geq 0.75\}} \left| \frac{-\omega_0^2 x_1 - 1.4\omega_0 x_2 - \hat{a}}{\hat{b}} \right| < M,$$

where  $M = 22$ . So the control law (19) is given by

$$u(t) = M \operatorname{sat} \left( -\frac{\omega_0^2 \hat{x}_1 + 1.4\omega_0 \hat{x}_2 + \hat{a}}{\hat{b}M} \right), \quad \forall t \geq 0$$

where  $\{\hat{x}_1, \hat{x}_2\}$  and  $\{\hat{a}, \hat{b}\}$  are provided by the ESO and the projected gradient estimator respectively. To be specific, the ESO is given by

$$\dot{\hat{x}}_1 = \hat{x}_2 + \bar{\beta}_1 e_1, \quad \dot{\hat{x}}_2 = \hat{x}_3 + \bar{\beta}_2 e_1, \quad \dot{\hat{x}}_3 = \bar{\beta}_3 e_1$$

where  $e_1 = x_1 - \hat{x}_1$ , and  $\bar{\beta}_i = \frac{\beta_i}{\epsilon^i}$  ( $i = 1, 2, 3$ ). We choose  $\beta_1 = \beta_2 = 3$ ,  $\beta_3 = 1$  to have the Hurwitz polynomial  $(s + 1)^3$ . Let  $w = \hat{x}_3 - \hat{a} - \hat{b}u$ , the projected gradient adaptive law for  $a(x, z, t)$  and  $b$  is given as follows:

$$\begin{aligned} \dot{\hat{a}} &= \begin{cases} 0 & \text{if } (\hat{a} = 8 \text{ and } w \geq 0) \\ & \text{or } (\hat{a} = -8 \text{ and } w \leq 0) \\ \gamma w & \text{otherwise} \end{cases} \\ \dot{\hat{b}} &= \begin{cases} 0 & \text{if } (\hat{b} = 1.75 \text{ and } wu \geq 0) \\ & \text{or } (\hat{b} = 0.75 \text{ and } wu \leq 0) \\ \gamma wu & \text{otherwise.} \end{cases} \end{aligned}$$

We take  $\epsilon = 0.01$  and  $\gamma = 20$ . Figs. 1–2 show the tracking errors, the corresponding tracking error deviations from the target response, as well as the control signals for  $b = p = 1$ ,  $r(t) = 0.1 \sin t$ ,  $d(t) = 1$  and  $\omega_0 = 1$ , respectively corresponding to the adaptive control based on both observers and estimators in this paper and the EGHO based control given in Freidovich and Khalil (2008). It can be seen that the tracking performance of the closed-loop system, as well as the transient response of the tracking error can meet the requirements. Moreover, comparing with the EGHO

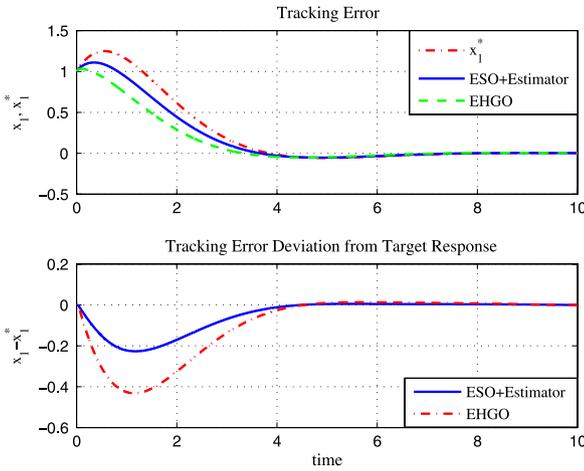


Fig. 1. The tracking error response and the derivation of the tracking error from the target system ( $b = 1$ ).

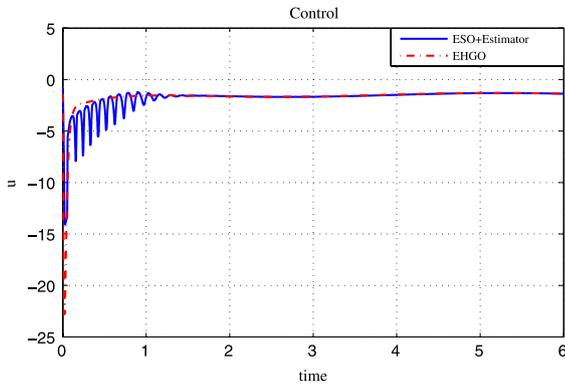


Fig. 2. Control signals ( $b = 1$ ).

based control method, our simulation result shows a noticeable improvement in the transient response.

Furthermore, in order to illustrate the capability of our proposed method in dealing with uncertainties in the input channel, we consider the system with the non-parametric uncertainty  $b(\xi_1, t) = 2 + \sin(10\xi_1^2 t^2)$ , instead of the unknown constant  $b \in [0.75, 1.75]$ . In this case, the tracking performances and the control signals are shown in Figs. 3–4. As expected, such uncertainty can also be dealt with by our adaptive control law, leading to the better transient and steady response than the EGHO based control method. What is more, to illustrate the validity of the ESO and the projected gradient estimator, Figs. 5–6 show the estimation error of total uncertainty  $a(x, z, t) + b(x, t)u - \hat{x}_3$ , the time-varying parameter estimation errors  $\tilde{a} = a(x, z, t) - \hat{a}(t)$ ,  $\tilde{b} = b(x, t) - \hat{b}(t)$ , and the prediction error  $\tilde{a} + \tilde{b}u$  respectively. From Fig. 5, we see that the total uncertainty  $a(x, z, t) + b(x, t)u$  can be well estimated by the ESO. Fig. 6 shows that the prediction error  $\tilde{a} + \tilde{b}u$ , which is exactly the term that effects the tracking error  $\|e^*(t)\|$ , can be arbitrarily close to zero, but the parameter estimation error ( $\tilde{a}$ ,  $\tilde{b}$ ) cannot, and which may even be not small. All of the above is consistent with the theoretical results, illustrating the effect of the proposed control method. Furthermore, we remark that in both cases where we decrease  $\epsilon$  and inject measurement noises, the superiority of our controller can also be observed.

## 5. Conclusion

The main purpose of this paper is to control a class of affine nonlinear uncertain systems with zero dynamics by output

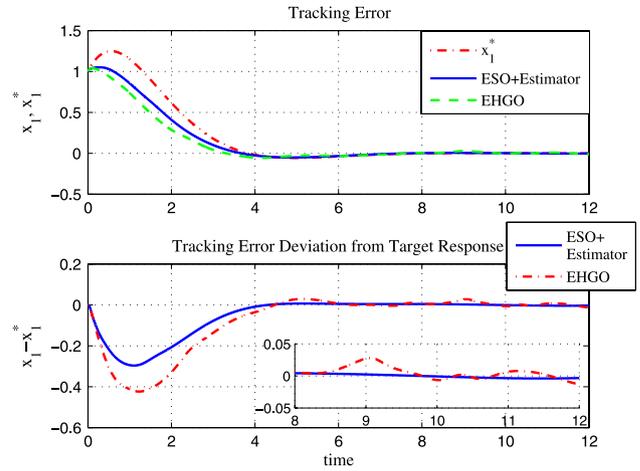


Fig. 3. The tracking error response and the derivation of the tracking error from the target system ( $b = 2 + \sin(10\xi_1^2 t^2)$ ).

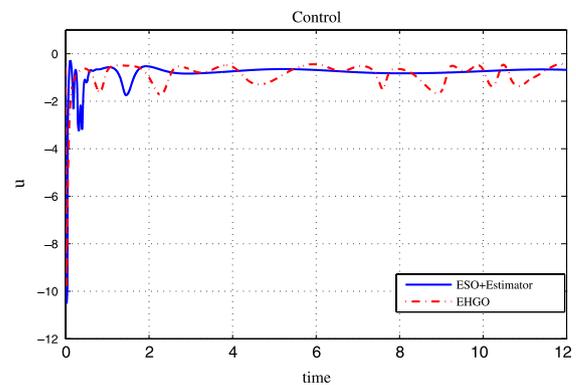


Fig. 4. Control signals ( $b = 2 + \sin(10\xi_1^2 t^2)$ ).

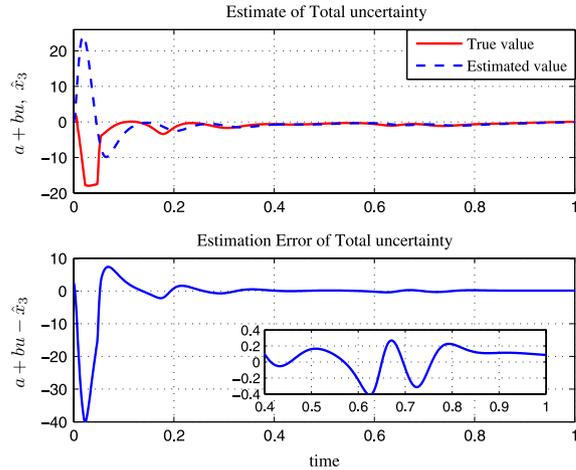


Fig. 5. The total uncertainty estimate and estimation error ( $b = 2 + \sin(10\xi_1^2 t^2)$ ).

feedback when there is no prior information about the uncertain dynamics in the input channel, except for some bounds. This is realized in this paper by firstly estimating the total uncertainties via the extended state observer, and then by estimating the unknown nonparametric dynamics treated as “time-varying parameters” via the projected gradient estimator. To the best of the authors’ knowledge, this work appears to be the first to adaptively estimate the dynamical uncertainties in the input channel for nonlinear uncertain systems with zero dynamics in output feedback control designs. Moreover, the tracking performance

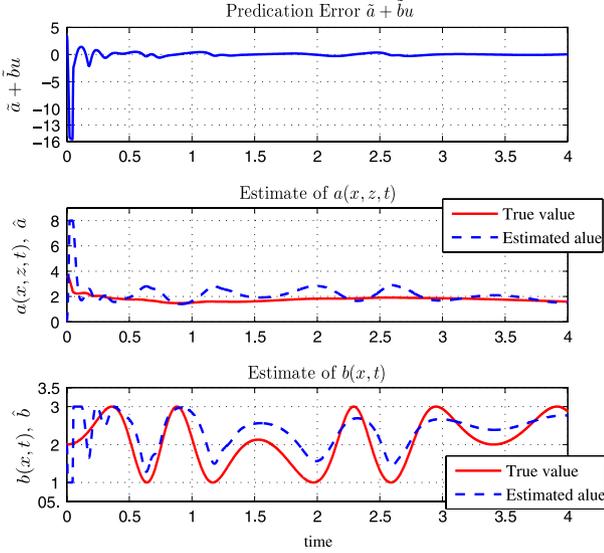


Fig. 6. Parameter estimation errors ( $b = 2 + \sin(10\xi_1^2 t^2)$ ).

is demonstrated via both theoretical analysis and a numerical example. Our simulation also indicates the superiority over the existing related methods at least for the example studied in the paper. Of course, there are still many problems remain to be solved concerning more general nonlinear systems, which belong to further investigation.

### Acknowledgments

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### Appendix A. Proof of Lemma 3.1

Let  $\kappa_1 \triangleq \alpha_2(\rho_*)$ , and consider the time-dependent set  $\Omega_{t, \kappa_1} \triangleq \{z \in \mathbb{R}^m \mid V_0(t, z) \leq \kappa_1\}$ . Then from (7), we have that for all  $t \geq t_0$ ,

$$\{z : \|z\| \leq \alpha_2^{-1}(\kappa_1)\} \subset \Omega_{t, \kappa_1} \subset \{z : \|z\| \leq \alpha_1^{-1}(\kappa_1)\}.$$

Since by (8) and according to the condition  $\|x(t)\| \leq c$  ( $\forall t \geq t_0$ ), we have that  $\dot{V}(t, z) \leq 0$  holds on set  $\{z : \|z\| \geq \alpha_0(c)\}$ . In addition, from the fact  $\kappa_1 \geq \alpha_2(\alpha_0(c))$ , we get the set  $\{z : \|z\| \geq \alpha_0(c)\}$  contains the boundary of  $\Omega_{t, \kappa_1}$  for any  $t \geq t_0$ . This together with the continuity of the internal dynamics solution concludes that: for any  $x_0 \in \Omega_{t_0, \kappa_1}$ , the solution starting at  $(t_0, x_0)$  stays in  $\Omega_{t, \kappa_1}$  for all  $t \geq t_0$ . Therefore, any solution starting in the set  $\{z : \|z\| \leq \alpha_2^{-1}(\kappa_1)\}$  stays in  $\Omega_{t, \kappa_1}$ , and consequently in the set  $\{z : \|z\| \leq \alpha_1^{-1}(\kappa_1)\}$  for all future time. Moreover, since

$$\|z(t_0)\| \leq \rho_0 \Rightarrow \|z(t_0)\| \leq \alpha_2^{-1}(\kappa_1),$$

it must have  $\|z(t)\| \leq \alpha_1^{-1}(\kappa_1) = \alpha_1^{-1}(\alpha_2(\rho_*))$  holds for all  $t \geq t_0$ . Thus, this lemma is true.

### Appendix B. Proof of Lemma 3.2

Since  $\dot{V}_1(\tilde{\theta}) = \tilde{\theta}^T(\dot{\tilde{\theta}} - \hat{\dot{\theta}})$ , what we need to prove is  $\tilde{\theta}^T \gamma w \varphi \leq \tilde{\theta}^T \hat{\dot{\theta}}$ , that is

$$\gamma w \tilde{a} + \gamma w \tilde{b}u \leq \tilde{a} \hat{\dot{a}} + \tilde{b} \hat{\dot{b}}. \quad (\text{B.1})$$

According to Assumptions (A1)–(A2), for any  $(x, z) \in D$ , we have  $|a(x, z, t)| \leq \mu_1(\rho_r)$ ,  $\underline{b} \leq b(x, z, t) \leq \mu_2(\rho_r)$ . This together with the estimation law (16)–(17) guarantees the following four claims are true:

- (1) If  $\hat{\dot{a}} = \gamma w$ , then  $\gamma w \tilde{a} = \tilde{a} \hat{\dot{a}}$ .
- (2) If  $\hat{\dot{a}} = 0$ , then from (16) the inequality  $\gamma w \tilde{a} = \gamma w(a - \hat{a}) \leq 0$  holds.
- (3) If  $\hat{\dot{b}} = \gamma w u$ , then  $\gamma w \tilde{b}u = \tilde{b} \hat{\dot{b}}$ .
- (4) If  $\hat{\dot{b}} = 0$ , then from (17) the inequality  $\gamma w \tilde{b}u = \gamma w u(b - \hat{b}) \leq 0$  holds.

So from (1)–(2) we get that  $\gamma w \tilde{a} \leq \tilde{a} \hat{\dot{a}}$ , and from (3)–(4) we get that  $\gamma w \tilde{b}u \leq \tilde{b} \hat{\dot{b}}$ , which guarantees the inequality (B.1) holds. Thus (38) holds and this lemma is true.

### Appendix C. Proof of Lemma 3.3

First of all, let us consider the following quadratic Lyapunov function

$$V_2(\xi_e) = \xi_e^T P_1 \xi_e, \quad (\text{C.1})$$

where  $P_1$  is a positive definite matrix satisfying  $P_1 A_e + A_e^T P_1 = -I$ , in which  $A_e$  is the Hurwitz matrix defined by (26). Then the derivative of  $V_2(\xi_e)$  along the trajectory of (25) is as follows:

$$\begin{aligned} \dot{V}_2(\xi_e) &= -\frac{1}{\epsilon} \|\xi_e\|^2 + \frac{2}{\epsilon} B_1^T P_1 \xi_e \Delta_u(x, z, \xi_e, \hat{a}, \hat{b}, t) \\ &\quad + 2B_2^T P_1 \xi_e \left( df_e(x, z, \hat{a}, \hat{b}, t)/dt \right) \\ &= -\frac{1}{\epsilon} \|\xi_e\|^2 + \frac{2}{\epsilon} B_1^T P_1 \xi_e \Delta_u(x, z, \xi_e, \hat{a}, \hat{b}, t) \\ &\quad + 2B_2^T P_1 \xi_e \eta(x, z, \xi_e, \hat{a}, \hat{b}, t), \end{aligned} \quad (\text{C.2})$$

where  $\Delta_u(\cdot)$  and  $\eta(\cdot)$  are functions given by (28) and (30). Hence, in order to analyze the boundedness of  $\xi_e$ , we need to estimate the upper bounds for  $\Delta_u(\cdot)$  and  $\eta(\cdot)$ . Now, by (28), (31) and the properties of  $g_e(\cdot)$ , we have

$$\begin{aligned} |\Delta_u| &\leq |b|M \left| \text{sat} \left( \psi(\hat{x}, \hat{a}, \hat{b}, t)/M \right) - g_e \left( \psi(\hat{x}, \hat{a}, \hat{b}, t)/M \right) \right| \\ &\quad + |b|M \left| g_e \left( \psi(\hat{x}, \hat{a}, \hat{b}, t)/M \right) - g_e \left( \psi(x, \hat{a}, \hat{b}, t)/M \right) \right| \\ &\leq \epsilon |b|M \left( 1/2 + |K_e^T \xi_e| / (|\hat{b}|M) \right). \end{aligned}$$

So under the condition (39) and according to Assumptions (A1)–(A2), we get  $\forall t \in [t_0, t_s]$ ,

$$|\Delta_u| \leq \epsilon \tau(\rho_s) \left( M/2 + (1/\underline{b}) |K_e^T \xi_e| \right), \quad (\text{C.3})$$

where  $\tau(\cdot)$  and  $\underline{b}$  are given in Assumption (A1) and (A2) respectively. Furthermore, we proceed to get an upper bound for  $|\eta(\cdot)|$ . To achieve this, we first need to derive upper bounds for  $|\hat{a}|$  and  $|\hat{b}|$ , which depend on the prediction error  $w$  given by (15). In fact, by (15), (22), (24) and (28), we know

$$\begin{aligned} w &= \hat{x}_{n+1} - \hat{a} - \hat{b}u \\ &= \left[ a + bu - (\hat{a} + \hat{b}u) \right] - \left[ f_e(x, z, \hat{a}, \hat{b}, t) - \hat{x}_{n+1} \right] \\ &\quad - \left[ (a + bu) - f_e(x, z, \hat{a}, \hat{b}, t) \right] \\ &= \tilde{\theta}^T \varphi - \xi_{n+1} - \Delta_u(x, z, \xi_e, \hat{a}, \hat{b}, t), \end{aligned} \quad (\text{C.4})$$

where  $\tilde{\theta}$ ,  $\varphi$  are vectors defined by (35) and (34) respectively. Hence, by (19), (39), Assumption (A1), the boundedness of  $\{\hat{a}, \hat{b}\}$

and (C.3), we know that there exist finite monotonic increasing functions  $\delta_{0i}(\cdot) : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$  ( $i = 1, 2$ ) such that for any  $t \in [t_0, t_s]$ ,

$$|w| \leq \delta_{01}(\rho_s, M) + \delta_{02}(\rho_s, M) \|\xi_e\|. \quad (\text{C.5})$$

By substituting (C.5) into (16) and (17), we further have for any  $t \in [t_0, t_s]$ ,

$$\begin{cases} |\dot{a}| \leq \gamma [\delta_{01}(\rho_s, M) + \delta_{02}(\rho_s, M) \|\xi_e\|], \\ |\dot{b}| \leq \gamma M [\delta_{01}(\rho_s, M) + \delta_{02}(\rho_s, M) \|\xi_e\|]. \end{cases} \quad (\text{C.6})$$

As a result, by (30) and combining (18), (19), (C.6), (39), Assumptions (A1)–(A2) and the properties of  $g_e(\cdot)$ , we get that there exist finite monotonic increasing functions  $\delta_{1i}(\cdot) : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$  ( $i = 1, 2, 3$ ) such that for any  $t \in [t_0, t_s]$ ,

$$|\eta| \leq \delta_{11}(\rho_s, M) + \gamma \delta_{12}(\rho_s, M) + \gamma \delta_{13}(\rho_s, M) \|\xi_e\|. \quad (\text{C.7})$$

Thus, by substituting (C.3) and (C.7) into (C.2), and when  $\gamma > 1$  and satisfies  $\gamma\epsilon = o(1)$  as  $\epsilon \rightarrow 0$ , we have there exists  $0 < \epsilon_0 < 1$  such that for any  $0 < \epsilon < \epsilon_0$ , and for any  $t \in [t_0, t_s]$ ,

$$\begin{aligned} \dot{V}_2(\xi_e) &\leq -\frac{1}{\epsilon} \|\xi_e\|^2 + \frac{2\|P_1\|}{\epsilon} \left( \frac{\epsilon}{b} \|K_e\| \tau(\rho_s) + \gamma \epsilon \delta_{13} \right) \|\xi_e\|^2 \\ &\quad + 2\|P_1\| \left( \frac{1}{2} \tau(\rho_s) M + \delta_{11} + \gamma \delta_{12} \right) \|\xi_e\|, \\ &\leq -\frac{1}{2\epsilon} \|\xi_e\|^2 + \delta_*(\rho_s, M, \gamma) \|\xi_e\|, \end{aligned}$$

where

$$\delta_*(\rho_s, M, \gamma) \triangleq 2\|P_1\| (\tau(\rho_s)M/2 + \delta_{11} + \gamma \delta_{12}). \quad (\text{C.8})$$

By simple calculations, we have

$$\frac{d}{dt} \sqrt{V_2(\xi_e)} \leq -\frac{\sqrt{V_2(\xi_e)}}{4c_{12}\epsilon} + \frac{\delta_*(\rho_s, M, \gamma)}{2\sqrt{c_{11}}},$$

where  $c_{11} \triangleq \lambda_{\min}(P_1)$  and  $c_{12} \triangleq \lambda_{\max}(P_1)$  are the minimum and maximum eigenvalues of the positive definite matrix  $P_1$  respectively. Then according to the Comparison Principle (Khalil, 2002), we get for any  $t \in [t_0, t_s]$ ,

$$\sqrt{V_2(\xi_e(t))} \leq \sqrt{V_2(\xi_e(t_0))} e^{-\frac{t-t_0}{4c_{12}\epsilon}} + \frac{2c_{12}\delta_*(\rho_s, M, \gamma)}{\sqrt{c_{11}}} \epsilon. \quad (\text{C.9})$$

Since  $\xi_e(t_0) = T^{-1}(\epsilon)E_e(t_0)$  and  $\|E_e(t_0)\| \leq \rho_e$ , the second term of (C.9) satisfies

$$\sqrt{V_2(\xi_e(t_0))} e^{-\frac{t-t_0}{4c_{12}\epsilon}} \leq \sqrt{c_{12}} \|\epsilon^n T^{-1}(\epsilon)\| \frac{\rho_e}{\epsilon^n} e^{-\frac{t-t_0}{4c_{12}\epsilon}}$$

where  $\epsilon^n T^{-1}(\epsilon)$  is analytical with respect to  $\epsilon$ . By taking  $t_c(\epsilon) \triangleq t_0 - 4(n+1)c_{12}\epsilon \ln \epsilon$ , it is not difficult to see for any  $t \geq t_c(\epsilon)$ ,

$$\begin{aligned} \sqrt{V_2(\xi_e(t_0))} e^{-\frac{t-t_0}{4c_{12}\epsilon}} &\leq \sqrt{c_{12}} \|\epsilon^n T^{-1}(\epsilon)\| \rho_e \epsilon \\ &\leq \sqrt{c_{12}} \rho_e \epsilon. \end{aligned} \quad (\text{C.10})$$

Note that  $\lim_{\epsilon \rightarrow 0} t_c(\epsilon) = t_0$ , so there exists  $\epsilon_1 \in (0, \epsilon_0]$  such that for any  $\epsilon \in (0, \epsilon_1)$ ,  $t_c(\epsilon) < t_s$  holds. Therefore, from (C.8)–(C.10), we have  $\forall t \in [t_c(\epsilon), t_s]$ ,

$$\begin{aligned} \|\xi_e(t)\| &\leq \zeta_{01}(\rho_s, M) \gamma \epsilon + \zeta_{02}(\rho_s, \rho_e, M) \epsilon \\ &\leq c_e(\rho_s, \rho_e, M) \gamma \epsilon, \end{aligned} \quad (\text{C.11})$$

where

$$\begin{aligned} \zeta_{01}(\rho_s, \rho_e, M) &= 4c_{12}^2 \delta_{12} / c_{11}, \\ \zeta_{02}(\rho_s, \rho_e, M) &= \frac{4c_{12}^2}{c_{11}} (\tau(\rho_s)M/2 + \delta_{11}) + \sqrt{\frac{c_{12}}{c_{11}}} \rho_e, \end{aligned}$$

and

$$c_e(\rho_s, \rho_e, M) = \zeta_{01} + \zeta_{02}. \quad (\text{C.12})$$

Noticing that  $\gamma = \epsilon^{-\frac{2}{3}}$ , hence from (C.11) we have

$$\|\xi_e(t)\| \leq c_e \epsilon^{\frac{1}{3}}. \quad (\text{C.13})$$

To sum up, by (C.12) and (C.13), we know (40) holds, thus this lemma is true.

#### Appendix D. Proof of Lemma 3.4

We adopt the contradiction argument. Suppose this lemma is not true, then there must exist a time  $t_e > t_s$  such that

$$(x(t_e), z(t_e), \xi_e(t_e)) \notin \Omega_r.$$

Since the triple  $(x(t_s), z(t_s), \xi_e(t_s)) \in \Omega_r$ , and the solution of (29) is continuous, there exists at least one time when the trajectory is reaching the boundary of  $\Omega_r$ . Let  $t_r$  be the first time the trajectory reaching the boundary of  $\Omega_r$ . So when  $t \in [t_s, t_r]$ , we have

$$\|x(t)\| \leq \rho_r, \|z(t)\| \leq \gamma_z(\rho_r), \|\xi_e(t)\| \leq \gamma_e(\rho_r) \epsilon^{\frac{1}{3}}, \quad (\text{D.1})$$

and at time  $t_r$ , there are only three cases given below:

$$\begin{aligned} &\|x(t_r)\| = \rho_r, \\ &\text{or } \|z(t_r)\| = \gamma_z(\rho_r), \\ &\text{or } \|\xi_e(t_r)\| = \gamma_e(\rho_r) \epsilon^{\frac{1}{3}}. \end{aligned} \quad (\text{D.2})$$

Moreover, in all above three cases, by (18), (31) and the boundedness of  $\xi_e$ , it can be easily verified that: there exists  $0 < \epsilon_2 < 1$  such that for any  $\epsilon \in (0, \epsilon_2)$  and for any  $t \in [t_s, t_r]$ , we have

$$|\psi(\hat{x}, \hat{a}, \hat{b}, t)| < M \quad \text{and} \quad |\psi(x, \hat{a}, \hat{b}, t)| < M,$$

where  $M$  is given by (20). Hence, by (19) and (23), we have for any  $t \in [t_s, t_r]$ ,

$$u(t) = (-K^T \hat{x} - \hat{a} + r(t)) / \hat{b},$$

and

$$Mg_e(\psi(x, \hat{a}, \hat{b}, t) / M) = (-K^T x - \hat{a} + r(t)) / \hat{b}.$$

Hence the corresponding closed-loop equation satisfies Eq. (36), in which the functions  $\Delta_u(\cdot)$  and  $\eta(\cdot)$  given by (28) and (30) can be simplified as

$$\Delta_u(x, z, \xi_e, \hat{a}, \hat{b}, t) = \epsilon K_e^T \xi_e b(x, z, t) / \hat{b}, \quad (\text{D.3})$$

and

$$\begin{aligned} \eta(x, z, \xi_e, \hat{a}, \hat{b}, t) &= \frac{\partial a}{\partial t} + \left( \frac{\partial a}{\partial z} + \frac{\partial b}{\partial z} \psi(x, \hat{a}, \hat{b}, t) \right) f_0 \\ &\quad + \frac{\partial b}{\partial t} \psi(x, \hat{a}, \hat{b}, t) + \left( \frac{\partial a}{\partial x} + \frac{\partial b}{\partial x} \psi(x, \hat{a}, \hat{b}, t) \right) \left[ A_m x \right. \\ &\quad \left. + B \left( \epsilon K_e^T \xi_e + \tilde{\theta}^T \varphi + r(t) \right) \right] + \frac{b}{\hat{b}} \left[ -\dot{\hat{b}} \psi(x, \hat{a}, \hat{b}, t) \right. \\ &\quad \left. - K^T A_m x - k_n \left( \epsilon K_e^T \xi_e + \tilde{\theta}^T \varphi + r(t) \right) - \dot{\hat{a}} + \dot{r} \right]. \end{aligned} \quad (\text{D.4})$$

Now we demonstrate the time  $t_r$  given in (D.2) does not exist, which means the trajectory of the closed-loop system will never reach the boundary of  $\Omega_r$ . Actually, to achieve this, we need to calculate the upper bounds of  $\|x(t_r)\|$ ,  $\|z(t_r)\|$  and  $\|\xi_e(t_r)\|$ , which will be carried out by analyzing the three cases in (D.2) one by one.

**Case (1)**  $\|x(t_r)\| = \rho_r$ ;

We will show if taking the parameters  $\epsilon$  and  $\gamma$  properly,

$$\|x(t_r)\| < \rho_r = 2 \left( \sqrt{\frac{c_{02}}{c_{01}}} \rho_s + \frac{2c_{02}\sqrt{2c_{02}}\bar{r}}{\sqrt{c_{01}}} \right), \quad (\text{D.5})$$

where  $\rho_s$  and  $\bar{r}$  are constants given by (41) and (3) respectively,  $c_{01}$  and  $c_{02}$  are given by (10).

First of all, we consider the following quadratic Lyapunov function

$$V_3(x) = x^T P_0 x, \quad (\text{D.6})$$

where  $P_0$  is a positive definite matrix defined by (5), then the derivative of  $V_3(x)$  along the trajectory of closed-loop equation (36) is

$$\dot{V}_3(x) = -\|x\|^2 + 2(\epsilon K_e^T \xi_e + \tilde{\theta}^T \varphi + r(t)) B^T P_0 x. \quad (\text{D.7})$$

Thus, in order to get an upper bound for  $\|x(t)\|$ , we need to consider simultaneously the properties of the state  $x(t)$  and the parameters estimation error  $\tilde{\theta}$ . We now introduce the following Lyapunov function

$$V_4(\tilde{\theta}, x) = \frac{1}{k} V_1(\tilde{\theta}) + V_3(x) = \frac{1}{2k} \tilde{\theta}^T \tilde{\theta} + x^T P_0 x, \quad (\text{D.8})$$

where  $k$  is a constant satisfying  $k = \frac{\gamma}{8\|P_0\|^2}$ , and  $V_1(\cdot)$ ,  $V_3(\cdot)$  are Lyapunov functions defined by (37) and (D.6) respectively. Next, we calculate the derivative of  $V_4(\tilde{\theta}, x)$  along the trajectory of (36). First, according to Lemma 3.2, substituting (C.4) into (38), and by (34), we know  $\forall t \in [t_s, t_r]$ ,

$$\begin{aligned} \dot{V}_1 &\leq -\gamma w \tilde{\theta}^T \varphi + \tilde{\theta}^T \dot{\tilde{\theta}} \\ &= -\gamma (\tilde{\theta}^T \varphi)^2 + \gamma (\tilde{\theta}^T \varphi) \xi_{n+1} + \gamma (\tilde{\theta}^T \varphi) \Delta_u \\ &\quad + \Delta_0(x, z, \xi_e, t) \end{aligned} \quad (\text{D.9})$$

where  $\Delta_u(\cdot)$  is given by (D.3) and

$$\begin{aligned} \Delta_0(x, z, \xi_e, t) &\triangleq \tilde{\theta}^T \left[ \frac{\partial \theta}{\partial t} + \frac{\partial \theta}{\partial z^T} f_0 \right] + \tilde{\theta}^T \frac{\partial \theta}{\partial x^T} \\ &\quad \times \left[ A_m x + B \left( \epsilon K_e^T \xi_e + \tilde{\theta}^T \varphi + r(t) \right) \right], \end{aligned} \quad (\text{D.10})$$

in which  $\partial \theta / \partial x^T = [\partial \theta / \partial x_1 \ \dots \ \partial \theta / \partial x_n] \in \mathbb{R}^{2 \times n}$ , and  $\partial \theta / \partial z^T = [\partial \theta / \partial z_1 \ \dots \ \partial \theta / \partial z_m] \in \mathbb{R}^{2 \times m}$ . Moreover, from (D.8) we know

$$\|x\|^2 \geq \frac{1}{\|P_0\|} \left( V_4 - \frac{\tilde{\theta}^T \tilde{\theta}}{2k} \right). \quad (\text{D.11})$$

Substituting (D.11) into (D.7) and by (D.8), (D.9) and  $k = \frac{\gamma}{8\|P_0\|^2}$ , we have  $\forall t \in [t_s, t_r]$ ,

$$\begin{aligned} \dot{V}_4 &\leq -\frac{1}{2\|P_0\|} V_4 - \frac{1}{2} \|x\|^2 + 2\|P_0\| \left( |\tilde{\theta}^T \varphi| + |r(t)| \right) \|x\| \\ &\quad - \frac{\gamma}{k} (\tilde{\theta}^T \varphi)^2 + \frac{\gamma}{k} |\tilde{\theta}^T \varphi| (|\xi_{n+1}| + |\Delta_u|) + \frac{\tilde{\theta}^T \tilde{\theta}}{4k\|P_0\|} \\ &\quad + 2\epsilon \|P_0\| \cdot |K_e^T \xi_e| \cdot \|x\| + \frac{1}{k} |\Delta_0(x, z, \xi_e, t)| \\ &\leq -\frac{1}{2\|P_0\|} V_4 + 4\|P_0\|^2 \bar{r}^2 + \frac{\gamma}{k} \xi_{n+1}^2 + \frac{\gamma}{k} \Delta_u^2 \\ &\quad + 2\epsilon \|P_0\| \cdot |K_e^T \xi_e| \cdot \|x\| + \frac{1}{k} \Delta_1(x, z, \xi_e, t), \end{aligned} \quad (\text{D.12})$$

where

$$\begin{aligned} \Delta_1(x, z, \xi_e, t) &\triangleq \frac{\tilde{\theta}^T \tilde{\theta}}{4\|P_0\|} + |\Delta_0(x, z, \xi_e, t)| \\ &\leq \frac{\tilde{\theta}^T \tilde{\theta}}{4\|P_0\|} + \left| \tilde{\theta}^T \frac{\partial \theta}{\partial z^T} f_0 \right| + \left| \tilde{\theta}^T \frac{\partial \theta}{\partial t} \right| + \left| \tilde{\theta}^T \frac{\partial \theta}{\partial x^T} A_m x \right| \\ &\quad + \left| \tilde{\theta}^T \frac{\partial \theta}{\partial x^T} B \right| \left( \epsilon |K_e^T \xi_e| + |\tilde{\theta}^T \varphi| + \bar{r} \right). \end{aligned} \quad (\text{D.13})$$

Now we calculate an upper bound of  $\|x(t_r)\|$ , for this we need to analyze upper bounds of the last four terms in (D.12). Actually, by (D.1), (D.3) and according to Assumptions (A1)–(A2), we know for any  $t \in [t_s, t_r]$ ,

$$\begin{aligned} \frac{\gamma}{k} (\xi_{n+1}^2 + \Delta_u^2) + 2\epsilon \|P_0\| \cdot |K_e^T \xi_e| \cdot \|x\| \\ \leq 8 \left[ 1 + (\epsilon \mu_2(\rho_r) \|K_e\| / \underline{b})^2 \right] \|P_0\|^2 \gamma_e^2(\rho_r) \epsilon^{\frac{2}{3}} \\ + 2\|P_0\| \rho_r \|K_e\| \gamma_e(\rho_r) \epsilon^{\frac{4}{3}} \\ \leq \delta_{21}(\rho_r) \epsilon^{\frac{2}{3}}, \end{aligned} \quad (\text{D.14})$$

where  $\delta_{21}(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a monotonic increasing function defined by

$$\begin{aligned} \delta_{21}(\rho_r) &\triangleq \max_{\epsilon \in (0, \epsilon_2]} \left\{ 8 \left[ 1 + (\epsilon \mu_2(\rho_r) \|K_e\| / \underline{b})^2 \right] \|P_0\|^2 \gamma_e^2(\rho_r) \right. \\ &\quad \left. + 2\|P_0\| \rho_r \|K_e\| \gamma_e(\rho_r) \epsilon^{\frac{2}{3}} \right\}. \end{aligned}$$

Furthermore, by (D.13), (D.1) and Assumptions (A1)–(A2), it is not difficult to see that there exists a monotonic increasing function  $\delta_{22}(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that

$$\Delta_1(x, z, \xi_e, t) \leq \delta_{22}(\rho_r), \quad \forall t \in [t_s, t_r]. \quad (\text{D.15})$$

Consequently, by combining (D.12), (D.14) and (D.15), and according to the Comparison Principle, we have

$$\begin{aligned} V_4(t_r) &\leq V_4(t_s) + \int_{t_s}^{t_r} 2\epsilon \|P_0\| |K_e^T \xi_e| \|x\| e^{-\frac{1}{2\|P_0\|}(t_r-\tau)} d\tau \\ &\quad + \int_{t_s}^{t_r} \frac{\gamma}{k} (\xi_{n+1}^2 + \Delta_u^2) e^{-\frac{1}{2\|P_0\|}(t_r-\tau)} d\tau \\ &\quad + \int_{t_s}^{t_r} \left( \frac{1}{k} \Delta_1 + 4\|P_0\|^2 \bar{r}^2 \right) e^{-\frac{1}{2\|P_0\|}(t_r-\tau)} d\tau \\ &\leq V_4(t_s) + 2\|P_0\| \delta_{21}(\rho_r) \epsilon^{\frac{2}{3}} + \frac{2}{k} \|P_0\| \delta_{22}(\rho_r) + 8\|P_0\|^3 \bar{r}^2 \\ &= V_4(t_s) + 8\|P_0\|^3 \bar{r}^2 + O\left(\epsilon^{\frac{2}{3}}\right). \end{aligned} \quad (\text{D.16})$$

Moreover, by (41), (D.8), Assumption (A1) and the boundedness of  $\hat{\theta}$ , we have

$$\begin{aligned} V_4(t_s) &\leq \frac{2}{k} (\mu_1^2(\rho_s) + \mu_2^2(\rho_s)) + \|P_0\| \rho_s^2 \\ &= \|P_0\| \rho_s^2 + O\left(\epsilon^{\frac{2}{3}}\right), \end{aligned} \quad (\text{D.17})$$

where  $\mu_i(\cdot)$  ( $i = 1, 2$ ) are given in (16)–(17). Hence, from (D.8) and (D.16)–(D.17) we get

$$\|x(t_r)\| \leq \frac{\sqrt{c_{02}}}{\sqrt{c_{01}}} \rho_s + \frac{2c_{02}\sqrt{2c_{02}}\bar{r}}{\sqrt{c_{01}}} \bar{r} + O\left(\epsilon^{\frac{1}{3}}\right), \quad (\text{D.18})$$

where  $c_{01}$  and  $c_{02}$  are defined by (10) and  $O$  is a constant equal to  $\sqrt{(2c_{02}/c_{01})(\delta_{21}(\rho_r) + 8c_{02}^2\delta_{22}(\rho_r)) + 4c_{02}\sqrt{(\mu_1^2(\rho_s) + \mu_2^2(\rho_s))/c_{01}}}$ . Thus the last term of (D.18) can be arbitrarily small by taking  $\epsilon$  small enough. Therefore, there exists  $0 < \epsilon_3 \leq \epsilon_2$  such that for

any  $\epsilon \in (0, \epsilon_3)$ ,

$$\|x(t_r)\| < 2 \left( \frac{\sqrt{c_{02}}}{\sqrt{c_{01}}} \rho_s + \frac{2c_{02}\sqrt{2c_{02}}}{\sqrt{c_{01}}} \bar{r} \right) = \rho_r, \quad (\text{D.19})$$

which contradicts the claim  $\|x(t_r)\| = \rho_r$ .

**Case (2)**  $\|z(t_r)\| = \gamma_z(\rho_r)$ ;

First of all, by (D.1) and (41), we know that  $\|z(t_s)\| \leq \rho_s$  and the state  $\|x(t)\| \leq \rho_r$  holds for any  $t \in [t_s, t_r]$ . Hence, according to Lemma 3.1, we have that for any  $t \in [t_s, t_r]$ ,

$$\|z(t)\| \leq \kappa(\rho_r), \quad (\text{D.20})$$

where  $\kappa(\rho_r) = \alpha_1^{-1}(\alpha_2(\bar{\rho}))$  and  $\bar{\rho} \triangleq \max\{\alpha_0(\rho_r), \rho_s\}$ . Furthermore, let

$$\gamma_z(\rho_r) > \kappa(\rho_r), \quad (\text{D.21})$$

and we get  $\|z(t_r)\| < \gamma_z(\rho_r)$ , which contradicts the claim  $\|z(t_r)\| = \gamma_z(\rho_r)$ .

**Case (3)**  $\|\xi_e(t_r)\| = \gamma_e(\rho_r)\epsilon^{\frac{1}{3}}$ ;

Consider the Lyapunov function  $V_2(\xi_e) = \xi_e^T P_1 \xi_e$  defined by (C.1), then the derivative of  $V_2(\xi_e)$  along the trajectory of (36) is as follows:

$$\begin{aligned} \dot{V}_2(\xi_e) = & -\frac{1}{\epsilon} \|\xi_e\|^2 + \frac{2}{\epsilon} B_1^T P_1 \xi_e \Delta_u(x, z, \xi_e, \hat{a}, \hat{b}, t) \\ & + 2B_2^T P_1 \xi_e \eta(x, z, \xi_e, \hat{a}, \hat{b}, t), \end{aligned} \quad (\text{D.22})$$

where  $\Delta_u(\cdot)$  and  $\eta(\cdot)$  are given in (D.3) and (D.4). Next, we calculate an upper bound of  $\|\xi_e(t_r)\|$  by utilizing the similar analysis given in Appendix C. First, from (D.1), (D.3)–(D.4), and according to Assumptions (A1)–(A2), it can be verified that by the analysis similar to (C.3)–(C.7), there exist finite monotonic increasing functions  $\delta_{3i}(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  ( $i = 1, 2, 3$ ), such that

$$\left| \Delta_u(x, z, \xi_e, \hat{a}, \hat{b}, t) \right| \leq \epsilon \mu_2(\rho_r) |K_e^T \xi_e| / \underline{b}, \quad (\text{D.23})$$

$$\left| \eta(x, z, \xi_e, \hat{a}, \hat{b}, t) \right| \leq \gamma \delta_{31}(\rho_r) \|\xi_e\| + \gamma \delta_{32}(\rho_r) + \delta_{33}(\rho_r). \quad (\text{D.24})$$

Substituting (D.23) and (D.24) into (D.22) and when  $\gamma \epsilon = o(1)$  as  $\epsilon \rightarrow 0$ , we know that there exists  $0 < \epsilon_4 \leq \epsilon_3$  such that for any  $\epsilon \in (0, \epsilon_4)$ ,

$$\dot{V}_2(\xi_e) \leq -\frac{1}{2\epsilon} \|\xi_e\|^2 + \zeta_1(\rho_r, \gamma) \|\xi_e\|, \quad \forall t \in [t_s, t_r]$$

where  $\zeta_1(\rho_r, \gamma) \triangleq 2c_{12}(\gamma \delta_{32}(\rho_r) + \delta_{33}(\rho_r))$ . Furthermore, by simple calculations, we have  $\forall t \in [t_s, t_r]$ ,

$$\sqrt{V_2(\xi_e(t))} \leq \sqrt{V_2(\xi_e(t_s))} e^{-\frac{t-t_s}{4c_{12}\epsilon}} + \frac{2c_{12}\zeta_1(\rho_r, \gamma)}{\sqrt{c_{11}}} \epsilon.$$

Thus, by (41) and  $\gamma = \epsilon^{-\frac{2}{3}}$ , we have

$$\begin{aligned} \|\xi_e(t_r)\| & \leq \sqrt{\frac{c_{12}}{c_{11}}} \|\xi_e(t_s)\| + \frac{2c_{12}\zeta_1(\rho_r, \gamma)}{c_{11}} \epsilon \\ & \leq \left[ \sqrt{\frac{c_{12}}{c_{11}}} c_e + \frac{4c_{12}^2 \delta_{32}(\rho_r)}{c_{11}} \right] \epsilon^{\frac{1}{3}} + O(\epsilon). \\ & < \gamma_e(\rho_r) \epsilon^{\frac{1}{3}}, \end{aligned} \quad (\text{D.25})$$

where

$$\gamma_e(\rho_r) \triangleq 2 \left[ \sqrt{\frac{c_{12}}{c_{11}}} c_e + \frac{4c_{12}^2 \delta_{32}(\rho_r)}{c_{11}} \right], \quad (\text{D.26})$$

and with some small  $\epsilon \in (0, \epsilon_*)$  satisfying  $\epsilon_* \in (0, \epsilon_4)$ . Thus we get a contradiction with  $\|\xi_e(t_r)\| = \gamma_e(\rho_r)\epsilon^{\frac{1}{3}}$ .

To sum up, by (D.19), (D.20)–(D.21) and (D.25), we know that if the parameter  $\epsilon$  is small enough, the trajectory of the closed-loop system (36) cannot reach the boundary of  $\Omega_r$ , defined by (42). More specifically, for any triple  $(x(t_s), z(t_s), \xi_e(t_s))$  belongs to the compact set  $\Omega_0$  given by (41), there exists  $\epsilon_* > 0$  such that for any

$\epsilon \in (0, \epsilon_*)$ , we have  $\forall t \geq t_s$ ,

$$\|x(t)\| \leq \rho_r, \quad \|z(t)\| \leq \gamma_z(\rho_r), \quad \|\xi_e(t)\| \leq \gamma_e(\rho_r)\epsilon^{\frac{1}{3}},$$

where  $\rho_r$ ,  $\gamma_z(\rho_r)$  and  $\gamma_e(\rho_r)$  are given by (D.5), (D.21), (D.26) respectively. Thus, this lemma is true.

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