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Controllability of Nash Equilibrium in Game-Based Control Systems

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***Abstract*—Controlling complex systems to desired states is of primary importance in science and engineering. In the classical control framework, the plants to be controlled usually do not have their own payoff or objective functions, however, this is not the case in many practical situations in, for examples, social, economic, and "intelligent" engineering systems. This motivates our introduction of the game-based control system (GBCS), which has a hierarchical decision-making structure: one regulator and multiple agents. The regulator is regarded as the global controller that makes decision first, and then the agents try to optimize their respective objective functions to reach a possible Nash equilibrium as a result of non-cooperative dynamic game. A fundamental issue in GBCS is: Is it possible for the regulator to change the macro-states by regulating the Nash equilibrium formed by the agents at the lower level? This leads to the investigation of controllability of Nash equilibrium of GBCS. In this paper, we will first formulate this new problem in general nonlinear framework, and then focus on linear systems. Some explicit necessary and sufficient algebraic conditions on the controllability of Nash equilibrium are given for linear GBCS, by solving the controllability problem of the associated forward and backward dynamic equations, which is a key technical issue and has rarely been explored in the literature.**

***Index Terms*—Non-cooperative differential games, hierarchical structure, Nash equilibrium, controllability, maximum principle.**

I. Introduction

In the traditional control theoretical framework, the plants to be controlled are usually modelled by physical laws, which do not have their own payoff or objective functions, such as the control of a car, an air plane, and an industrial process, etc. However, this may well not be the situation in many practical systems such as social, economic and the now rapidly developing "intelligent" engineering systems [1]-[4]. The common characteristic of these systems is that the objects to be controlled involve multiple active agents whose behaviors are not only driven by physical laws, but also by their own interests or willingness, which may not be the same as the global control objective.

One example comes from the demand response management (DRM) system of the smart grid (see, e.g., [5]-[6]), which can effectively reduce the cost of power generation and users, and can balance the demand and supply in the power market through real-time pricing. In the DRM system, the utility companies (UCs) are interested in maximizing their benefits by setting the prices in a finite time period, and the end-users will correspondingly maximize their benefits by choosing the amount of power supplied by UCs. Hence, the UCs can be regarded as active agents who play a non-cooperative price selection game, while the market regulator may apply some macro-policies including pricing policy, to the DRM system to achieve some macro-goals such as participation incentives and dangerous emissions reduction. In this case, the regulator is faced with controlling the DRM system consisting of active agents whose behaviors are driven by their own interests, and the goals of the regulator may well be different from that of the agents.

Many other examples exist widely in social, economic and engineering systems. Moreover, many problems that are previously investigated by using the game theoretic framework may also be investigated by introducing a higher level regulator to induce the Nash equilibrium to a desired one. These include distributed game theoretic control [7]-[9], coverage optimization problem for mobile sensors [10], team formation control [11]-[12], intelligent transportation system [13], ecological systems [14]-[15], and multi-phase systems in chemical engineering [16], among others.

In all the above-mentioned systems, the objects to be regulated involve multiple active agents whose behaviors are driven by their own interests or willingness which are interdependent and may conflict with each other, resulting in strategic behaviors of the agents to pursue their own interests. If the strategic behaviors of the agents involved are ignored, the system dynamics may be seriously distorted and lead to some misunderstandings, see, e.g. [4] and [17]. Hence, it turns out to be necessary to incorporate game theoretical methods in the modeling of such control systems. This leads to the introduction and investigation of the game-based control system (GBCS), which has a hierarchical decision-making structure: one regulator and multiple agents. The regulator is regarded as the global controller and makes decision first, and then the agents try to optimize their respective objective functions to reach a possible Nash equilibrium as a result of non-cooperative dynamic game. We will delineate the details of GBCS in Section II. To the best of our knowledge, the first attempt devoted to the introduction and investigation of GBCS was given in [18] and [19].

The main contributions of this paper are as follows. First, we formulate the general game-based control system (GBCS) as a two-level hierarchical structure, where the lower level is a noncooperative differential game among multiple agents, and the higher level is a macro-regulator which can intervene in the lower level differential game to achieve a desire macro-state. This framework has broad practical background, and

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is different from both the classical game theory and the classical control theory. Second, some explicit necessary and sufficient algebraic conditions on the controllability of the Nash equilibrium are obtained for general linear time-varying GBCS, by solving the controllability problem of the associated forward-backward dynamic equations. This is a key technical difficulty of the current paper, which makes our analysis quite different from that in the classical theory.

The remainder of this paper is organized as follows. In section II, we introduce the game-based control systems (GBCS) together with some related concepts and examples. Section III gives the complete solution to the controllability study of linear GBCS. The proofs of theorems are given in section IV. Section V concludes the paper with some remarks.

II. Problem formulation

In this section, we will first give a general nonlinear framework of GBCS, and then introduce the concept of controllability of GBCS, followed by two illustrative examples where the structure applies.

First, we introduce some notations to be used throughout this paper. All vectors are column vectors. The identity matrix of size $m \times m$ is denoted by I_m and the null matrix of size $m \times n$ is denoted by $0_{m\times n}$ (or 0_m if $m = n$). For a matrix A, the operator rank$\{A\}$ means the rank of A, A^T denotes the transposition of A, Im$\{A\}$ represents the image space of A and $\Lambda(A)$ denotes the set of all eigenvalues of A. We use the notation $< \cdot, \cdot >$ for the inner product in R^n and f_x for the partial derivative of a function f with respect to x.

A. Game-Based Control Systems

Consider the following hierarchical control systems with one regulator and L agents:

$$\begin{cases} \dot{x}(t) = f(t, x(t), x_1(t), \ldots, x_L(t), u_1(t), \ldots, u_L(t), u(t)), \\ \dot{x}_i(t) = f_i(t, x(t), x_1(t), \ldots, x_L(t), u_1(t), \ldots, u_L(t), u(t)), \\ x(0) = x_0, \ x_i(0) = x_{i,0}, \ i = 1, 2, \ldots, L, \ t \in [0, T], \end{cases} \tag{1}$$

where $x(t) \in R^n$ stands for the macro-state of the system, $x_i(t) \in R^{n_i}$ the state of the agent $i (i = 1, 2, \ldots L)$, $u_i(t) \in D_i \subset R^{m_i}$ the strategy or control of the agent i, and $u(t) \in D \subset R^m$ stands for the strategy or control of the macro-regulator.

For convenience, we always use the shorthand $x^F(t) = (x_1(t)^T, \cdots, x_L(t)^T)^T$ for all the micro-states of the agents, and $X(t) = (x^T(t), x_1(t)^T, \cdots, x_L(t)^T)^T$ for the extended states including the macro-states of the regulator.

For each agent, there is a corresponding payoff function which represents its own interests. Let the payoff function of agent i be denoted by $J_i(u_i(\cdot), u_{-i}(\cdot))$, where $u_i(\cdot)$ is the strategy of agent i and $u_{-i}(\cdot)$ represents the strategy profile of all agents except for agent i. We assume that any agent i wants to minimize $J_i(u_i(\cdot), u_{-i}(\cdot))$ by selecting $u_i(\cdot)$ from its admissible set, given $u_{-i}(\cdot)$.

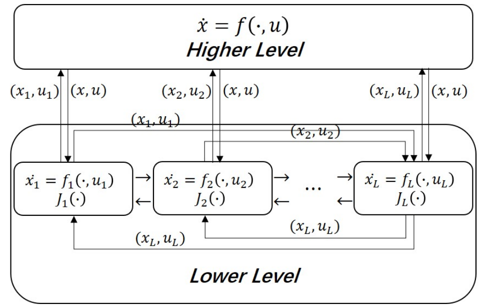

Fig. 1. Structure of the GBCS.

The commonly used payoff function for finite time period T is

$$\begin{aligned} J_i(u_1(\cdot), u_2(\cdot), \ldots, u_L(\cdot)) = \\ K_i(x^F(T)) + \int_0^T L_i(X(\cdot), u_1(\cdot), u_2(\cdot), \ldots, u_L(\cdot))\, dt, \end{aligned} \tag{2}$$

where $K_i(\cdot)$ and $L_i(\cdots)$ $(i = 1, \cdots, L)$ are continuous functions of their variables. For each agent, the states of other agents and the macro-state may affect the running cost of its payoff, but the rational agent may not care about the finial macro-state value $x(T)$. This is reflected by the fact that the running cost function L_i is a function of X, but the terminal cost function K_i is only a function of x^F.

It is well known that information structures play a crucial role in differential games [20]. In our GBCS, the regulator will first make and announce his macro-decision, then each agent has the access to the information of the regulator's input but does not know other agents' inputs when making his own decision. Therefore, given the decision of the regulator, the agents in GBCS will form a non-cooperative differential game at the lower level. The Figure 1 illustrates the structure of the GBCS. The higher level block represents the dynamics of the macro-state. Each sub-block in the lower level block represents the dynamics of the micro-state of an agent. An arrow represents the influence of the state and input from one block to another.

In this paper, we always assume that the information $\{f, f_i, J_i, i = 1, 2, \ldots, L\}$ and the initial state $\{x(0), x_i(0), i = 1, 2, \ldots, L\}$ of the system are common knowledge [21, Chapter 14]. For simplicity, we only consider the open-loop strategies of agents, so the Nash equilibrium formed by the agents is also called open-loop Nash equilibrium [22, Definition 5.6, Theorem 6.11].

If the Nash equilibrium of the non-cooperative dynamical game (1) exists and is unique for some given input $u(t) \in U$ and $x(0) = x_0$, then, under some mild conditions, there is a unique state evolution process $X^*(t) (t \in [0, T])$ of the system, satisfying the following ordinary differential equation:

$$\begin{cases} \dot{x}^*(t) = f(t, X^*(t), u_1^*(t), \cdots, u_L^*(t), u(t)), \\ \dot{x}_i(t) = f_i(t, X^*(t), u_1^*(t), \cdots, u_L^*(t), u(t)), \\ x^*(0) = x_0, \ x_i(0) = x_{i,0}, \ i = 1, 2, \cdots, L, \end{cases} \tag{3}$$

where $(u_1^*, u_2^*, \ldots, u_L^*)$ is the Nash equilibrium corresponding to the regulator's strategy $u(t) (t \in [0, T])$. Hence, the system state dynamics is essentially determined by the regulator and thus can be regarded as a control system where the regulator can change the macro-state by regulating the Nash equilibrium form at the lower level.

The above problem formulation can be directly extended to more general settings with hybrid dynamics and multi-layers.

B. Controllability Problem

As a control system, there are many interesting problems to be investigated. Here we are only interested in whether or not the macro-state can be driven from any initial state to any desired state by the influence of the regulator, which can be captured by the concept of controllability.

Definition 1. The system (1) is called controllable, if for any given initial states $x(0) = x_0 \in R^n$, $x_i(0) = x_{i,0} \in R^{n_i}, i = 1, 2, \ldots, L$ and any terminal macro state $x(T) = x_T \in R^n$, there is a strategy $u(t)(t \in [0, T])$ of the regulator, under which the Nash equilibrium exists and is unique, and the solution $x^*(t)$ of the (3) satisfies $x^*(T) = x_T$.

We remark that the initial states include both the macro state $x(0)$ and the agents' states $x_1(0), \cdots, x_L(0)$, but in the final states we are only interested in the macro state $x(T)$ in our controllability definition.

C. Examples of GBCS

We will give an example where the GBCS framework maybe applied, other more complicated examples such as smart grid will not be discussed here.

Example 1. Consider the problem of optimal economic stabilization policies under decentralized control and conflicting objectives, which has been studied in the literature [23]-[25]. In many countries, macroeconomic policy is made and implemented by different authorities, where different authorities may control different sets of policy instruments with different objectives [23].

The linear-quadratic difference games have been used to model this kind of systems around a certain nominal state [23]. The continuous-time analogy of the discrete-time econometric dynamical model in state-variable in [23] is

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{k} B_i u_i(t) + Cz(t), \tag{4}$$

where $u_i(i = 1, \cdots, k)$ are control variables, each of which can be manipulated by an authority which chooses its strategy u_i to minimize its own cost function

$$J_i = \int_0^T \Big(x^T(t)Q_i x(t) + u_i^T(t)R_i u_i(t)\Big)dt. \tag{5}$$

where $z(t)$ can be taken as a higher level regulator input, such as law and policy, which can not be changed by the lower level authorities. In different economic situations, the higher level regulator may need to regulate the equilibrium state $x(T)$ of the macro-system by regulating the input $z(t)$. This is another typical problem on the controllability of GBCS, which can be solved by using Theorem 2 in Section III directly, and the details will not be repeated in this paper.

III. Main results

In this part, we will focus on linear dynamic systems to give some explicit necessary and sufficient algebraic conditions on the controllability of Nash equilibrium.

A. General Linear-Quadratic Systems

Consider the following general non-cooperative linear-quadric differential game with one regulator and L agents:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + \sum_{i=1}^{L} A_i(t)x_i(t) + \sum_{i=1}^{L} D_i(t)u_i(t) \\ \qquad + B(t)u(t), \\ \dot{x}_i(t) = E_i(t)x(t) + \sum_{j=1}^{L} F_{ij}(t)x_i(t) + \sum_{j=1}^{L} B_{ij}(t)u_j(t) \\ \qquad + B_i(t)u(t), \\ x(0) = x_0,\ x_i(0) = x_{i,0}\ (i = 1, 2, \ldots, L). \end{cases} \tag{6}$$

The payoff function to be minimized by $u_i(\cdot)$ of any agent $i(i = 1, 2, \ldots, L)$ is

$$\begin{aligned} J_i(u_1(\cdot), u_2(\cdot), \ldots, u_L(\cdot)) = \frac{1}{2}x^F(T)^T Q_{iT} x^F(T) \\ + \frac{1}{2}\int_0^T [X^T(t)Q_i(t)X(t) + u_i^T(t)R_i(t)u_i(t)]\, dt, \end{aligned} \tag{7}$$

where for any $t \in [0, T]$, $R_i(t) > 0$, $Q_i(t)$ and Q_{iT} are symmetric, all entries of the matrices $A(t)$, $B(t)$, $A_i(t)$, $B_i(t)$, $C_i(t)$, $E_i(t)$, $F_i(t)$, $Q_i(t)$, $R_i(t)$, $i = 1, 2, \cdots, L$ are piecewise smooth functions of time, and the dimensions of vectors x, x_i, u, u_i are n, n_i, m, m_i, respectively.

We introduce the following notations:

$$\begin{aligned} &\overline{A}(t) = \\ &\begin{bmatrix} \widetilde{A}(t) & \widetilde{B}_1(t)R_1^{-1}(t)\widetilde{B}_1^T(t) & \ldots & \widetilde{B}_L(t)R_L^{-1}(t)\widetilde{B}_L^T(t) \\ Q_1(t) & -\widetilde{A}^T(t) & \ldots & 0_{N\times N} \\ \vdots & \vdots & \ddots & \vdots \\ Q_L(t) & 0_{N\times N} & \ldots & -\widetilde{A}^T(t) \end{bmatrix}, \\ &\overline{B}(t) = \begin{bmatrix} \widetilde{B}(t) \\ 0_{NL\times m} \end{bmatrix},\ N = n + \sum_{i=1}^{L} n_i, \end{aligned} \tag{8}$$

where

$$\begin{aligned} &\widetilde{A}(t) = \begin{bmatrix} A(t) & A_1(t) & \ldots & A_L(t) \\ E_1(t) & F_{11}(t) & \ldots & F_{1L}(t) \\ \vdots & \vdots & \ddots & \vdots \\ E_L(t) & F_{L1}(t) & \ldots & F_{LL}(t)(t) \end{bmatrix}, \\ &\widetilde{B}(t) = \begin{bmatrix} B(t) \\ B_1(t) \\ \vdots \\ B_L(t) \end{bmatrix},\ \widetilde{B}_i(t) = \begin{bmatrix} D_i(t) \\ B_{1i}(t) \\ \vdots \\ B_{Li}(t) \end{bmatrix}, \\ &\widetilde{Q}_{iT} = \begin{bmatrix} 0_{n\times n} & 0_{n\times(N-n)} \\ 0_{(N-n)\times n} & Q_{iT} \end{bmatrix}. \end{aligned} \tag{9}$$

Assumption 1.1. When the control of regulator is 0, i.e. $u(t) = 0 (t \in [0,T])$, for any initial state $x_0, x_{i,0} (i = 1,2,\ldots,L)$, the L-person linear-quadratic differential game of (6)-(7) admits an open-loop Nash equilibrium.

Assumption 1.2. The following Riccati differential equations have a set of symmetric solutions K_j on $[0,T]$ for $j = 1,2,\cdots,L$:

$$\begin{cases} \dot{K}_j(t) = -\widetilde{A}^T K_j - K_j \widetilde{A} - Q_j + K_j \widetilde{S}_j K_j, \\ K_j(T) = \widetilde{Q}_{jT}, \end{cases} \tag{10}$$

where $\widetilde{S}_j = \widetilde{B}_j R_j^{-1} \widetilde{B}_j^T$.

Remark 1.1. It has been shown in [26, Page 269, Note 2] that if Assumption 1.1 holds, then the L Riccati equations above have a set of symmetric solutions K_j on $(0,T]$, but may not be on $[0,T]$.

Remark 1.2. If the matrixes $Q_i(t)$, Q_{iT}, $(i = 1,2,\cdots,L)$ are positive semi-definite, then Assumption 1.2 holds automatically [27, Chapter 6.1.4].

We now introduce the transition matrix $\Phi(t)$ defined by

$$\begin{cases} \dfrac{d\Phi(t)}{dt} = -\Phi(t)\overline{A}(t), \\ \Phi(0) = I_{(L+1)N}, \end{cases} \tag{11}$$

together with controllability Gramian matrix W defined by

$$W(T) = \int_0^T \Phi(t)\overline{B}(t)\overline{B}^T(t)\Phi^T(t)\, dt. \tag{12}$$

Theorem 1. Suppose that Assumptions 2.1 and 2.2 hold. Then, the GBCS (6)-(7) is controllable if and only if the following matrix is of full rank

$$\left[\begin{bmatrix} 0_{N\times(LN)} \\ I_{NL} \end{bmatrix}, \Phi(T) Q_T \begin{bmatrix} 0_{n\times(N-n)} \\ I_{N-n} \end{bmatrix}, W(T) \right], \tag{13}$$

where

$$Q_T = \begin{bmatrix} I_N \\ -\widetilde{Q}_{1T} \\ \vdots \\ -\widetilde{Q}_{LT} \end{bmatrix} \in R^{(L+1)N\times N}. \tag{14}$$

In the special case studied in [28] where there is no micro-states (i.e., $n_i = 0$ in (6)) and no finial costs (i.e., $Q_{iT} = 0$), the controllability criterion in the above theorem can be considerably simplified.

Corollary 1.1 [28]. Under Assumptions 2.1 and 2.2 with $n_i = 0$ and $Q_{iT} = 0$ for $i = 1,\cdots,L$, the GBCS (6)-(7) is controllable if and only if the following matrix is of full rank

$$\left[\begin{bmatrix} 0_{N\times(LN)} \\ I_{NL} \end{bmatrix}, W(T) \right]. \tag{15}$$

Proof: If $n_i = 0$, $i = 1,\cdots,L$, then $N = n$ and there is no the matrix

$$\Phi(T) Q_T \begin{bmatrix} 0_{n\times(N-n)} \\ I_{N-n} \end{bmatrix}.$$

Hence, the full rank condition of matrix (13) is equivalent to the full rank of matrix (15).

Remark 1.3. Under the assumptions of Corollary 2.1, if the matrixes $A_i = 0, D_i = 0$, $i = 1,\cdots,L$, then the above rank condition degenerates to the classical criterion for the controllability of linear control systems.

B. Time-Invariant Linear-Quadratic System

When the system (6)-(7) is time-invariant, i.e., $A(t), B(t), A_i(t), B_i(t), C_i(t), E_i(t), F_i(t), Q_i(t), F_i(t) (i = 1,2,\ldots,L)$ are independent of time t, we can denote them by $A, B, A_i, B_i, C_i, E_i, F_i, Q_i, R_i (i = 1,2,\ldots,L)$, respectively, and can get much simpler and explicit criterion for controllability.

Theorem 2. Let the GBCS (6)-(7) be time invariant and Assumptions 2.1 and 2.2 hold. Then, the system is controllable if and only if the following rank condition holds:

$$rank(Q_C) = N, \tag{16}$$

where

$$\begin{aligned} Q_C &= \left[Q_{C1},\ Q_{C2}\right], \\ Q_{C1} &= \left[\widetilde{B}, \widetilde{A}\widetilde{B}, \widetilde{A}^2\widetilde{B} + P_1, \cdots, \widetilde{A}^{(L+1)N-1}\widetilde{B} + P_{(L+1)N-2}\right], \\ P_k &= \left[I_N\ 0\right] \overline{A}^{k+1}\overline{B} - \widetilde{A}^{k+1}\widetilde{B}, \\ Q_{C2} &= \left[I_N\ 0\right] e^{-\overline{A}T} Q_T \begin{bmatrix} 0_{n\times(N-n)} \\ I_{N-n} \end{bmatrix}, \end{aligned}$$

in which $\overline{A}$ and $\overline{B}$ are the corresponding time-invariant matrices of (8).

Corollary 2.1. Let us assume that the linear GBCS (6)-(7) is time invariant with $n_i = 0$, $i = 1,\cdots,L$. Then, under Assumptions 2.1 and 2.2, the GBCS is controllable if and only if

$$rank(Q_{C1}) = n.$$

In this case the following result holds:

$$\begin{aligned} &\text{for any } s \in \Lambda(\overline{A}) \Rightarrow \\ &rank(\left[A - sI_n, B_1 R_1^{-1} B_1^T, \cdots, B_L R_L^{-1} B_L^T, B\right]) = n. \end{aligned} \tag{17}$$

IV. Proofs of the Theorems

In this section, we will present the main proofs of theorems with some auxiliary material given in the Appendices.

A. Proof of Theorem 1

To start with, let us temporarily assume that for any $u(t) (t \in [0,T])$ and any initial states $x_0, x_{i,0} (i = 1,2,\ldots,L)$, the open-loop Nash equilibrium $(u_1^*(\cdot), u_2^*(\cdot), \cdots, u_L^*(\cdot))$ exists and is unique. Then, according to the maximum principle, we get the following equations defining the dynamics of the Nash equilibrium reached by the agents inputs' $u_i^*, i = 1,\cdots,L$:

$$\begin{cases} u_i^*(t) = R_i^{-1}(t)\widetilde{B}_i^T(t)\phi_i(t), \\ \dot{X}(t) = \widetilde{A}(t)X(t) + \sum_{i=1}^{L} \widetilde{B}_i(t)u_i^*(t) + \widetilde{B}(t)u(t), \\ \dot{\phi}_i(t) = Q_i(t)X(t) - \widetilde{A}^T(t)\phi_i(t), \\ X(0) = X_0,\ \phi_i(T) = -\widetilde{Q}_{iT}X(T),\ i = 1,2,\cdots,L \end{cases} \tag{18}$$

where $X(t) = (x^T(t), x_1^T(t), \cdots, x_L^T(t))^T$.

To get a compact form, we introduce the following notations:

$$\begin{aligned}\widetilde{S}(t) &= [\widetilde{S}_1(t), \cdots, \widetilde{S}_L(t)]\\ &= [\widetilde{B}_1(t)R_1^{-1}(t)\widetilde{B}_1^T(t), \cdots, \widetilde{B}_L(t)R_L^{-1}(t)\widetilde{B}_L^T(t)],\\ \widetilde{P}(t) &= -\widetilde{A}^T(t)\otimes I_L,\\ \widetilde{Q}(t) &= [Q_1^T(t), \cdots, Q_L^T(t)]^T,\\ \widetilde{Q}_T &= [-\widetilde{Q}_{1T}^T, \cdots, -\widetilde{Q}_{LT}^T]^T,\\ \phi(t) &= [\phi_1^T(t), \cdots, \phi_L^T(t)]^T.\end{aligned} \tag{19}$$

Then, (18) can be rewritten as follows:

$$\begin{cases}\dot{X}(t) = \widetilde{A}(t)X(t) + \widetilde{S}(t)\phi(t) + \widetilde{B}(t)u(t),\\ \dot{\phi}(t) = \widetilde{Q}(t)X(t) + \widetilde{P}(t)\phi(t),\\ X(0) = X_0,\ \phi(T) = \widetilde{Q}_T X(T),\end{cases} \tag{20}$$

which is a coupled forward-backward differential equation (FBDE). FBDE (20) is called partially controllable, if for any given initial state X_0 and any terminal state $x_T \in R^n$, there is an admissible input $u(t)$ of regulator, under which the trajectory of the equation exists and is unique and satisfies $x(T) = x_T$.

To prove Theorem 1, we first give some lemmas.

Lemma 1. Consider the forward-backward differential equation

$$\begin{cases}\dot{X}(t) = A(t)X(t) + B(t)Y(t) + c(t),\\ \dot{Y}(t) = C(t)X(t) + D(t)Y(t) + d(t),\\ X(0) = X_0,\ Y(T) = Q_T X_T,\ 0 \leqslant t \leqslant T,\end{cases} \tag{21}$$

where $X(t) \in R^n$, $Y(t) \in R^m$, then equation (21) has a unique solution for any X_0 if and only if the following matrix is non-singular:

$$\begin{bmatrix}I_n, 0\end{bmatrix}\Phi^{-1}(T,0)\begin{bmatrix}I_n\\ Q_T\end{bmatrix}, \tag{22}$$

where $\Phi(t,s)$ is the transition matrix, i.e.,

$$\begin{cases}\dfrac{\partial\Phi(t,s)}{\partial t} = E(t)\Phi(t,s)\\ \Phi(s,s) = I_{(n+m)}\end{cases}, E(t) = \begin{bmatrix}A(t) & B(t)\\ C(t) & D(t)\end{bmatrix}. \tag{23}$$

The proof of this lemma can be carried out by using similar methods as in [26, Page 267], and details will be omitted due to space limitations.

Lemma 2. If Assumption 1.2 holds, then for any given input $u(t)(t \in [0,T])$ of the regulator and any initial states $x_0, x_{i,0}(i = 1, 2, \ldots, L)$, the non-cooperative differential game of (6)-(7) admits an open-loop Nash equilibrium if and only if the forward-backward differential equation (20) has a solution. Moreover, if both Assumptions 1.1 and 1.2 hold, then the solution is unique.

Proof: We only give the sketch of the proof. Denote the solution of (10) by $K_i(t), i = 1, \cdots, L$ (Assumption 1.2).

If the game admits an open-loop Nash equilibrium $(u_1^*, \cdots, u_L^*)$, then the equation (20) has a solution by the maximum principle. Conversely, if Assumption 1.2 holds and the equation (20) has a solution $(x(t), \phi_1(t), \cdots, \phi_L(t))$, then we can verify that the following strategy profile is an open-loop Nash equilibrium

$$u_i(t) = -R_i^{-1}B_i^T(t)\phi_i(t), i = 1, \cdots, L.$$

Hence, the first part of the lemma is proved. Moreover, Assumption 1.1 means that the equation (20) has a solution for any x_0 when $u(t) = 0$. Hence, according to Lemma 1, the solution of the equation (20) is unique, and the proof is complete.

Define a linear space

$$\begin{aligned}W = \{ &\int_0^T \Phi(T,t)\overline{B}(t)u(t)\,dt \mid\\ &u(t)(t \in [0,T]) \text{ is an admissible control}\}\end{aligned} \tag{24}$$

and the matrix

$$W(T,0) = \int_0^T \Phi(T,s)\overline{B}(s)\overline{B}^T(s)\Phi^T(T,s)\,ds, \tag{25}$$

where $\Phi(t,s)$ is defined by

$$\begin{cases}\dfrac{\partial\Phi(t,s)}{\partial t} = \overline{A}(t)\Phi(t,s)\\ \Phi(s,s) = I_{(L+1)N}.\end{cases} \tag{26}$$

Lemma 3. $Im(W(T,0)) = W$.

Similar methods as in [29, Page 17] can be used to prove this lemma, and details are omitted.

The next lemma is close to Theorem 1.

Lemma 4. The GBCS (6)-(7) is controllable, if and only if the following matrix is of full rank

$$\left[\Phi(T,0)\begin{bmatrix}0\\ I_{NL}\end{bmatrix},\ W(T,0),\ Q_T\begin{bmatrix}0\\ I_{N-n}\end{bmatrix}\right], \tag{27}$$

where Q_T is defined in (14), which is

$$Q_T = \begin{bmatrix}I_N\\ \widetilde{Q}_T\end{bmatrix} \in R^{(L+1)N\times N}.$$

Proof: By Lemmas 1 and 2, we know that the controllability of Nash equilibrium of GBCS (6)-(7) is equivalent to the partial controllability of FBDE (20), so we only need to prove that (20) is partially controllable if and only if the matrix (27) in Lemma 4 is of full rank.

The sufficiency and necessity will be proved separately.

(1) Sufficiency.

Assume the matrix (27) of dimension $(L+1)N \times ((2L+2)N - n)$ in Lemma 4 is of full rank, then it has at lease one right inverse matrix, denoted by M^{-1}. Let

$$\begin{aligned}&M_1 = \Phi(T,0)\begin{bmatrix}0\\ I_{NL}\end{bmatrix}, M_2 = W(T,0), M_3 = Q_T\begin{bmatrix}0\\ I_{N-n}\end{bmatrix},\\ &M = [M_1, M_2, M_3],\end{aligned}$$

and partition the right inverse matrix M^{-1} as

$$M^{-1} = \begin{bmatrix}\overline{M}_1\\ \overline{M}_2\\ \overline{M}_3\end{bmatrix},$$

so we have

$$\begin{aligned}&MM^{-1} = I_{(L+1)N}, \text{or}\\ &M_1\overline{M}_1 + M_2\overline{M}_2 + M_3\overline{M}_3 = I_{(L+1)N}.\end{aligned} \tag{28}$$

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For any given initial state X_0 and terminal state $x_T \in R^n$, we construct the input of the regulator and the initial value of ϕ as follows:

$$\overline{u}(t) = \overline{B}^T(t)\Phi^T(T,t)\overline{M}_2(\Phi(T,0)\begin{bmatrix}-X_0\\0\end{bmatrix} + \begin{bmatrix}x_T\\x_F\\\overline{Q}_T v\end{bmatrix}),$$
$$\phi_0 = \overline{M}_1(\Phi(T,0)\begin{bmatrix}-X_0\\0\end{bmatrix} + \begin{bmatrix}x_T\\x_F\\\overline{Q}_T v\end{bmatrix}) \in R^{LN}, \tag{29}$$

where

$$\overline{Q}_T \triangleq \widetilde{Q}_T \begin{bmatrix}0_{n\times(N-n)}\\I_{N-n}\end{bmatrix} \in R^{LN\times(N-n)},$$

$x_F \in R^{N-n}$ is any arbitrarily selected vector and $v \in R^{N-n}$ is a vector to be determined. Because $\widetilde{Q}_T$ in (19) has the following form:

$$\widetilde{Q}_T = [-\widetilde{Q}_{1T}^T, \cdots, -\widetilde{Q}_{LT}^T]^T, \tag{30}$$

and the first n columns of matrixes $\widetilde{Q}_{iT}, i = 1, \cdots, L$ are zeros from the definition of $\widetilde{Q}_{iT}$ in (9), we have the following relation:

$$\widetilde{Q}_T = [0_{LN\times n}, \overline{Q}_T]. \tag{31}$$

What we are going to do next is to prove that the forward-backward equations (20) are solvable and $x(T) = x_T$ when $u(t) = \overline{u}(t)$ and $X(0) = X_0$.

By (20), the expression of $\overline{u}(t)$ in (29), and the definition of $\overline{W}_i$, $i = 1, 2, 3$, we can get

$$\begin{aligned}\begin{bmatrix}X(T)\\\phi(T)\end{bmatrix} &= \Phi(T,0)\begin{bmatrix}X_0\\\phi_0\end{bmatrix} + \int_0^T \Phi(T,t)\overline{B}(t)u(t)\,dt\\
&= \Phi(T,0)\begin{bmatrix}X_0\\0\end{bmatrix} + \Phi(T,0)\begin{bmatrix}0\\I_{NL}\end{bmatrix}\phi_0 +\\
&\quad M_2\overline{M}_2\Big(\Phi(T,0)\begin{bmatrix}-X_0\\0\end{bmatrix} + \begin{bmatrix}x_T\\x_F\\\overline{Q}_T v\end{bmatrix}\Big)\\
&= \Phi(T,0)\begin{bmatrix}X_0\\0\end{bmatrix} +\\
&\quad M_1\overline{M}_1\Big(\Phi(T,0)\begin{bmatrix}-X_0\\0\end{bmatrix} + \begin{bmatrix}x_T\\x_F\\\overline{Q}_T v\end{bmatrix}\Big) +\\
&\quad M_2\overline{M}_2\Big(\Phi(T,0)\begin{bmatrix}-X_0\\0\end{bmatrix} + \begin{bmatrix}x_T\\x_F\\\overline{Q}_T v\end{bmatrix}\Big)\\
&= \Phi(T,0)\begin{bmatrix}X_0\\0\end{bmatrix} +\\
&\quad (M_1\overline{M}_1 + M_2\overline{M}_2)\Big(\Phi(T,0)\begin{bmatrix}-X_0\\0\end{bmatrix} + \begin{bmatrix}x_T\\x_F\\\overline{Q}_T v\end{bmatrix}\Big)\\
&= \Phi(T,0)\begin{bmatrix}X_0\\0\end{bmatrix} +\\
&\quad (I_{(L+1)N} - M_3\overline{M}_3)\Big(\Phi(T,0)\begin{bmatrix}-X_0\\0\end{bmatrix} + \begin{bmatrix}x_T\\x_F\\\overline{Q}_T v\end{bmatrix}\Big)\\
&= \Phi(T,0)\begin{bmatrix}X_0\\0\end{bmatrix} + \Phi(T,0)\begin{bmatrix}-X_0\\0\end{bmatrix} + \begin{bmatrix}x_T\\x_F\\\overline{Q}_T v\end{bmatrix} -\\
&\quad M_3\overline{M}_3\Big(\Phi(T,0)\begin{bmatrix}-X_0\\0\end{bmatrix} + \begin{bmatrix}x_T\\x_F\\\overline{Q}_T v\end{bmatrix}\Big)\\
&= \begin{bmatrix}x_T\\x_F\\\overline{Q}_T v\end{bmatrix} - M_3\overline{M}_3\Big(\Phi(T,0)\begin{bmatrix}-X_0\\0\end{bmatrix} + \begin{bmatrix}x_T\\x_F\\\overline{Q}_T v\end{bmatrix}\Big).\end{aligned}$$

Now we calculate the terminal value $\phi(T)$ and verify its terminal constrain in (20).

Denote

$$\begin{bmatrix}y_T\\y_F\\y_\phi\end{bmatrix} = \Phi(T,0)\begin{bmatrix}-X_0\\0\end{bmatrix},\ w = \begin{bmatrix}y_T + x_T\\y_F + x_F\\y_\phi + \overline{Q}_T v\end{bmatrix},$$

where $y_T \in R^n$, $y_F \in R^{N-n}$ and $y_\phi \in R^{LN}$. Because

$$M_3\overline{M}_3 = Q_T\begin{bmatrix}0\\I_{N-n}\end{bmatrix}\overline{M}_3 = \begin{bmatrix}0_{n\times(L+1)N}\\\overline{M}_3\\\overline{Q}_T\overline{M}_3\end{bmatrix}$$

and

$$\begin{bmatrix}x(T)\\\overline{x}_F\\\phi(T)\end{bmatrix} = \begin{bmatrix}x_T\\x_F\\\overline{Q}_T v\end{bmatrix} - \begin{bmatrix}0\\\overline{M}_3\\\overline{Q}_T\overline{M}_3\end{bmatrix} w = \begin{bmatrix}x_T\\x_F - \overline{M}_3 w\\\overline{Q}_T(v - \overline{M}_3 w)\end{bmatrix}, \tag{32}$$

where $(x(T)^T, \overline{x}_F^T)^T = X(T)^T$, we have $x(T) = x_T$ and can get the following equation if we let $v = x_F$:

$$\phi(T) = \overline{Q}_T\overline{x}_F. \tag{33}$$

Furthermore, we have

$$\phi(T) = \overline{Q}_T\overline{x}_F = \left[0_{LN\times n}, \overline{Q}_T\right]\begin{bmatrix}x_T\\\overline{x}_F\end{bmatrix} = \widetilde{Q}_T X(T),$$

here the second equality follows from the relation (31)

$$\widetilde{Q}_T = \left[0_{LN\times n}, \overline{Q}_T\right],$$

so the terminal constrain $\phi(T) = \widetilde{Q}_T X(T)$ holds.

The above proof demonstrates that the forward-backward equation (20) is partially controllable, so the GBCS (6)-(7) is controllable. The proof of sufficiency is complete.

(2) Necessity.

If the system is controllable, then for any initial state X_0, there is an input $u(t)$, under which the solution of equations in (20) exists and $x(T) = 0$, i.e., for any X_0, there is $\phi_0 \in R^{NL}, u(t)$ such that

$$\Phi(T,0)\begin{bmatrix}X_0\\\phi_0\end{bmatrix} + \int_0^T \Phi(T,t)\overline{B}(t)u(t)\,dt = Q_T\begin{bmatrix}0_{n\times 1}\\x_F\end{bmatrix},$$

where $x_F \in R^{N-n}$. By simple algebraic manipulations, we know that for any $X_0 \in R^N$, there are $\phi_0, u(t), x_F$, such that

$$\Phi(T,0)\begin{bmatrix} X_0 \\ 0 \end{bmatrix} = \Phi(T,0)\begin{bmatrix} 0 \\ \phi_0 \end{bmatrix} + \int_0^T \Phi(T,t)\overline{B}(t)u(t)\,dt + Q_T \begin{bmatrix} 0_{n\times 1} \\ x_F \end{bmatrix}.$$

This leads to

$$Im(\Phi(T,0)\ \begin{bmatrix} I_N \\ 0 \end{bmatrix}) \subseteq Im(\left[\Phi(T,0)\begin{bmatrix} 0 \\ I_{NL} \end{bmatrix}, \overline{W},\ Q_T\begin{bmatrix} 0 \\ I_{N-n} \end{bmatrix}\right]),$$

where $\overline{W}$ is a basis matrix of subspace W. Because $\Phi(T,0)$ is invertible, we obtain

$$\begin{aligned} &Im(\Phi(T,0)\begin{bmatrix} I_N \\ 0 \end{bmatrix})\ \cap\ Im(\Phi(T,0)\begin{bmatrix} 0 \\ I_{NL} \end{bmatrix}) = \{0\}, \\ &Im(\Phi(T,0)\begin{bmatrix} I_N \\ 0 \end{bmatrix}) + Im(\Phi(T,0)\begin{bmatrix} 0 \\ I_{NL} \end{bmatrix}) = R^{(L+1)N}. \end{aligned} \tag{34}$$

Thus, the matrix

$$\left[\Phi(T,0)\begin{bmatrix} 0 \\ I_{NL} \end{bmatrix},\ \overline{W},\ Q_T\begin{bmatrix} 0 \\ I_{N-n} \end{bmatrix}\right]$$

is of full rank. By Lemma 2, we know that the following matrix is also of full rank:

$$\left[\Phi(T,0)\begin{bmatrix} 0 \\ I_{NL} \end{bmatrix},\ W(T,0),\ Q_T\begin{bmatrix} 0 \\ I_{N-n} \end{bmatrix}\right],$$

and so the proof of necessity is complete.

Now, we give the proof of Theorem 1.

Because of Assumptions 1.1 and 1.2 and Lemma 2, we know that for any input $u(t) (t \in [0,T])$ of the regulator and any initial states $x_0, x_{i,0} (i = 1, 2, \ldots, L)$, the open-loop Nash equilibrium exists and is unique.

If we define the matrix $\Psi(t)$ by

$$\begin{cases} \dfrac{d\Psi(t)}{dt} = \overline{A}(t)\Psi(t), \\ \Psi(0) = I_{(L+1)N}, \end{cases} \tag{35}$$

then

$$\begin{aligned} \Psi^{-1}(t) &= \Phi(t), \\ \Phi(t,s) &= \Psi(t)\Psi^{-1}(s), \end{aligned} \tag{36}$$

where $\Phi(t)$ and $\Phi(t,s)$ are defined in (11) and (26), respectively. Thus we have

$$\begin{aligned} W(T,0) &= \Psi(T)\int_0^T \Psi^{-1}(t)\overline{B}(t)\overline{B}^T(t)\Psi^{-T}(t)\,dt\Psi^T(T) \\ &= \Psi(T)\int_0^T \Phi(t)\overline{B}(t)\overline{B}^T(t)\Phi^T(t)\,dt\Psi^T(T) \\ &= \Psi(T)W(T)\Psi^T(T), \end{aligned} \tag{37}$$

and moreover,

$$\begin{aligned} &Im(\left[\Phi(T,0)\begin{bmatrix} 0 \\ I_{NL} \end{bmatrix},\ W(T,0),\ Q_T\begin{bmatrix} 0 \\ I_{N-n} \end{bmatrix}\right]) \\ &= Im(\left[\Psi(T)\begin{bmatrix} 0 \\ I_{NL} \end{bmatrix},\ \Psi(T)W(T)\Psi^T(T),\ Q_T\begin{bmatrix} 0 \\ I_{N-n} \end{bmatrix}\right]) \\ &= Im(\Psi(T) \\ &\quad \left[\begin{bmatrix} 0 \\ I_{NL} \end{bmatrix},\ W(T)\Psi^T(T), \Psi^{-1}(T)Q_T\begin{bmatrix} 0 \\ I_{N-n} \end{bmatrix}\right]) \\ &= Im(\Psi(T)\left[\begin{bmatrix} 0 \\ I_{NL} \end{bmatrix},\ W(T),\ \Phi(T)Q_T\begin{bmatrix} 0 \\ I_{N-n} \end{bmatrix}\right]) \\ &= Im(\Psi(T)\left[\begin{bmatrix} 0 \\ I_{NL} \end{bmatrix},\ \Phi(T)Q_T\begin{bmatrix} 0 \\ I_{N-n} \end{bmatrix},\ W(T)\right]), \end{aligned} \tag{38}$$

where the third equality follows from the fact that $\Psi(T)$ is nonsingular. By Lemma 4, we know that Theorem 1 holds, and so the proof is complete.

B. Proofs of Theorem 2 and Corollary 2.1

We need the following result on linear time-invariant control systems, which can be found in [30, Corollary 3.2].

Lemma 5. The linear space W defined in (24) is an invariant subspace of $\overline{A}$ and the following holds:

$$Im(\left[\overline{B},\ \overline{AB},\ \overline{A}^2\overline{B},\ \cdots,\ \overline{A}^{(L+1)N-1}\overline{B}\right]) = W. \tag{39}$$

Now, we prove Theorem 2 as follows.

In the time-invariant case, the matrix $\Phi(T,0) = e^{\overline{A}T}$ and $\Phi^{-1}(T,0) = e^{-\overline{A}T}$. There is a finite representation of the matrix exponential function as follows [29, Proposition 1.2.1]:

$$e^{\overline{A}T} = \sum_{k=0}^{(L+1)N-1} a_k\overline{A}^k, \tag{40}$$

where $a_k, k = 1, 2, \cdots, (L+1)N-1$ are some real numbers which depend on $\overline{A}T$. This implies that the linear space W is an invariant subspace of $\Phi(T,0)$. Because of the invertibility of $\Phi(T,0)$, we get $\Phi(T,0)W = W$. Denote

$$\overline{W} = \left[\overline{B},\ \overline{AB},\ \overline{A}^2\overline{B},\ \cdots,\ \overline{A}^{(L+1)N-1}\overline{B}\right],$$

then we have

$$\begin{aligned} &Im(\left[\Phi(T,0)\begin{bmatrix} 0 \\ I_{NL} \end{bmatrix},\ W(T,0),\ Q_T\begin{bmatrix} 0 \\ I_{N-n} \end{bmatrix}\right]) \\ &= Im(\left[\Phi(T,0)\begin{bmatrix} 0 \\ I_{NL} \end{bmatrix},\ \overline{W},\ Q_T\begin{bmatrix} 0 \\ I_{N-n} \end{bmatrix}\right]) \\ &= Im(\left[\Phi(T,0)\begin{bmatrix} 0 \\ I_{NL} \end{bmatrix},\ \Phi(T,0)\overline{W},\ Q_T\begin{bmatrix} 0 \\ I_{N-n} \end{bmatrix}\right]) \\ &= Im(\Phi(T,0)\left[\begin{bmatrix} 0 \\ I_{NL} \end{bmatrix},\ \overline{W},\ \Phi^{-1}(T,0)Q_T\begin{bmatrix} 0 \\ I_{N-n} \end{bmatrix}\right]) \\ &= Im(\Phi(T,0)\left[\begin{bmatrix} 0 \\ I_{NL} \end{bmatrix},\ \overline{W},\ e^{-\overline{A}T}Q_T\begin{bmatrix} 0 \\ I_{N-n} \end{bmatrix}\right]). \end{aligned}$$

By Lemma 4, the system is controllable if and only if the matrix

$$\Phi(T,0)\left[\begin{bmatrix} 0 \\ I_{NL} \end{bmatrix},\ \overline{W},\ e^{-\overline{A}T}Q_T\begin{bmatrix} 0 \\ I_{N-n} \end{bmatrix}\right]$$

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is of full rank, which is equivalent to the full rank property of

$$[I_N, 0]\left[\begin{array}{cc} \overline{W}, & e^{-\overline{A}T}Q_T\begin{bmatrix}0\\ I_{N-n}\end{bmatrix}\end{array}\right],$$

i.e., the matrix

$$Q_C = \begin{bmatrix}Q_{C1}, \ Q_{C2}\end{bmatrix}$$

is of full rank. The proof is complete.

The proof of Corollary 2.1 can be found in [28].

V. Conclusions

In this paper, we have investigated the controllability of Nash equilibrium of a class of game-based control systems, called GBCSs. The motivation for studying GBCS comes from rich situations in the real world, where the macro-states of the global system are determined by the Nash equilibrium formed at lower level via non-cooperative differential games. Thus the GBCS is beyond the framework of both the classical control theory and the game theory. In this paper, we have first described a general framework for the controllability of GBCS, and then presented some necessary and sufficient conditions for the controllability of linear-quadratic GBCS. Compared with the controllability of classical control systems, the key difficulty in this work lies in the fact that we have to analyze the controllability of the associated forward-backward differential equations, which has rarely been explored in the literature. For future investigation, it would be interesting to generalize the results of the paper to more complicated situations, including multi-layer games, robust and adaptive control problems, and dynamic systems with various constraints and different objectives, etc.

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