Extended PID Control of Nonlinear Uncertain Systems

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Abstract

Since the classical proportional-integral-derivative (PID) controller is the most widely and successfully used ones in industrial processes, it is of vital importance to investigate theoretically the rationale of this ubiquitous controller in dealing with nonlinearity and uncertainty. Recently, we have investigated the capability of the classical PID control for second order nonlinear uncertain systems and provided some analytic design methods for the choices of PID parameters, where the system is assumed to be in the canonical form of cascade integrators. In this paper, we will consider the natural extension of the classical PID control for high order affine-nonlinear uncertain systems. In contrast to most of the literature on controller design of nonlinear systems, we do not require such special system structures as pure-feedback form, thanks to the strong robustness of the extend PID controller. To be specific, we will show that under some suitable conditions on nonlinearity and uncertainty of the systems, the extended PID controller can globally(or semi-globally) stabilize the nonlinear uncertain systems, and at the same time the regulation error converges to 0 exponentially fast, as long as the control parameters are chosen from an open unbounded parameter manifold.

Keywords: PID control, affine nonlinear uncertain systems, stabilization, canonical form, diffeomorphism, differential observers.

1 Introduction

Over the past 60 years, remarkable progresses in modern control theory have been made, e.g., numerous advanced control techniques including optimal control, robust control, adaptive control, nonlinear control, intelligent control, etc have been introduced and investigated. However, the classical PID (proportional-integral-derivative) controller (or its variations), which has nearly 100 years of

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history, is still the most widely and successfully used one in engineering systems by far(see e.g., [2, 20]), which exhibits its lasting vitality.

In fact, a recent survey [20] shows that the PID controller has much higher impact rating than the advanced control technologies and that we still have nothing that compares with PID. However, it has also been reported that most of the practical PID loops are poorly tuned, and there is strong evidence that PID controllers remain poorly understood [19]. Therefore, as pointed out in [1], better understanding of the PID control may considerably improve its widespread practice, and so contribute to better product quality. This is the primary motivation of our theoretical investigation of the PID controller.

As is well-known, the PID controller has been investigated extensively in the literature by various scientists and engineers, but most focus on linear systems (e.g., [1, 5, 22]), albeit almost all practical systems are nonlinear with uncertainties. Therefore, to justify the remarkable effectiveness of the PID controllers for real world systems, we have to face with nonlinear uncertain dynamical systems and to understand the rationale and capability of this controller.

Recently, we have given a theoretical investigation for the convergence and design of PID controller for a basic class of nonlinear uncertain systems (see [27], [28] and ([16]). For example, in [27] we have shown that for second order nonlinear uncertain dynamical systems, one can select the three PID parameters to globally stabilize the closed-loop systems and at the same time to make the output of the controlled system converge to any given setpoint, provided that the nonlinear uncertain functions satisfy a Lipschitz condition. Moreover, necessary and sufficient conditions for the selection of the PID parameters have also been discussed and provided in [28]. These results have demonstrated theoretically that the classical PID controller does indeed have large-scale robustness with respect to both the uncertain system structure and the selection of the controller parameters. However, in the work of [27] and [28], we have only provided global convergence results for second-order nonlinear uncertain systems of integral cascades and there is no uncertainty in the controller channel.

Actually, in the area of nonlinear control, extensive researches have been conducted on the controller design (e.g.,[17, 8, 13, 6, 12]). For examples, the feedback linearization method by using full knowledge of the system nonlinear functions (e.g., [4], [8]), the backstepping approach for pure feedback forms in [17], the extremum seeking methods for nonlinear uncertain systems(see e.g., [15, 23]), and many other interesting design methods for certain triangular forms(see, e.g.,[25, 9, 10, 11, 6]), as well as for feedforward nonlinear systems, see e.g., [18].

In this paper, we will consider a general class of single input single output(SISO) affine nonlinear uncertain systems. We will introduce a generalized concept called the extended PID controller(which is the high-order extension the classical PID controller). We will show that for a large class of n-dimensional SISO affine nonlinear uncertain systems, an $(n+1)$-dimensional parameter manifold can be constructed, from which the extended PID controller parameters can be arbitrarily chosen to globally or semi-globally stabilize the nonlinear uncertain systems with the regulation error converging to 0 exponentially, even if the system may not be globally transformed into the canonical form by coordinate transformation, thanks to the strong robustness of the extended PID controller as will be demonstrated in the paper. Moreover, in the case where the derivatives of the regulation error are not available, the same results will also be established by incorporating a high-gain differential observer.

The rest of the paper is organized as follows. In Section II, we will introduce the problem formulation. The main results are presented in section III. Section IV contains the proofs of the main theorems. Section V will conclude the paper with some remarks.

2 Problem formulation

2.1 Notations

We first introduce some notations and definitions to be used throughout this paper:

Let x be a vector in the n-dimensional Euclidean space $\mathbb{R}^n$, P be an $m \times n$ matrix, and x^T, P^T denotes the transpose of x and P respectively.

Also, let $\|x\|$ denote the Euclidean norm of x, and $\|P\|$ denote the operator norm of the matrix P induced by the Euclidean norm, i.e., $\|P\| = \sup_{x\in\mathbb{R}^n, \|x\|=1} \|Px\|$, which is known to be the largest singular value of P.

Let $z(t)$ be a function of time t, then $\dot{z}(t)$ denotes the time derivative of $z(t)$. For simplicity, we oftentimes omit the variable t whenever there is no ambiguity in the sequel.

Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$. For $z \in \mathbb{R}^n$, we denote $\Phi^{-1}(z) \triangleq \{x \in \mathbb{R}^n : \Phi(x) = z\}$.

A map $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is called injective if $\Phi(x) \neq \Phi(y)$ for any $x \neq y$ and is called surjective if $\Phi^{-1}(z) \neq \emptyset$ for any $z \in \mathbb{R}^n$. Moreover, a map Φ is called a global diffeomorphism on $\mathbb{R}^n$ if it is both injective and surjective, and both Φ and its inverse mapping (also denoted by Φ^{-1} for simplicity) are continuously differentiable.

Consider the following single-input-single-output(SISO) affine nonlinear system,

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \tag{1}$$

where $f, g : \mathbb{R}^n \to \mathbb{R}^n$, $h : \mathbb{R}^n \to \mathbb{R}$ are sufficiently smooth mappings.

The mappings $f, g : \mathbb{R}^n \to \mathbb{R}^n$ are called smooth vector fields on $\mathbb{R}^n$. Let the coordinates of x be x_i and the components of f and g be f_i and g_i respectively, $i = 1, \cdots, n$. Define $L_f h(x) \triangleq \sum_{i=1}^n \frac{\partial h}{\partial x_i}(x) f_i(x)$, which is called the Lie derivative of h along with the vector field f. Let us further denote $L_g L_f h(x) \triangleq \sum_{i=1}^n \frac{\partial L_f h}{\partial x_i} g_i(x)$, $L_f^k h(x) \triangleq L_f L_f^{k-1} h(x)$, $k \geq 1$, with $L_f^0 h(x) \triangleq h(x)$. Moreover, the Lie bracket of the two vector fields f and g at x is defined by $[f, g](x) = \frac{\partial g}{\partial x}(x) f(x) - \frac{\partial f}{\partial x}(x) g(x)$, which is a new vector field on $\mathbb{R}^n$. We also need the notation for vector fields $ad_f^k g$, which is defined recursively by $ad_f^k g = [f, ad_f^{k-1} g]$, $k \geq 1$, with $ad_f^0 g = g$.

2.2 Background

Let us consider the system (1), where $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}$ are sufficiently smooth unknown vector fields or unknown functions.

Let $y^* \in \mathbb{R}$ be a given setpoint. Our control objective is to design a controller $u(t)$ to globally (or semi-globally) stabilize the system (1) and to achieve asymptotic regulation $\lim_{t\to\infty} y(t) = y^*$.

To start with, let us assume that the relative degree of (1) is n at some point $x_0 \in \mathbb{R}^n$, i.e.(see, e.g., [8]),

$$L_g L_f^k h(x) = 0, 0 \le k \le n-2; L_g L_f^{n-1} h(x_0) \neq 0$$

in a neighborhood of x_0. Let us define

$$\Phi(x) = (h(x), L_f h(x), \cdots, L_f^{n-1} h(x))^T. \tag{2}$$

Then there exists a neighborhood U_{x_0} of x_0, such that Φ is a diffeomorphism on U_{x_0}. Under the coordinates transformation $z \stackrel{\triangle}{=} (z_1, \cdots, z_n)^T = \Phi(x)$, we have

$$\begin{cases} \dot{z}_1 = z_2 \\ \quad \vdots \\ \dot{z}_n = a(z) + b(z)u \end{cases} \tag{3}$$

where $y(t) = z_1(t)$, and for $z \in \Phi(U_{x_0})$,

$$a(z) = L_f^n h(x) = L_f^n h(\Phi^{-1}(z)), \tag{4}$$

$$b(z) = L_g L_f^{n-1} h(x) = L_g L_f^{n-1} h(\Phi^{-1}(z)), \tag{5}$$

see, e.g., [8].

If f, g, h were known, then one can chose $k_1, \cdots, k_n$ such that the following matrix

$$\begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -k_1 & -k_2 & \cdots & -k_n \end{bmatrix}$$

is Hurwitz. Then under the state feedback control law

$$u(x) = \frac{-L_f^n h(x) + v}{L_g L_f^{n-1} h(x)}, \tag{6}$$

with

$$v = -k_1(z_1 - y^*) - k_2 z_2 - \cdots - k_n z_n \tag{7}$$

the system (1) could be linearized to

$$\begin{cases} \dot{z}_1 &= z_2 \\ &\vdots \\ \dot{z}_n &= -k_1(z_1 - y^*) - k_2 z_2 - \cdots - k_n z_n \\ y &= z_1 \end{cases} \tag{8}$$

If $(y^*, 0, \cdots, 0) \in \mathbb{R}^n$ is sufficiently close to $\Phi(x_0)$, then it is not difficult to show that the closed-loop system (1),(6) and (7) will satisfy $\lim_{t\to\infty} y(t) = y^*$ whenever the initial state $x(0)$ lies in some neighborhood $U \subset U_{x_0}$.

As is well-known, there are some fundamental limitations of the above method: The state feedback control law (6) needs the exact information about both the structure and states of (1), which is unrealistic in most practical situations. Moreover, even if we know the exact information of (1), the robustness of such designed controller is still a concern. Furthermore, the established theoretical results are usually local and it is hard to estimate the corresponding local region.

Thus, a challenging fundamental problems is: Can we establish a theory on global(or semi-global) stabilization and regulation by designing an output feedback controller without using the precise mathematical model information?

2.3 Problem Formulation

Let us reconsider the system (1), where $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^n$ are smooth vector fields and $h : \mathbb{R}^n \to \mathbb{R}$ is a smooth function, which are all assumed to be unknown.

It is well known that the structure of the classical PID controller is simple and its implementation does not need precise mathematical models. In fact, in our previous work [28], we have shown that for second order nonlinear uncertain systems, the classical PID control $u(t) = k_1 e(t) + k_0 \int_0^t e(s)ds + k_2 \frac{de(t)}{dt}$ has the ability to achieve global stabilization, where $e(t) = y^* - y(t)$ is the regulation error. We have also shown in [27] that the classical PID controller cannot stabilize dynamical systems which are described by ordinary differential equations with order ≥ 3 even for linear time invariant systems. Therefore, it inspires us to introduce the concept "extended PID controller", which is defined by

$$u(t) = k_1 e(t) + k_0 \int_0^t e(s)ds + k_2 \dot{e}(t) + \cdots + k_n e^{(n-1)}(t) \tag{9}$$

where $e(t) = y^* - y(t)$ is the regulation error, $\dot{e}(t), \cdots, e^{(n-1)}(t)$ are the time derivatives of $e(t)$ up to the $(n-1)^{\text{th}}$ order.

From the definition (9), we know that the extended PID controller is an output feedback whose design does not need the precise model of the plant (1), the control variable $u(t)$ is simply a weighted linear combination of the proportional, integral and derivative terms of the system regulation error, where the weighting parameters $(k_0, k_1, \cdots, k_n)$ are called extended PID parameters.

We will in this paper investigate the capability of the extended PID controller (9). To be specific, the control objective is to understand when and how the extended PID controller can guarantee that the output $y(t)$ converges to a given reference value y^* globally (or semi-globally) with an exponential rate of convergence, under the dynamic uncertainty $f(\cdot)$, the control channel uncertainty $g(\cdot)$ and the state-output uncertainty $h(\cdot)$.

In this paper, we will rigrously show that the extended PID controller does indeed have the abovementioned nice properties, even if the systems may not be transformed into the normal form globally by the coordinate transformation.

3 The main results

3.1 Assumptions

First, we introduce some notations that will be used throughout this paper. Let y^* be a setpoint. Denote

$$z^* \triangleq (y^*, 0, \cdots, 0)^T \in \mathbb{R}^n \tag{10}$$

and denote

$$H(x) \triangleq (F(x), G(x)), \tag{11}$$

where

$$F(x) \triangleq L_f^n h(x), \quad G(x) \triangleq L_g L_f^{n-1} h(x). \tag{12}$$

Assume that the system (1) has uniform relative degree n, i.e.,

$$L_g L_f^i h(x) = 0, i = 0, \cdots, n-2; \quad G(x) \neq 0, \forall x \in \mathbb{R}^n.$$

Since $G(x)$ is continuous and $G(x) \neq 0,\ \forall x \in \mathbb{R}^n$, we know that the sign of $G(x)$ cannot change. Therefore we introduce the following assumption.

Assumption (A): System (1) has uniform relative degree n. Furthermore, the sign of $G(\cdot)$ is known and $G(x)$ is uniformly bounded away from zero. Without loss of generality, we assume that $G(x) \geq \underline{b} > 0$ for any $x \in \mathbb{R}^n$.

Remark 1 *From Assumption (A), we know that $J_\Phi(x)$ is invertible for any $x \in \mathbb{R}^n$, where $J_\Phi(x)$ denotes the Jacobian matrix of Φ, (see e.g., [8]). Under the new coordinates $z = \Phi(x)$, the system (1) transforms into the canonical form of cascade integrators (3) locally. Since the Jacobian matrix $J_\Phi(x)$ is invertible for any $x \in \mathbb{R}^n$, we conclude that $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is a locally injective map, but may not be a global diffeomorphism. Therefore, the system (1) may not be globally transformed into the normal form (3) in general, unless the n vector fields $(-1)^{i-1} ad_{\tilde{f}}^{i-1} \tilde{g}(x), i = 1, \cdots, n$ are complete, where $\tilde{f}(x) = f(x) - \frac{F(x)g(x)}{G(x)}$ and $\tilde{g}(x) = \frac{g(x)}{G(x)}$, (see [8]).*

Let $\tau_1, \tau_2 : [0, \infty) \to [0, \infty)$ be two increasing functions with $\limsup_{\rho \to 0} \frac{\tau_2(\rho)}{\rho} < \infty$, which will be used to describe the model uncertainty quantitatively.

Specifically, let us introduce the following assumption.

Assumption (B): The functions Φ and H defined respectively by (2) and (11) satisfy

(i) $\|\Phi(x)\| \leq \tau_1(\|x\|), \|H(x)\| \leq \tau_1(\|\Phi(x)\|)$ for any $x \in \mathbb{R}^n$,

(ii) There exists $x^* \in \Phi^{-1}(z^*)$ such that the "gap" of H at x^* is bounded by that of Φ in the sense that

$$\|H(x) - H(x^*)\| \leq \tau_2(\|\Phi(x) - \Phi(x^*)\|)$$

holds for any $x \in \mathbb{R}^n$.

Remark 2 *If the system is of the following normal form,*

$$\begin{cases} \dot{x}_1 & = x_2 \\ & \vdots \\ \dot{x}_{n-1} & = x_n \\ \dot{x}_n & = a(x_1, \cdots, x_n) + b(x_1, \cdots, x_n)u(t) \\ y(t) & = x_1(t) \end{cases}, \tag{13}$$

where $a(\cdot)$ and $b(\cdot)$ are unknown smooth functions on $\mathbb{R}^n$. Then for any setpoint $y^* \in \mathbb{R}$, we can verify Assumptions (A) and (B) as long as the functions satisfies the following simplified assumption:

Assumption (B'): $b(x) \geq \underline{b} > 0$, $|a(x)| + |b(x)| + \|\nabla a(x)\| + \|\nabla b(x)\| \leq \rho(\|x\|), \quad \forall x \in \mathbb{R}^n$, *for some constant $\underline{b} > 0$ and some increasing function $\rho(\cdot)$, where $\nabla a(x)$ denotes the gradient of $a(\cdot)$.*

Remark 3 *For the system (13), one can show that under Assumption (B'), the extended PID controller (9) has the ability to semi-globally stabilize the system(see Corollary 2). However, for the general affine-nonlinear uncertain system (1), Assumptions (A) and (B) are not sufficient. In fact, we will give an example in Appendix B to illustrate that for any $R > 0$ and for any choices of the extended PID controller parameters, there always exists initial states $\|x(0)\| \leq R$, such that the the solution of the closed-loop system will have finite escape time, even though both Assumptions (A) and (B) are satisfied. Therefore, we need to introduce certain additional assumptions. It turns out that either of the following two conditions can ensure the solution of the closed-loop system exists in the entire time interval $[0, \infty)$, under appropriate choice of the controller parameters and initial states.*

Assumption (C): The growth rate of the inverse of the Jacobian matrix $J_\Phi(x)$ satisfies

$$\left\|J_\Phi^{-1}(x)\right\| \leq N_1\|x\| \log \|x\| + N_2, \forall x \in \mathbb{R}^n \tag{14}$$

for some constants $N_1 > 0, N_2 > 0$ (possibly unknown).

Assumption (C'): The coordinates transformation map Φ is a global diffeomorphism on $\mathbb{R}^n$.

Remark 4 *We remark that for systems (13), both Assumptions (C) and (C') are satisfied since the coordinate transformation map Φ is the identity map, i.e., $\Phi(x) = x, \ \forall x \in \mathbb{R}^n$. On the other hand, we point out that the growth rate of $\left\|J_\Phi^{-1}(x)\right\|$ in Assumption (C) cannot be relaxed slightly to, e.g.,*

$$\left\|J_\Phi^{-1}(x)\right\| \leq N_1\|x\| \log^{1+\eta} \|x\| + N_2, \ \forall x \in \mathbb{R}^n \tag{15}$$

for any $\eta > 0$. We will give an example to illustrate this in Appendix B.

3.2 Semi-global stabilization

We will first show that the extended PID controller (9) can stabilize the system (1) semi-globally under the above assumptions in the following sense: For any $R > 0$, there exists an open unbounded set $\Omega(R) \subset \mathbb{R}^{n+1}$, such that whenever the extended PID parameters $(k_0, \cdots, k_n) \in \Omega(R)$, then the closed-loop system will satisfy $\lim_{t\to\infty} y(t) = y^*$ for any initial state $x(0)$ satisfying $\|x(0)\| \leq R$. To be specific, we have the following theorem.

Theorem 1 *Consider the SISO affine-nonlinear uncertain system* (1) *with the extended PID controller defined by (9). Suppose that Assumptions (A), (B) and (C) are satisfied, where* $\underline{b} > 0$ *and the increasing functions* $\tau_1(\cdot), \tau_2(\cdot)$ *are known. Then for any* $R > 0$*, there exists an* $(n+1)$*-dimensional unbounded parameter manifold* $\Omega(R) \subset \mathbb{R}^{n+1}$*, such that whenever* $(k_0, \cdots, k_n) \in \Omega(R)$*, the solution of the the closed-loop system with initial state* $\|x(0)\| \leq R$ *will exist in* $[0, \infty)$ *and the regulation error* $e(t)$ *will converge to zero exponentially. Moreover, if Assumption (C) is replaced by Assumption (C'), then all the above statements still hold and there exists* $x^* \in \mathbb{R}^n$ *depending on* y^* *and* Φ *only, such that the state* $x(t)$ *is bounded and* $\lim_{t\to\infty} x(t) = x^*$.

Remark 5 *The concrete construction of the parameter manifold* $\Omega(R)$ *can be found in the proof of Theorem 1 to be given in Section IV. Moreover, from the proof of Theorem 1, we can also get an upper bound of* $e(t)$*: whenever* $(k_0, \cdots, k_n) \in \Omega(R)$*, we have for any* $t \geq 0$,

$$|e(t)| \leq \|\Phi(x(t)) - z^*\| \leq ce^{-\alpha t}\left\{\|\Phi(x(0)) - z^*\| + \tau_1(|y^*|)/(k_0\underline{b})\right\}, \qquad (16)$$

for any initial states satisfying $\|x(0)\| \leq R$*, where* $\alpha > 0$ *is a constant depends on* $(k_0, \cdots, k_n)$ *and* $c > 0$ *is a constant only depends on* n*. As a consequence, we have* $\lim_{t\to\infty} \Phi(x(t)) = z^*$*. Furthermore, if the initial state satisfies* $\Phi(x(0)) = z^*$*, then from (16), it is easy to see that* $\sup_{t\geq 0} |e(t)| \leq c\tau_1(|y^*|)/(k_0\underline{b})$*. Therefore, for any* $\epsilon > 0$*, we can get* $\sup_{t\geq 0} |e(t)| \leq \epsilon$ *as long as the integral parameter* k_0 *is suitably large.*

We remark that Assumption (B) is crucial for stabilization of (1) by the extended PID controller, though it may not be easy to be verified in general. However, if the norm of the coordinate transformation $\Phi(x)$ has a lower bound function and $\|\Phi(x)\|, |F(x)|, |G(x)|, \|\nabla F(x)\|, \|\nabla G(x)\|$ have some upper bound functions, then the verification can be considerably simplified. To this end, we introduce the following assumption on Φ and H to replace assumption (B), which does not depend on the setpoint y^*.

Assumption (B0): The functions Φ and H defined by (2) and (11) satisfy the following inequalities:

$$\|\Phi(x)\| + \|H(x)\| + \|J_\Phi^{-1}(x)\| + \|J_H(x)\| \leq \rho_1(\|x\|), \ \|\Phi(x)\| \geq \rho_0(\|x\|), \quad \forall x \in \mathbb{R}^n$$

where $\rho_0(\cdot), \rho_1(\cdot)$ are two known continuous increasing functions with $\lim_{r\to\infty} \rho_0(r) = \infty$.

By Theorem 1, we can get the following corollary which does not need Assumptions (C) or (C'):

Corollary 1 *Consider the SISO affine-nonlinear uncertain system* (1) *with the extended PID controller defined by (9). Suppose that Assumptions (A), (B0) are satisfied. Then for any setpoint* $y^* \in \mathbb{R}$ *and for any* $R > 0$*, there exists* $x^* \in \mathbb{R}^n$ *and an* $(n+1)$*-dimensional parameter manifold* $\Omega \subset \mathbb{R}^{n+1}$*, such that whenever* $(k_0, \cdots, k_n) \in \Omega$*, the solution of the closed-loop system with initial state* $\|x(0)\| \leq R$ *will exist in* $[0, \infty)$ *and the system state* $x(t)$ *will be bounded with* $\lim_{t\to\infty} x(t) = x^*$ *exponentially, and at the same time the regulation error* $e(t)$ *converges to* 0 *exponentially.*

We remark that the conditions used in the above corollary can be further simplified for the basic class of n^{th}-order uncertain chain of integrators (13), since in this case the coordinate transformation map $\Phi(x) = x$. The following corollary can be deduced from Corollary 1 immediately.

Corollary 2 *Consider the nonlinear uncertain system (13) with the extended PID controller (9). Then for any setpoint* y^* *and for any* $R > 0$*, there exists an open unbounded set* $\Omega \subset \mathbb{R}^{n+1}$*, such that whenever* $(k_0, \cdots, k_n) \in \Omega$*, the solution of the closed-loop system with initial state* $\|x(0)\| \leq R$ *will satisfy* $\lim_{t\to\infty} e(t) = 0$ *and* $\lim_{t\to\infty} x(t) = z^*$ *exponentially, provided that the nonlinear uncertain functions* $a(\cdot)$ *and* $b(\cdot)$ *satisfy Assumption (B').*

3.3 Global stabilization

In this section, we will show that the extended PID controller can globally stabilize the system (1) under some additional assumptions on the mappings Φ, F and G. Specifically, if $G(x)$(which is defined in (12)) has a constant upper bound $\overline{b}$ and the increasing function τ_2 in Assumption (B) is a linear function, i.e., $\tau_2(r) = Lr$, then the extended PID controller can globally stabilize (1) in the sense that: there exists an open unbounded parameter set $\Omega \subset \mathbb{R}^{n+1}$, such that the closed-loop system will satisfy $\lim_{t\to\infty} e(t) = 0$ for any initial state $x(0) \in \mathbb{R}^n$ as long as $(k_0, \cdots, k_n) \in \Omega$.

Let $\tau : [0, \infty) \to [0, \infty)$ be an increasing function and $L > 0$, $\overline{b} > 0$ be two constants. We introduce the following assumption to describe the model uncertainty.

Assumption (B1): The functions Φ and H defined respectively by (2) and (11) satisfy:

(i) $G(x) \leq \overline{b}$, $\|H(x)\| \leq \tau(\|\Phi(x)\|)$ for any $x \in \mathbb{R}^n$,

(ii) There exists $x^* \in \Phi^{-1}(z^*)$ such that

$$\|H(x) - H(x^*)\| \leq L\|\Phi(x) - \Phi(x^*)\|, \ \forall x \in \mathbb{R}^n.$$

Now, we have the following global results on the extended PID controller:

Theorem 2 *Consider the SISO affine-nonlinear uncertain system* (1) *with the extended PID controller defined by (9). Suppose that Assumptions (A), (B1) and (C) are satisfied. Then there exists an* $(n+1)$*-dimensional unbounded parameter manifold* $\Omega \subset \mathbb{R}^{n+1}$*, such that whenever* $(k_0, \cdots, k_n) \in \Omega$*, the solution of the closed-loop system will exist in* $[0, \infty)$ *for any initial state* $x(0) \in \mathbb{R}^n$ *and that the regulation error* $e(t)$ *will converge to* 0 *exponentially. Moreover, if Assumption (C) is replaced by Assumption (C'), then all the above statements still hold and there exists* $x^* \in \mathbb{R}^n$ *depending on* y^* *and* Φ *only, such that the state* $x(t)$ *is bounded and* $\lim_{t\to\infty} x(t) = x^*$.

Remark 6 *We first remark that, in contrast to the semi-global results established in Theorem 1, the global results in Theorem 2 do not require the parameters* $(k_0, \cdots, k_n)$ *of the extended PID depends on the range of initial state* $x(0)$*. Second, we note that from the proof of Theorem 2, the second condition (ii) in Assumption (B1) can be slightly weakened to* $|F(x)G(x^*) - F(x^*)G(x)| \leq L\|\Phi(x) - \Phi(x^*)\|,\ \forall x \in \mathbb{R}^n$*, for some* $L > 0$.

Now, let us give three typical examples to show how the conditions used in Theorem 2 can be further simplified if some additional information on the systems structure are available.

Example 1 ($Nonlinear\ canonical\ form$): Consider the nonlinear uncertain system (13) with the extended PID controller (9). Then for any setpoint $y^* \in \mathbb{R}$, there exists an $(n+1)$-dimensional unbounded parameter manifold $\Omega \subset \mathbb{R}^{n+1}$, such that whenever $(k_0, \cdots, k_n) \in \Omega$, the closed-loop system will satisfy

$$\lim_{t\to\infty} e(t) = 0,\ \ \lim_{t\to\infty} x(t) = z^*$$

for any initial state $x(0) \in \mathbb{R}^n$, provided that the nonlinear uncertain functions $a(\cdot)$ and $b(\cdot)$ satisfy the following conditions respectively:

$$\text{(i) } |a(x) - a(y)| \leq L\|x - y\|,\ \forall x, y \in \mathbb{R}^n \text{ and } \ |a(0)| \leq M$$

$$\text{(ii) } 0 < \underline{b} \leq b(x) \leq \overline{b},\ |b(x) - b(y)| \leq L\|x - y\|,\ \forall x, y \in \mathbb{R}^n,$$

where $\underline{b}, \overline{b}, L$ and M are known positive constants.

In the following two typical examples, we give some simple conditions for the classical PID to globally stabilize second order nonlinear uncertain systems, which generalize the nonlinear models investigated in [28].

Example 2 ($Second\ order\ system\ with\ strict-feedback\ form$): The following nonlinear uncertain system

$$\begin{cases} \dot{x}_1 = f_1(x_1) + x_2 \\ \dot{x}_2 = f_2(x_1, x_2) + u(t) \\ y(t) = x_1(t) \end{cases} \tag{17}$$

can be globally stabilized by the classical PID controller, provided that the nonlinear uncertain functions f_1 and f_2 satisfies $|f_1'(x_1)| \leq L, \forall x_1 \in \mathbb{R}$ and

$|f_2(x) - f_2(y)| \leq L\|x - y\|, \forall x, y \in \mathbb{R}^2$. This example also generalizes the corresponding results in [27].

Example 3 (*Second order system with pure − feedback form*): The following 2-dimensional SISO system

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) + g(x_1, x_2)u(t) \\ y(t) = x_1(t) \end{cases} \tag{18}$$

can be globally stabilized by the classical PID controller as long as the following conditions are satisfied: $\|\nabla f_i(x)\| \leq L$, $i = 1, 2$, $\|\nabla g(x)\| \leq L$, $\|\mathrm{Hess}(f_1)(x)\| \leq L$, $\frac{\partial f_1}{\partial x_2}(x) \geq \underline{b}, b_1 \leq g(x) \leq b_2$ for any $x = (x_1, x_2) \in \mathbb{R}^2$, and $|f_1(0)| + |f_2(0)| \leq M$, where $\mathrm{Hess}(f_1)$ is the Hessian matrix of f_1 and the constants $0 < \underline{b} \leq L$, $0 < b_1 \leq b_2$ and $M > 0$ are known.

3.4 Extended PID controller with differential trackers

From the definition of the extended PID controller (9), we know that the implementation of (9) needs the derivatives information $\dot{e}(t), \cdots, e^{(n-1)}(t)$ of the regulation error, if the system order $n \geq 2$. However, in most practical situations, these derivatives may not be available directly. Therefore, we need to construct a differential observer to obtain an online estimation of the derivatives of the regulation error.

In this subsection, we introduce the following high gain differential observers(see e.g., [7],[14]):

$$\begin{cases} \dot{\hat{z}}_1 &= \hat{z}_2 + \frac{\beta_1}{\epsilon}(e - \hat{z}_1) \\ &\vdots \\ \dot{\hat{z}}_{n-1} &= \hat{z}_n + \frac{\beta_{n-1}}{\epsilon^{n-1}}(e - \hat{z}_1) \\ \dot{\hat{z}}_n &= \frac{\beta_n}{\epsilon^n}(e - \hat{z}_1) \end{cases} \tag{19}$$

where the parameters $(\beta_1, \cdots, \beta_n) \in \mathbb{R}^n$ are given such that the polynomial $s^n + \beta_1 s^{n-1} + \cdots + \beta_n$ is Hurwitz and $\epsilon > 0$ is the observer gain parameter to be determined. We introduce the following differential observer-based extended PID controller:

$$u(t) = k_1 e(t) + k_0 \int_0^t e(s)ds + k_2 \hat{z}_2(t) + \cdots + k_n \hat{z}_n(t) \tag{20}$$

where $\hat{z}_i(t), i = 2, \cdots, n$ are the estimators of the derivatives $\frac{de(t)}{dt}, \cdots, \frac{d^{n-1}e(t)}{dt^{n-1}}$ of the regulation error respectively.

Now, let us consider the closed-loop system (1) with the extended PID controller defined by (19)-(20).

Theorem 3 *Consider the SISO affine-nonlinear uncertain system* (1)*, where* $u(t)$ *is the differential observer-based extended PID controller. Suppose that Assumptions (A), (B1) and (C) are satisfied. Then there exists an open unbounded* *set $\Omega \subset \mathbb{R}^{n+1}$, such that for any $(k_0, \cdots, k_n) \in \Omega$, there exists $\epsilon^* > 0$, such that for any $0 < \epsilon < \epsilon^*$, and for any initial states $x(0) \in \mathbb{R}^n$, $\hat{z}(0) \in \mathbb{R}^n$, the solution of the closed-loop system will exist in $[0, \infty)$ and the regulation error $e(t)$ will converge to zero exponentially. Moreover, if Assumption (C) is replaced by Assumption (C'), then all the above statements still hold and there exists $x^* \in \mathbb{R}^n$ depending on y^* and Φ only, such that the state $x(t)$ is bounded and $\lim_{t\to\infty} x(t) = x^*$.*

Remark 7 *First, we remark that from the proof of Theorem 3 to be given in the next section, a precise upper bound on the regulation error can be obtained. Second, we remark that semi-global results like Theorem 1 can also obtained, which will be discussed in details elsewhere.*

4 Proofs of the main results

Before proving the theorems, we first list some lemmas, whose proofs are presented in Appendix A.

Denote $\lambda \triangleq (\lambda_0, \cdots, \lambda_n) \in \mathbb{R}^{n+1}$ and define an open unbounded set $\Omega_1 \subset \mathbb{R}^{n+1}$ as follows:

$$\Omega_1 = \{\lambda \big| 2 < \lambda_i - 2i < 3, i = 0, \cdots, n-1; \lambda_n > 2n+2\} \tag{21}$$

and for $\lambda \in \Omega_1$, we define a $(n+1) \times (n+1)$ matrix $P = P(\lambda)$ as follows (see [3]):

$$P = \begin{bmatrix} (-\lambda_0)^{-n} & \cdots & (-\lambda_n)^{-n} \\ \vdots & & \vdots \\ (-\lambda_0)^{-1} & \cdots & (-\lambda_n)^{-1} \\ 1 & \cdots & 1 \end{bmatrix} \tag{22}$$

and denote $(d_0, \cdots, d_n)^T$ be the last column of P^{-1}, i.e.,

$$(d_0, \cdots, d_n)^T = P^{-1}(0, \cdots, 0, 1)^T \tag{23}$$

Lemma 1 *Under the above notations, let us define*

$$c_1 \triangleq \sup_{\lambda\in\Omega_1} \|P\|, \quad c_2 \triangleq \sup_{\lambda\in\Omega_1} \|P\|\|P^{-1}\|, c_3 \triangleq \sup_{\lambda\in\Omega_1} \sqrt{n}(2n+1)d_n \tag{24}$$

$$c_4(i) \triangleq \sup_{\lambda\in\Omega_1} |(2n+1)n\lambda_n d_i| \, , i = 0, \cdots, n-1, \tag{25}$$

and denote

$$c_0 = \max\{c_1, c_2, c_3, c_4(i), i = 0, \cdots, n-1\}, \tag{26}$$

then c_0 is a positive finite number.

Note that Lemma 1 is nontrivial because λ_n together with Ω_1 are unbounded. To introduce other lemmas, we now define a parameter manifold first. Let $c \geq c_0$ be any constant. For $L > 0$ and $0 < \underline{b} \leq \overline{b}$, we define the following $n+1$ dimensional parameter manifold $\Omega_{L,\underline{b},\overline{b},c} \subset \mathbb{R}^{n+1}$(which is open and unbounded in $\mathbb{R}^{n+1}$),

$$\Omega_{L,\underline{b},\overline{b},c} \triangleq \left\{ \begin{bmatrix} k_0 \\ \vdots \\ k_n \end{bmatrix} \middle| \begin{bmatrix} k_0 \\ \vdots \\ k_{n-1} \\ k_n \end{bmatrix} = \frac{1}{\underline{b}} \begin{bmatrix} \prod_{i=0}^{n} \lambda_i \\ \vdots \\ \sum_{i<j}^{n} \lambda_i \lambda_j \\ \sum_{i=0}^{n} \lambda_i \end{bmatrix}, \lambda \in \Omega_\Lambda \right\} \tag{27}$$

where Ω_Λ is defined by

$$\Omega_\Lambda = \left\{ \lambda \in \Omega_1 \middle| \lambda_n > \left(Lc^2 + \frac{(\overline{b} - \underline{b})\, c}{\underline{b}} \right)^2 + Lc^2 \right\} \tag{28}$$

In the following lemmas, the constant T can be a finite positive number $0 < T < \infty$ or an infinity $T = \infty$.

Lemma 2 *Let $\overline{Y}(t) = (y_0(t), \cdots, y_n(t))^T$ be a continuously differentiable vector valued function on $[0, T)$. Suppose that there exists a_t and b_t such that the following equalities hold for $t \in [0, T)$,*

$$\begin{cases} \dot{y}_0 & = y_1 \\ & \vdots \\ \dot{y}_{n-1} & = y_n \\ \dot{y}_n & = a_t - b_t(k_0 y_0 + \cdots + k_n y_n) \end{cases} \tag{29}$$

where $|a_t| \leq L \left\| \overline{Y}(t) \right\|$ and $0 < \underline{b} \leq b_t \leq \overline{b}$, for any $t \in [0, T)$. Then for any $(k_0, \cdots, k_n) \in \Omega_{L,\underline{b},\overline{b},c}$(where $\Omega_{L,\underline{b},\overline{b},c}$ is defined in (27)), there exists $\alpha > 0$, such that $\overline{Y}(t)$ satisfies

$$\left\| \overline{Y}(t) \right\| \leq c e^{-\alpha t} \left\| \overline{Y}(0) \right\|, \quad \forall t \in [0, T)$$

Lemma 3 *Consider the system of equalities (29) again, but where $|a_t| \leq \tau_2(\left\| \overline{Y}(t) \right\|)$ and $0 < \underline{b} \leq b_t \leq \tau_1(\left\| \overline{Y}(t) \right\|)$, for any $t \in [0, T)$ and where $\tau_1, \tau_2 : \mathbb{R}^+ \to \mathbb{R}^+$ are two increasing functions with $\limsup_{\rho \to 0} \frac{\tau_2(\rho)}{\rho} < \infty$. Then for any $R > 0$, and any $(k_0, \cdots, k_n) \in \Omega_{L_0,\underline{b},b_0,c}$ with $L_0 = \sup_{0 \leq \rho \leq cR} \frac{\tau_2(\rho)}{\rho}$, $b_0 = \tau_1(cR)$, there exists $\alpha > 0$, such that $\overline{Y}(t)$ satisfies*

$$\left\| \overline{Y}(t) \right\| \leq c e^{-\alpha t} \left\| \overline{Y}(0) \right\|, \forall t \in [0, T)$$

provided that $\left\| \overline{Y}(0) \right\| \leq R$.

Lemma 4 *Let $\overline{Y}(t) = (y_0(t), \cdots, y_n(t))^T$, $\xi(t) = (\xi_1(t), \cdots, \xi_n(t))^T$ be two continuously differentiable vector valued functions on $[0, T)$. Suppose that $\forall t \in [0, T)$, there exists a_t and b_t such that the following equalities hold,*

$$\begin{cases} \dot{y}_0 &= y_1 \\ &\vdots \\ \dot{y}_n &= a_t - b_t\left(\sum_{i=0}^n k_i y_i - \sum_{i=2}^n k_i \epsilon^{n-i}\xi_i\right) \\ \dot{\xi} &= \frac{1}{\epsilon}B\xi + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_t - b_t\left(\sum_{i=0}^n k_i y_i - \sum_{i=2}^n k_i \epsilon^{n-i}\xi_i\right) \end{bmatrix} \end{cases} \tag{30}$$

where $|a_t| \leq L\left\|\overline{Y}(t)\right\|$ and $0 < \underline{b} \leq b_t \leq \overline{b}$ for any $t \in [0, T)$, and where B is a Hurwitz matrix. Then for any $(k_0, \cdots, k_n) \in \Omega_{L,\underline{b},\overline{b},c}$, there exists $\epsilon^* > 0$, such that for any $0 < \epsilon < \epsilon^*$, $Y(t)$ and $\xi(t)$ satisfy

$$\|Y(t)\| \leq ce^{-\beta t}(\|Y(0)\| + |y_0(0)| + \sqrt{2\lambda_{\max}(Q)}\|\xi(0)\|),$$

$$\|\xi(t)\| \leq \frac{c\left\|\overline{Y}(0)\right\| + \sqrt{\lambda_{\max}(Q)}\|\xi(0)\|}{\sqrt{\lambda_{\min}(Q)}}e^{-\beta t}, \forall t \in [0, T),$$

for some $\beta > 0$, where Q is the unique positive definite matrix satisfying $B^TQ + QB = -I$.

Remark 8 *Note that in Lemmas 2-4, we do not need the uncertain functions a_t and b_t to have a fixed form, nor other conditions excepts some upper bound functions, thanks to the design of the extended PID parameters and to the strong robustness of the PID controller. This enables us to establish global or semi-global results on output regulation of general SISO affine nonlinear uncertain systems by using its (local) nonlinear canonical form in the analysis, as will be demonstrated shortly in the proofs of the theorems.*

To prove our theorems, we also need the following result:

Theorem A1. Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable. Then Φ is a global diffeomorphism if and only if the Jacobian matrix $J_\Phi(x)$ is nonsingular for all $x \in \mathbb{R}^n$ and $\lim_{\|x\|\to\infty} \|\Phi(x)\| = \infty$.

See [21] and [24] for detailed discussion.

Proof of Theorem 1.

Step 1. First, let us denote

$$z(t) = (z_1(t), \cdots, z_n(t))^T \stackrel{\triangle}{=} \Phi(x(t)) \tag{31}$$

By Assumption (A), we know that the system (1) has uniform relative degree

n, thus we have (see e.g., [8]),

$$\begin{cases} \dot{z}_1 & = z_2 \\ & \vdots \\ \dot{z}_{n-1} & = z_n \\ \dot{z}_n & = F(x) + G(x)u \end{cases} \tag{32}$$

where $z_1(t) = h(x(t)) = y(t)$. Denote $y_0(t) \triangleq -\int_0^t e(s)ds - \frac{F(x^*)}{k_0 G(x^*)}$, $y_1(t) \triangleq -e(t) = -(y^* - y(t)) = z_1(t) - y^*$, $y_2(t) \triangleq -\dot{e}(t) = z_2(t)$, $\cdots$, $y_n(t) \triangleq -e^{(n-1)}(t) = z_n(t)$. Then we have

$$\begin{cases} \dot{y}_0 & = y_1 \\ & \vdots \\ \dot{y}_{n-1} & = y_n \\ \dot{y}_n & = F(x) + G(x)u \end{cases} \tag{33}$$

where $y_0(0) = -\frac{F(x^*)}{k_0 G(x^*)}$. Since $e(t) = y^* - y(t) = y^* - z_1(t)$, we can rewrite $u(t)$ as follows:

$$\begin{aligned} u(t) =& k_1 e(t) + k_0 \int_0^t e(s)ds + k_2 \dot{e}(t) + \cdots + k_n e^{(n-1)}(t) \\ =& -\left(\sum_{i=0}^n k_i y_i(t) + \frac{F(x^*)}{G(x^*)} \right) \end{aligned} \tag{34}$$

Denote $a_t \triangleq F(x(t)) - \frac{F(x^*)}{G(x^*)} G(x(t))$, $b_t \triangleq G(x(t))$. From (33)-(34), we have

$$\begin{cases} \dot{y}_0 & = y_1 \\ & \vdots \\ \dot{y}_{n-1} & = y_n \\ \dot{y}_n & = a_t - b_t \sum_{i=0}^n k_i y_i. \end{cases} \tag{35}$$

Denote

$$Y(t) \triangleq (y_1(t), \cdots, y_n(t))^T, \ \overline{Y}(t) \triangleq (y_0(t), \cdots, y_n(t))^T,$$

it is easy to see that $Y(t) = z(t) - z^* = \Phi(x(t)) - \Phi(x^*)$, where z^* is defined in (10).

Step 2. Next, we will apply Lemma 3 to prove that if the initial state $\|x(0)\| \leq R$ and the parameters $(k_0, \cdots, k_n) \in \Omega_{L_0, \underline{b}, b_0, c}$, where $b_0 \triangleq \tau_1(cR_0 + |y^*|)$, $L_0 \triangleq \sup_{0 \leq \rho \leq R_0} \frac{\tau_1(|y^*|) + b}{\underline{b}} \frac{\tau_2(\rho)}{\rho}$ and $R_0 \triangleq \tau_1(R) + |y^*| + \tau_1(|y^*|)$, then there exists $\alpha > 0$, such that

$$\left\|\overline{Y}(t)\right\| \leq c e^{-\alpha t} \left\|\overline{Y}(0)\right\|, \quad t \in [0, T), \tag{36}$$

where $[0,T)$ is the maximal existence interval of the solution of the closed-loop system (1) and (9) with initial state $x(0)$.

By Assumption (B), we have $\|H(x^*)\| \le \tau_1(\|\Phi(x^*)\|) = \tau_1(\|z^*\|) = \tau_1(|y^*|)$. Therefore, $|F(x^*)| \le \|H(x^*)\| \le \tau_1(|y^*|)$ and we can obtain

$$\begin{aligned}
&|a_t| = |F(x(t)) - F(x^*)G(x(t))/G(x^*)| \\
\le& |F(x(t)) - F(x^*)| + |F(x^*)||G(x(t)) - G(x^*)|/G(x^*) \\
\le& |F(x(t)) - F(x^*)| + \tau_1(|y^*|)|G(x(t)) - G(x^*)|/\underline{b} \\
\le& (\underline{b} + \tau_1(|y^*|))\|H(x(t)) - H(x^*)\|/\underline{b} \le (\underline{b} + \tau_1(|y^*|))\tau_2(\|\Phi(x(t)) - \Phi(x^*)\|)/\underline{b} \\
=& (\underline{b} + \tau_1(|y^*|))\tau_2(\|Y(t)\|)/\underline{b} \le (\underline{b} + \tau_1(|y^*|))\tau_2\left(\left\|\overline{Y}(t)\right\|\right)/\underline{b}.
\end{aligned} \tag{37}$$

On the other hand, by Assumptions (A) and (B), it is easy to see that

$$\begin{aligned}
0 < \underline{b} \le b_t =& G(x(t)) \le \|H(x(t))\| \le \tau_1(\|\Phi(x(t))\|) \\
=& \tau_1(\|Y(t) + z^*\|) \le \tau_1(\|Y(t)\| + |y^*|) \le \tau_1(\left\|\overline{Y}(t)\right\| + |y^*|).
\end{aligned} \tag{38}$$

Since $\|x(0)\| \le R$, then by Assumption (B), we have

$$\|Y(0)\| = \|\Phi(x(0)) - z^*\| \le \tau_1(\|x(0)\|) + \|z^*\| \le \tau_1(R) + |y^*|.$$

Recall that $y_0(0) = -\frac{F(x^*)}{k_0 G(x^*)}$, we have

$$\left\|\overline{Y}(0)\right\| \le \|Y(0)\| + |y_0(0)| \le \tau_1(R) + |y^*| + \frac{\tau_1(|y^*|)}{k_0 G(x^*)} \le \tau_1(R) + |y^*| + \tau_1(|y^*|),$$

where the last inequality holds since $k_0 = \frac{\prod_{i=0}^{n}\lambda_i}{\underline{b}} \ge \frac{1}{\underline{b}}$ and $G(x^*) \ge \underline{b}$. By Lemma 3, we know that (36) holds for any $(k_0, \cdots, k_n) \in \Omega_{L_0,\underline{b},b_0,c}$ as long as the initial state $\|x(0)\| \le R$.

Step 3. In this step, we will show that if Assumption (C) holds, then the maximal existence interval of the solution of the closed-loop system (1) and (9) is $[0,\infty)$.

We use the contradiction argument. Suppose that the solution of the system (1) with controller (9) only exists in $[0,T)$ for $T < \infty$ for some initial state $\|x(0)\| \le R$. Denote $x_0(t) = \int_0^t y^* - h(x(s))ds$, then $x_0(0) = 0$ and by simple calculations, the extended PID controller (9) can be rewritten as $u(t) = k_0 x_0(t) + k_1(y^* - h(x(t))) - k_2 L_f h(x(t)) - \cdots - k_n L_f^{n-1} h(x(t))$. Therefore, the maximal existence interval of the solution of the following $(n+1)^{\text{th}}$ order autonomous differential equation is also finite:

$$\begin{cases} \dot{x}_0 &= y^* - h(x) \\ \dot{x} &= f(x) + g(x)(k_0 x_0 + k_1 y^* - \sum_{j=1}^{n} k_j L_f^{j-1} h(x)) \end{cases} \tag{39}$$

with the initial value $[0, x^T(0)] \in \mathbb{R}^{n+1}$. Then it is well-known from the theory of ordinary differential equations that

$$\limsup_{0 \le t < T} \left\|[x_0(t), x(t)^T]\right\| = \infty. \tag{40}$$

By step 2, we know that $\left\|\overline{Y}(t)\right\|$ is bounded on $t \in [0, T)$. Hence, by (37),(38), the right hand of (35) and the boundedness of $\left\|\overline{Y}(t)\right\|$, it is not difficult to conclude that

$$\left\|\dot{\overline{Y}}(t)\right\| \leq N, \quad \forall t \in [0, T)$$

for some constant $N > 0$(possibly depends on the initial state $x(0)$).

On the other hand, since $z(t) = \Phi(x(t))$, we have $\dot{z}(t) = J_\Phi(x(t))\dot{x}(t)$ and therefore we can obtain

$$\begin{aligned} \|\dot{x}(t)\| &= \left\|J_\Phi^{-1}(x(t))\dot{z}(t)\right\| \leq \|J_\Phi^{-1}(x(t))\|\|\dot{z}(t)\| = \|J_\Phi^{-1}(x(t))\| \left\|\dot{Y}(t)\right\| \\ &\leq \|J_\Phi^{-1}(x(t))\| \left\|\dot{\overline{Y}}(t)\right\| \leq N \left\|J_\Phi^{-1}(x(t))\right\|, \quad t \in [0, T). \end{aligned} \tag{41}$$

By Assumption (C), we have

$$\|\dot{x}(t)\| \leq \alpha_1 \|x(t)\| \log \|x(t)\| + \alpha_2 \tag{42}$$

for any $t \in [0, T)$, where $\alpha_1 = NN_1$ and $\alpha_2 = NN_2$. Denote $v(t) \overset{\triangle}{=} \|x(t)\|$ and $D^+v(t) \overset{\triangle}{=} \limsup_{h\to 0^+} \frac{v(t+h)-v(t)}{h}$ be the upper right-hand derivative of $v(t)$.

Then it is not difficult to obtain

$$\begin{aligned} D^+v(t) \leq & |D^+v(t)| = \limsup_{h\to 0^+} \left|\frac{\|x(t+h)\| - \|x(t)\|}{h}\right| \leq \limsup_{h\to 0^+} \frac{\|x(t+h) - x(t)\|}{h} \\ &= \limsup_{h\to 0^+} \left\|\frac{x(t+h) - x(t)}{h}\right\| = \|\dot{x}(t)\| \end{aligned}$$

Noticing that $v(t) = \|x(t)\|$, from (42), we have

$$D^+v(t) \leq \|\dot{x}(t)\| \leq \alpha_1 v(t) \log v(t) + \alpha_2, \quad t \in [0, T)$$

By the comparison lemma in ordinary differential equations(see e.g., [13]), we have

$$\int_{v(0)}^{v(t)} \frac{d\eta}{\alpha_1 \eta \log \eta + \alpha_2} \leq t < T, \quad \forall t \in [0, T),$$

which implies

$$\sup_{0\leq t<T} v(t) = \sup_{0\leq t<T} \|x(t)\| < \infty \tag{43}$$

since $\int_{v(0)}^{\infty} \frac{d\eta}{\alpha_1 \eta \log \eta + \alpha_2} = \infty$. By (43) and the fact $T < \infty$, it is not difficult to see that

$$\sup_{0\leq t<T} |x_0(t)| = \sup_{0\leq t<T} \left|\int_0^t y^* - h(x(s))ds\right| < \infty.$$

Therefore, the solution of (39) with initial state $[0, x^T(0)]$ satisfy

$$\sup_{0\leq t<T} \left\|[x_0(t), x^T(t)]\right\| < \infty,$$

which contradicts to (40).

Therefore, under Assumptions (A),(B) and (C), if $(k_0, \cdots, k_n) \in \Omega_{L_0,\underline{b},b_0,c}$, then for any initial state $\|x(0)\| \le R$, the solution of the closed-loop system will exist in $[0,\infty)$.

Step 4. If Assumption (C) is replaced by Assumption (C'), then the maximal existence interval of the solution of the closed-loop system (1) and (9) is also $[0,\infty)$.

We use the contradiction argument again. Suppose that the solution of the system (1) with controller (9) only exists in $[0,T)$ for $T<\infty$ and for some initial state $\|x(0)\| \le R$. By step 2, we know that $Y(t)$ is bounded on $[0,T)$. Notice that $Y(t) = \Phi(x(t)) - z^*$, we know that $\Phi(x(t))$ is bounded on $[0,T)$. Since Φ is a global diffemorphism on $\mathbb{R}^n$, by Theorem (A1), we know that $x(t)$ is bounded on $[0,T)$. Similarly, we have $\sup_{0\le t<T} \left\|[x_0(t), x(t)^T]\right\| < \infty$, which contradicts to (40).

Step 5. Since solution of the closed-loop equation exists in $[0,\infty)$, we conclude that (37) and (38) are satisfied in $[0,\infty)$. By using Lemma 3 again, we have $\left\|\overline{Y}(t)\right\| \le ce^{-\alpha t}\left\|\overline{Y}(0)\right\|$ for any $t \in [0,,\infty)$.

Therefore, we have

$$\begin{aligned}|e(t)| = |y_1(t)| &\le \|Y(t)\| = \|\Phi(x(t)) - z^*\| \le \left\|\overline{Y}(t)\right\| \\ &\le ce^{-\alpha t}\left\|\overline{Y}(0)\right\| \le ce^{-\alpha t}\left(\|Y(0)\| + \left|\frac{F(x^*)}{k_0 G(x^*)}\right|\right) \\ &\le ce^{-\alpha t}\left(\|\Phi(x(0)) - z^*\| + \left|\frac{F(x^*)}{k_0 G(x^*)}\right|\right) \\ &= ce^{-\alpha t}\left(\|\Phi(x(0)) - z^*\| + \frac{\tau_1(|y^*|)}{k_0\underline{b}}\right)\end{aligned}$$

for any $t \in [0,\infty)$.

If Assumption (C') is satisfied, then we can see that $\Phi^{-1}(z^*)$ only has one element, denote it as x^*. It is not difficult to obtain $\lim_{t\to\infty} x(t) = x^*$. As a consequence, the system state $x(t)$ is bounded. This completes the proof of Theorem 1. □

Proof of Corollary 1.

By Assumption (A), we know that $J_\Phi(x)$ is invertible for any $x \in \mathbb{R}^n$. From Assumption (B0), we have $\lim_{\|x\|\to\infty} \|\Phi(x)\| \ge \lim_{\|x\|\to\infty} \rho_0(\|x\|) = \infty$. By using Theorem A1, we know that Φ is a global diffeomorphism on $\mathbb{R}^n$, i.e., Assumption (C') is satisfied. Let y^* be any given setpoint, then $\Phi^{-1}(z^*)$ is not empty and $\Phi^{-1}(z^*)$ only has one element, we denote it as x^*.

Now, we proceed to verify Assumption (B), i.e., to find two increasing functions τ_1, τ_2 with $\limsup_{\rho\to 0} \frac{\tau_2(\rho)}{\rho} < \infty$ such that the following inequalities

$$\|\Phi(x)\| \le \tau_1(\|x\|),\ \|H(x)\| \le \tau_1(\|\Phi(x)\|), \tag{44}$$

$$\|H(x) - H(x^*)\| \le \tau_2(\|\Phi(x) - \Phi(x^*)\|) \tag{45}$$

hold for any $x \in \mathbb{R}^n$.

Since $\rho_0(\|x\|) \leq \|\Phi(x)\|$, it is easy to obtain

$$\|x\| \leq \rho_0^{-1}(\|\Phi(x)\|), \forall x \in \mathbb{R}^n, \tag{46}$$

where $\rho_0^{-1}(r) \triangleq \sup\{y \geq 0 | \rho_0(y) \leq r\}$ for $r \geq 0$. From (46), we have

$$\|\Phi^{-1}(z)\| \leq \rho_0^{-1}(\|z\|), \forall z \in \mathbb{R}^n. \tag{47}$$

By Assumption (B0) and (46), we have

$$\|H(x)\| \leq \rho_1(\|x\|) \leq \rho_1 \circ \rho_0^{-1}(\|\Phi(x)\|), \tag{48}$$

where $\rho_1 \circ \rho_0^{-1}$ denotes the composition of the functions ρ_1 and ρ_0^{-1}. Therefore, we conclude that (44) holds with

$$\tau_1(r) = \max\{\rho_1(r), \rho_1 \circ \rho_0^{-1}(r)\}.$$

From (46), we have $\|x^*\| \leq \rho_0^{-1}(\|\Phi(x^*)\|) = \rho_0^{-1}(\|z^*\|) \leq \rho_0^{-1}(|y^*|) \triangleq M_0$. Next, we proceed to estimate the upper bound of $\|H(x) - H(x^*)\|$. For this, we need to prove the following statement first: Let $U \subset \mathbb{R}^n$ be a convex open set. $f : U \to \mathbb{R}^m$ is a continuously differentiable vector valued function. Then for any $x_1, x_2 \in U$, we have

$$\|f(x_1) - f(x_2)\| \leq \sup_{0 \leq \theta \leq 1} \|J_f(x_2 + \theta(x_1 - x_2)\| \, \|x_1 - x_2\| \tag{49}$$

The proof is elementary. Since U is convex, then $g(\theta) = f(x_2 + \theta(x_1 - x_2))$ is a continuously differentiable function of $0 \leq \theta \leq 1$. Note that $f(x_1) - f(x_2) = g(1) - g(0) = \int_0^1 g'(\theta) d\theta = \int_0^1 J_f(x_2 + \theta(x_1 - x_2))(x_1 - x_2) d\theta$. Thus, we have

$$\begin{aligned}\|f(x_1) - f(x_2)\| &\leq \int_0^1 \|J_f(x_2 + \theta(x_1 - x_2))\| \, d\theta \|x_1 - x_2\| \\ &\leq \sup_{0 \leq \theta \leq 1} \|J_f(x_2 + \theta(x_1 - x_2))\| \, \|x_1 - x_2\|\end{aligned}$$

By (49), we know that

$$\|H(x) - H(x^*)\| \leq \sup_{0 \leq \theta \leq 1} \|J_H(x^* + \theta(x - x^*))\| \, \|x - x^*\|.$$

Since $\|J_H(x)\| \leq \rho_1(\|x\|)$, $\|x^*\| \leq M_0$ and $0 \leq \theta \leq 1$, therefore we have

$$\|H(x) - H(x^*)\| \leq \rho_1(M_0 + \|x - x^*\|)\|x - x^*\|. \tag{50}$$

On the other hand, from the identity $\Phi^{-1} \circ \Phi(x) = x$, we have $J_{\Phi^{-1}}(\Phi(x)) J_\Phi(x) = I$, i.e.,

$$J_{\Phi^{-1}}(\Phi(x)) = J_\Phi^{-1}(x), \quad \forall x \in \mathbb{R}^n, \tag{51}$$

where $J_{\Phi^{-1}}$ denotes the Jacobian matrix of the mapping Φ^{-1}. By using (49) again, we have

$$\begin{aligned}\|x-x^*\| =&\|\Phi^{-1}(\Phi(x))-\Phi^{-1}(\Phi(x^*))\| \\ \leq& \sup_{0\leq\theta\leq1}\|J_{\Phi^{-1}}(z_\theta)\|\,\|\Phi(x)-\Phi(x^*)\| \end{aligned} \tag{52}$$

where $z_\theta \triangleq \theta(\Phi(x)-\Phi(x^*))+\Phi(x^*)$. Combine (51) and (52), we have

$$\|x-x^*\| \leq \sup_{0\leq\theta\leq1}\left\|J_\Phi^{-1}(\Phi^{-1}(z_\theta))\right\|\|\Phi(x)-\Phi(x^*)\| \tag{53}$$

Since $\|z_\theta\| \leq \|\Phi(x)-\Phi(x^*)\|+\|\Phi(x^*)\| = \|\Phi(x)-\Phi(x^*)\|+|y^*|$, from (47), we have

$$\|\Phi^{-1}(z_\theta)\| \leq \rho_0^{-1}(\|\Phi(x)-\Phi(x^*)\|+|y^*|). \tag{54}$$

By Assumption (B0), we have $\left\|J_\Phi^{-1}(x)\right\| \leq \rho_1(\|x\|)$, combine (53) and (54), we can obtain

$$\|x-x^*\| \leq \rho_2(\|\Phi(x)-\Phi(x^*)\|), \tag{55}$$

where $\rho_2(r) \triangleq \left[\rho_1\circ\rho_0^{-1}(r+|y^*|)\right]r$.

Denote $\tau_2(r) \triangleq \rho_1(M_0+\rho_2(r))\rho_2(r)$, then from (50) and (55), we have

$$\|H(x)-H(x^*)\| \leq \tau_2(\|\Phi(x)-\Phi(x^*)\|). \tag{56}$$

Since

$$\limsup_{r\to0}\frac{\rho_2(r)}{r} = \limsup_{r\to0}\rho_1\circ\rho_0^{-1}(r+|y^*|) < \infty,$$

we arrive at $\limsup_{r\to0}\frac{\tau_2(r)}{r} < \infty$.

Therefore, all the conditions in Assumption (B) are satisfied. Consequently, all results in Theorem 1 hold.

Finally, we show that $x(t)$ has exponential convergence rate. By the proof of Theorem 1, we know that $\|\Phi(x(t))-z^*\| \leq ce^{-\alpha t}\big(\|\Phi(x(0))-z^*\|+\frac{\tau_1(|y^*|)}{k_0\underline{b}}\big)$. As a consequence, $\Phi(x(t))$ will converge to z^* exponentially. From (55), we have $\|x(t)-x^*\| \leq \rho_2(\|\Phi(x(t))-z^*\|)$. Recall that $\limsup_{r\to0}\frac{\rho_2(r)}{r} < \infty$, we see that the trajectory $x(t)$ is bounded and $\lim_{t\to\infty}x(t)=x^*$ exponentially.□

Proof of Theorem 2. We will use the same notations as in the proof of Theorem 1, then

$$\begin{cases}\dot{y}_0 &= y_1\\ &\vdots\\ \dot{y}_n &= a_t - b_t(k_0y_0+\cdots+k_ny_n),\end{cases} \tag{57}$$

where $a_t = F(x(t)) - \frac{F(x^*)}{G(x^*)}G(x(t))$, $b_t = G(x(t))$ and $y_0(0) = -\frac{F(x^*)}{k_0G(x^*)}$.

By Assumption (B1), we have $|F(x^*)| \leq \tau(\|\Phi(x^*)\|) = \tau(|y^*|)$. Denote $\tilde{L} = \frac{L(\tau(|y^*|)+\underline{b})}{\underline{b}}$. By $\|H(x) - H(x^*)\| \leq L\|\Phi(x) - \Phi(x^*)\|$, we can obtain

$$
\begin{aligned}
|a_t| \leq & |F(x(t)) - F(x^*)| + \left|\frac{F(x^*)}{G(x^*)}\right| |G(x(t) - G(x^*)| \\
\leq & \frac{\tau(|y^*|) + \underline{b}}{\underline{b}} \|H(x(t)) - H(x^*)\| \leq \tilde{L}\|\Phi(x(t)) - z^*\| \\
= & \tilde{L}\|z(t) - z^*\| = \tilde{L}\|Y(t)\| \leq \tilde{L} \left\|\overline{Y}(t)\right\|,
\end{aligned} \tag{58}
$$

and from $0 < \underline{b} \leq G(x) \leq \overline{b}$, we have

$$
0 < \underline{b} \leq b_t = G(x(t)) \leq \overline{b}.
$$

By Lemma 2, we know that $\left\|\overline{Y}(t)\right\| \leq c \left\|\overline{Y}(0)\right\|$ for $t \in [0, T)$ if the parameters $(k_0, \cdots, k_n) \in \Omega_{\tilde{L}, \underline{b}, \overline{b}, c}$, where $[0, T)$ is the maximal existence interval of the solution of the closed-loop system (1) and (9).

Similar to the proof of Theorem 1, we know that either Assumption (C) or (C') holds, then the solution of the closed-loop system will exist in $[0, \infty)$ for any initial state $x(0) \in \mathbb{R}^n$, whenever $(k_0, \cdots, k_n) \in \Omega_{\tilde{L}, \underline{b}, \overline{b}, c}$. By using Lemma 2 with $T = \infty$, we have $\left\|\overline{Y}(t)\right\| \leq ce^{-\alpha t} \left\|\overline{Y}(0)\right\|$ for any $t \geq 0$ and for some $\alpha > 0$. Therefore,

$$
|e(t)| \leq \left\|\overline{Y}(t)\right\| \leq ce^{-\alpha t}\|\overline{Y}(0)\| \leq ce^{-\alpha t}(\|Y(0)\| + \frac{\tau(|y^*|)}{k_0\underline{b}}),
$$

for any $t \in [0, \infty)$, which implies $e(t)$ converge to 0 exponentially.

The proof of the convergence of the state $x(t)$ under Assumption (C') is similar to that of Theorem 1, which will not be repeated .□

Proof of Theorem 3.

Denote $y_0(t) = -\int_0^t e(s)ds - \frac{F(x^*)}{k_0 G(x^*)}$, $y_1(t) = -e(t) = z_1(t) - y^*, \cdots, y_n(t) = -e^{(n-1)}(t) = z_n(t)$, $Y(t) = (y_1(t), \cdots, y_n(t))^T = z(t) - z^*$, $\xi_i(t) = \frac{y_i(t) + \hat{z}_i(t)}{\epsilon^{n-i}}$, $i = 1, \cdots, n$. Then the differential observer-based extended PID controller (19)-(20) can be rewritten as

$$
\begin{aligned}
u(t) = & k_1 e(t) + k_0 \int_0^t e(s)ds + k_2 \hat{z}_2(t) + \cdots + k_n \hat{z}_n(t) \\
= & -\sum_{i=0}^{n} k_i y_i(t) + \sum_{i=2}^{n} k_i \epsilon^{n-i} \xi_i(t) - \frac{F(x^*)}{G(x^*)}.
\end{aligned}
$$

Consequently, we have the following equation:

$$\begin{cases} \dot{y}_0 & = y_1 \\ & \vdots \\ \dot{y}_{n-1} & = y_n \\ \dot{y}_n & = a_t - b_t \left(\sum_{i=0}^{n} k_i y_i - \sum_{i=2}^{n} k_i \epsilon^{n-i} \xi_i \right) \\ \dot{\xi}_1 & = -\frac{\beta_1}{\epsilon}\xi_1 + \frac{\xi_2}{\epsilon} \\ & \vdots \\ \dot{\xi}_{n-1} & = -\frac{\beta_{n-1}}{\epsilon}\xi_1 + \frac{\xi_n}{\epsilon} \\ \dot{\xi}_n & = -\frac{\beta_n}{\epsilon}\xi_1 + a_t - b_t \left(\sum_{i=0}^{n} k_i y_i - \sum_{i=2}^{n} k_i \epsilon^{n-i} \xi_i \right) \end{cases} \tag{59}$$

where $a_t = F(x(t)) - \frac{F(x^*)}{G(x^*)} G(x(t))$, $b_t = G(x(t))$.

Denote $B \triangleq \begin{bmatrix} -\beta_1 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ -\beta_{n-1} & 0 & 0 & \cdots & 1 \\ -\beta_n & 0 & 0 & \cdots & 0 \end{bmatrix}$, and $\xi(t) \triangleq (\xi_1(t), \cdots, \xi_n(t))^T$.

Then (59) turns into a more compact form:

$$\begin{cases} \dot{y}_0 = y_1 \\ \quad \vdots \\ \dot{y}_n = a_t - b_t \left(\sum_{i=0}^{n} k_i y_i - \sum_{i=2}^{n} k_i \epsilon^{n-i} \xi_i \right) \\ \dot{\xi} = \frac{1}{\epsilon} B\xi + \left(0, \cdots, a_t - b_t \left(\sum_{i=0}^{n} k_i y_i - \sum_{i=2}^{n} k_i \epsilon^{n-i} \xi_i \right)\right)^T \end{cases} \tag{60}$$

By Assumption (B1), and from the proof of Theorem 2, we have

$$|a_t| \le \tilde{L}\|Y(t)\|, \ \underline{b} \le b_t = G(x(t)) \le \overline{b},$$

where $\tilde{L} = \frac{L(\tau(|y^*|)+\underline{b})}{\underline{b}}$. From Lemma 4, we know that for any $(k_0, \cdots, k_n) \in \Omega_{\tilde{L},\underline{b},\overline{b},c}$, there exists $\epsilon^* > 0$, such that for any $0 < \epsilon < \epsilon^*$, there exists $\beta > 0$, such that the following inequality holds on the interval where the solution exists:

$$\|Y(t)\| \le ce^{-\beta t}(\|Y(0)\| + |y_0(0)| + \sqrt{2\lambda_{\max}(Q)}\|\xi(0)\|).$$

The final thing we need to prove is: if the controller parameters $(k_0, \cdots, k_n) \in \Omega_{\tilde{L},\underline{b},\overline{b},c}$ and the observer gain $0 < \epsilon < \epsilon^*$, then for any initial states $x(0) \in \mathbb{R}^n$, $\hat{z}(0) \in \mathbb{R}^n$, the maximal existence interval of the solution of the closed-loop system (1),(19) and (20) is $[0, \infty)$.

We use the contradiction argument. Suppose that for some initial states $x(0) \in \mathbb{R}^n$, $\hat{z}(0) \in \mathbb{R}^n$, the solution of the closed-loop system only exists in $[0, T)$ for some $T < \infty$. Then we can see that the maximal interval of existence of the following $(2n+1)^{\text{th}}$-order autonomous differential equation composed of the control systems and the observers is also finite:

$$\begin{cases} \dot{x}_0 & = y^* - h(x) \\ \dot{x} & = f(x) + g(x)\big(k_0 x_0 + k_1(y^* - h(x)) + \sum_{j=2}^n k_j \hat{z}_j\big) \\ \dot{\hat{z}}_1 & = \hat{z}_2 + \frac{\beta_1}{\epsilon}(y^* - h(x) - \hat{z}_1) \\ & \vdots \\ \dot{\hat{z}}_n & = \frac{\beta_n}{\epsilon^n}(y^* - h(x) - \hat{z}_1) \end{cases} \tag{61}$$

for the initial conditions $\begin{bmatrix} 0 & x(0)^T & \hat{z}(0)^T \end{bmatrix} \in \mathbb{R}^{2n+1}$. Therefore, the solution of (61) with the initial value $\begin{bmatrix} 0 & x(0)^T & \hat{z}(0)^T \end{bmatrix}$ will satisfy

$$\limsup_{t \to T} \| \begin{bmatrix} x_0(t) & x(t)^T & \hat{z}(t)^T \end{bmatrix} \| = \infty. \tag{62}$$

From Lemma 4, we know that $\left\|\overline{Y}(t)\right\|$ and $\|\xi(t)\|$ are bounded on $[0, T)$. Therefore, by the right hand of (60), we can obtain

$$\left\|\dot{Y}(t)\right\| \leq N, \quad \forall t \in [0, T)$$

for some $N > 0$. Similar to the proof of Theorem 1, we know that $\limsup_{t \to T} \|x(t)\| < \infty$. As a consequence, $x_0(t) = \int_0^t y^* - h(x(s))ds$ is also bounded on $[0, T)$. On the other hand, by the boundedness of $\xi(t)$, we have $\sup_{0 \leq t < T} \|\hat{z}(t)\| < \infty$, which contradicts to (62).

Finally, by noticing $|y_0(0)| \leq \frac{\tau(|y^*|)}{k_0 \underline{b}}$ and applying Lemma 4 again, we conclude that

$$|e(t)| \leq \|Y(t)\| \leq ce^{-\beta t} \left(\|\Phi(x(0)) - z^*\| + \frac{\tau(|y^*|)}{k_0 \underline{b}} + \sqrt{2\lambda_{\max}(Q)}\|\xi(0)\| \right)$$

The proof of the convergence of the state $x(t)$ under Assumption (C') is similar to that of Theorem 1, which will not be repeated .□

5 Conclusions

In this paper, we have presented a theoretical investigation on the extended PID controller for a general class of SISO affine-nonlinear uncertain dynamical systems. It is shown that the extended PID controller can globally (or semi-globally) stabilize this class of systems under some fairly general conditions on nonlinearity and uncertainty of the systems, as long as the controller parameters are chosen from an open and unbounded parameter manifold. It is worth noting that the extended PID controller includes the classical PID controller as a special case, and that its design does not need the precise information about the mathematical model. It is also interesting to note that the nonlinear canonical form (which may not be global) in geometrical nonlinear control theory can be used in our theoretical analysis to get global(or semi-global) convergence results, thanks to the strong robustness of the extended PID controller. This enables us to avoid assuming special system structures like pure-feedback forms and to get general conditions that can be considerably simplified once the system structures are in certain special forms. Of course, many interesting problems still remain open. It would be interesting to consider extended PID control for multi-input-multi-output affine nonlinear uncertain systems, and to generalize our recent results on PID control of coupled multi-agent dynamical systems [26]. It would also be interesting to consider more complicated situations such as saturation, deadzone, time-delayed inputs, sampled-data PID controllers under a prescribed sampling rate, etc. These belong to further investigation.

6 Appendix

6.1 Appendix A

Proof of Lemma 1. From the definition of Ω_1, it is easy to see that $\lambda_n > 2n+2 > 2n+1 > \lambda_{n-1} > \cdots > \lambda_0 > 2$ and $|\lambda_j - \lambda_i| \geq 1$ for $i \neq j$ when $(\lambda_0, \cdots, \lambda_n) \in \Omega_1$.

First, we prove that P is invertible and $(\Pi_{i=0}^n \lambda_i)^n \det P = \Pi_{0\leq i<j\leq n}(\lambda_i - \lambda_j)$.

By the properties of determinant, we have $(\Pi_{i=0}^n \lambda_i)^n \det P = \det P'$, where P' is a Vandermonde matrix defined by

$$P' = \begin{bmatrix} 1 & \cdots & 1 \\ -\lambda_0 & \cdots & -\lambda_n \\ \vdots & & \vdots \\ & & \\ (-\lambda_0)^n & \cdots & (-\lambda_n)^{-n} \end{bmatrix}$$

Since P' is a Vandermonde matrix, we have $(\Pi_{i=0}^n \lambda_i)^n \det P = \Pi_{0\leq i<j\leq n}(\lambda_i - \lambda_j) \neq 0$, which implies that P is invertible.

Next, we proceed to prove that there exists $\delta > 0$, such that $|\det P| \geq \delta$ when $(\lambda_0, \cdots, \lambda_n) \in \Omega_1$.

It is easy to see that

$$\begin{aligned} \lambda_n^n (\Pi_{i=0}^{n-1} \lambda_i)^n |\det P| =& \Pi_{0\leq i<j\leq n-1}(\lambda_j - \lambda_i) \Pi_{i=0}^{n-1}(\lambda_n - \lambda_i) \\ \geq& \Pi_{i=0}^{n-1}(\lambda_n - \lambda_i) \geq \Pi_{i=0}^{n-1}(\lambda_n - (2i+3)) \\ \geq& \lambda_n^n \Pi_{i=0}^{n-1}(1 - \frac{2i+3}{\lambda_n}) \geq \lambda_n^n (\frac{1}{2n+2})^n \end{aligned}$$

Since $(\Pi_{i=0}^{n-1} \lambda_i)^n$ is bounded, we see that there exists $\delta > 0$ depends on n only, such that $|\det P| \geq \delta$.

By using the fact $|\det P| \geq \delta$ and the formula $P^{-1} = \frac{P^*}{\det P}$, where P^* is the adjoint matrix of P, we conclude that $\|P^{-1}\| \leq \frac{\|P^*\|}{\delta}$.

Since all $(n+1) \times (n+1)$ elements of P are bounded (bounded by 1), it is easy to see that there exists δ_2, such that $\|P\| \leq \delta_2$ and $\|P^*\| \leq \delta_2$ for any $\lambda \in \Omega_1$.

Hence, $c_1 = \sup_{\lambda\in\Omega_1} \|P\| < \infty$, and $c_2 = \sup_{\lambda\in\Omega_1} \|P\|\|P^{-1}\| \le c_1\frac{\delta_2}{\delta} < \infty$.

Recall that $(d_0, \cdots, d_n)^T$ is the last column of P^{-1}, by some simple calculations of determinants, we have $d_i = \frac{\lambda_i^n}{\prod_{j\neq i}(\lambda_i-\lambda_j)}$, $i = 0, \cdots, n$. Since $\lambda \in \Omega_1$, we have

$$0 < d_n = \frac{\lambda_n^n}{\prod_{i\neq n}(\lambda_n - \lambda_i)} \le \big(\frac{\lambda_n}{\lambda_n - (2n+1)}\big)^n < (2n+2)^n,$$

therefore $c_3 < \infty$. For $i = 0, \cdots, n-1$, it is easy to see that

$$\frac{\lambda_n \lambda_i^n}{(\lambda_n - \lambda_i)\left|\prod_{j\neq i, j\neq n}(\lambda_j - \lambda_i)\right|} \le \frac{(2n+1)^n \lambda_n}{\lambda_n - (2n+1)},$$

and so $c_4(i) < \infty$. Hence, $c_0 < \infty$ and therefore the proof of Lemma 1 is complete.

Proof of Lemma 2. Rewrite (29) as

$$\begin{cases} \dot{y}_0 = y_1 \\ \quad\vdots \\ \dot{y}_n = -\underline{b}\sum_{i=0}^n k_i y_i + a_t + (\underline{b} - b_t)\sum_{i=0}^n k_i y_i \end{cases}. \tag{63}$$

Suppose that $(k_0, \cdots, k_n) \in \Omega_{L,\underline{b},\overline{b},c}$ and denote

$$A \triangleq \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\underline{b}k_0 & -\underline{b}k_1 & -\underline{b}k_2 & \cdots & -\underline{b}k_n \end{bmatrix}.$$

Then (63) can be rewritten as

$$\dot{\overline{Y}} = A\overline{Y} + (0, \cdots, 0, a_t + (\underline{b} - b_t)(k_0 y_0 + \cdots + k_n y_n))^T \tag{64}$$

It is easy to see that the characteristic polynomial of A is $\det(sI - A) = s^{n+1} + \sum_{i=0}^n \underline{b}k_i s^i$.

By the definition (27) of $\Omega_{L,\underline{b},\overline{b},c}$, there exists $(\lambda_0, \cdots, \lambda_n) \in \Omega_\Lambda$ such that $(\underline{b}k_0, \cdots, \underline{b}k_n) = (\prod_{i=0}^n \lambda_i, \cdots, \sum_{i=0}^n \lambda_i)$. Therefore, by Vieta's formulas, we know that $-\lambda_i, i = 0, \cdots, n$ are $(n+1)$ distinct eigenvalues of A. Hence, A is similar to J, where J is a diagonal matrix defined by $J \triangleq \text{diag}(-\lambda_0, \cdots, -\lambda_n)$.

It is not difficult to get the relationship $AP = PJ$, where P is defined in (22). To simplify the analysis, we introduce an invertible linear transformation $\overline{Y}(t) = P\overline{w}(t)$, where $\overline{w} = (w_0, \cdots, w_{n-1}, w_n)^T$.

We first need to prove the following equality:

$$\sum_{i=0}^n k_n y_n = \frac{1}{\underline{b}}\sum_{i=0}^n \lambda_i w_i.$$

By the relationship $\overline{Y} = P\overline{w}$, we have $y_i = \sum_{k=0}^{n} (-\lambda_k)^{i-n} w_k$. Therefore

$$\begin{aligned}\sum_{i=0}^{n} \underline{b}k_i y_i =& \sum_{i=0}^{n} \underline{b}k_i \sum_{k=0}^{n} (-\lambda_k)^{i-n} w_k = \sum_{k=0}^{n} \big(\sum_{i=0}^{n} \underline{b}k_i (-\lambda_k)^{i-n} \big) w_k \\ =& \sum_{k=0}^{n} \big(\frac{\sum_{i=0}^{n} \underline{b}k_i (-\lambda_k)^i}{(-\lambda_k)^n} \big) w_k = \sum_{k=0}^{n} \frac{-(-\lambda_k)^{n+1}}{(-\lambda_k)^n} w_k = \sum_{k=0}^{n} \lambda_k w_k.\end{aligned}$$

The second to last equality holds since $-\lambda_k$ is the root of the polynomial $s^{n+1} + \sum_{i=0}^{n} \underline{b}k_i s^i$.

By the relationship $A = PJP^{-1}$, it is easy to see (64) can be transformed to

$$\dot{\overline{w}} = J\overline{w} + P^{-1} \left(0, \cdots, 0, a_t + (\underline{b} - b_t) \sum_{i=0}^{n} k_i y_i \right)^T \tag{65}$$

Since $(d_0, \cdots, d_n)^T$ is the last column of the matrix P^{-1}, by the equality $\sum_{i=0}^{n} k_i y_i = \frac{1}{\underline{b}} \sum_{i=0}^{n} \lambda_i w_i$, we see that (65) becomes

$$\begin{cases} \dot{w}_0 = -\lambda_0 w_0 + d_0 \left(a_t + \frac{\underline{b} - b_t}{\underline{b}} \sum_{i=0}^{n} \lambda_i w_i \right) \\ \quad \vdots \\ \dot{w}_n = -\lambda_n w_n + d_n \left(a_t + \frac{\underline{b} - b_t}{\underline{b}} \sum_{i=0}^{n} \lambda_i w_i \right). \end{cases} \tag{66}$$

Now, we consider the following quadratic function:

$$V(\overline{w}(t)) = \frac{1}{2} \sum_{i=0}^{n} w_i^2(t) = \frac{1}{2} \|\overline{w}(t)\|^2.$$

Then it is easy to compute the time derivative of V $\dot{V}(\overline{w}(t)) \triangleq \frac{dV(\overline{w}(t))}{dt}$ as follows:

$$\begin{aligned}\dot{V}(\overline{w}(t)) =& -\sum_{i=0}^{n} \lambda_i w_i^2 + \left(\sum_{i=0}^{n} d_i w_i \right) \left(a_t + \frac{\underline{b} - b_t}{\underline{b}} \sum_{i=0}^{n} \lambda_i w_i \right) \\ =& \underbrace{-\sum_{i=0}^{n} \lambda_i w_i^2}_{\text{I}} + \underbrace{\left(\sum_{i=0}^{n} d_i w_i \right) a_t}_{\text{II}} + \underbrace{\frac{\underline{b} - b_t}{\underline{b}} \sum_{i=0}^{n} d_i w_i \sum_{i=0}^{n} \lambda_i w_i}_{\text{III}}\end{aligned} \tag{67}$$

Next, we proceed to estimate (67) term by term.

Denote $(w_0, \cdots, w_{n-1})^T \triangleq w$. Obviously, the first term

$$\text{I} = -\sum_{i=0}^{n} \lambda_i w_i^2 \leq -2\|w\|^2 - \lambda_n w_n^2 \tag{68}$$

since $\lambda_i > 2, i = 0 \cdots, n-1$.

By Lemma 1, we have $\|P\| \leq c$, therefore $|a_t| \leq L\left\|\overline{Y}\right\| = L\|P\overline{w}\| \leq Lc\|\overline{w}\| \leq Lc(\|w\| + |w_n|)$.

On the other hand, by Lemma 1 and the fact $c \geq c_0$, we also have $|d_i| \leq \frac{c}{(2n+1)n\lambda_n} < \frac{c}{\sqrt{n}\lambda_n}, i = 0, \cdots, n-1$, and $|d_n| \leq \frac{c}{(2n+1)\sqrt{n}} < c$. Therefore, we have $|\sum_{i=0}^{n} d_i w_i| \leq c\big(\sum_{i=0}^{n-1} |\frac{w_i}{\sqrt{n}\lambda_n}| + |w_n|\big) \leq c\big(\frac{\|w\|}{\lambda_n} + |w_n|\big)$.

As a consequence, we have the following upper bound for the second term:

$$\begin{aligned}
\mathrm{II} \leq & \left|\left(\sum_{i=0}^{n} d_i w_i\right) a_t\right| \leq Lc^2(\|w\| + |w_n|)\big(\|w\|/\lambda_n + |w_n|\big) \\
\leq & Lc^2\big(\|w\|^2/\lambda_n + 2\|w\||w_n| + |w_n|^2\big). \qquad (69)
\end{aligned}$$

Finally, we proceed to estimate the third term. Since $d_n = \frac{\lambda_n^n}{\prod_{i=0}^{n-1}(\lambda_n - \lambda_i)} > 0$, it is easy to get

$$\begin{aligned}
\mathrm{III} = & \frac{\underline{b} - b_t}{\underline{b}} \sum_{i=0}^{n} d_i w_i \sum_{i=0}^{n} \lambda_i w_i \\
= & \frac{\underline{b} - b_t}{\underline{b}} \left\{ \sum_{i=0}^{n-1} d_i w_i \sum_{i=0}^{n-1} \lambda_i w_i + d_n w_n \sum_{i=0}^{n-1} \lambda_i w_i + \left(\sum_{i=0}^{n-1} d_i w_i\right) \lambda_n w_n + \lambda_n d_n w_n^2 \right\} \\
\leq & \frac{\underline{b} - b_t}{\underline{b}} \left\{ \sum_{i=0}^{n-1} d_i w_i \sum_{i=0}^{n-1} \lambda_i w_i + d_n w_n \sum_{i=0}^{n-1} \lambda_i w_i + \left(\sum_{i=0}^{n-1} d_i w_i\right) \lambda_n w_n \right\}.
\end{aligned}$$

Since we know that $|d_n| \leq \frac{c}{(2n+1)\sqrt{n}}$, $|d_i| \leq \frac{c}{(2n+1)n\lambda_n}$, $i = 0, \cdots, n-1$, and $0 < \lambda_i < 2n+1, i = 0, \cdots, n-1$, and $\sum_{i=0}^{n-1} |w_i| \leq \sqrt{n}\|w\|$, we can easily get the following three inequalities:

$$\begin{aligned}
\left|\sum_{i=0}^{n-1} d_i w_i \sum_{i=0}^{n-1} \lambda_i w_i\right| \leq & \sum_{i=0}^{n-1} \frac{c|w_i|}{(2n+1)n\lambda_n} \sum_{i=0}^{n-1} (2n+1)|w_i| \\
\leq & \frac{c}{n\lambda_n}\left(\sum_{i=0}^{n-1} |w_i|\right)^2 \leq \frac{c}{\lambda_n}\|w\|^2;
\end{aligned}$$

$$\left|d_n w_n \sum_{i=0}^{n-1} \lambda_i w_i\right| \leq \frac{c(2n+1)\sqrt{n}}{(2n+1)\sqrt{n}} |w_n|\|w\| = c|w_n|\|w\|;$$

$$\begin{aligned}
\left|\left(\sum_{i=0}^{n-1} d_i w_i\right) \lambda_n w_n\right| \leq & \left(\sum_{i=0}^{n-1} \frac{c}{(2n+1)n\lambda_n} |w_i|\right) \lambda_n |w_n| \\
\leq & \frac{c}{(2n+1)\sqrt{n}} \|w\||w_n| \leq c\|w\||w_n|.
\end{aligned}$$

Therefore, the upper bound of the third term can be estimated as

$$\text{III} = \frac{\underline{b} - b_t}{\underline{b}} \sum_{i=0}^{n} d_i w_i \sum_{i=0}^{n} \lambda_i w_i \leq \frac{\overline{b} - \underline{b}}{\underline{b}} \left(\frac{c}{\lambda_n} \|w\|^2 + 2c|w_n| \|w\| \right).$$

Denote $m = Lc^2 + \frac{(\overline{b} - \underline{b})c}{\underline{b}}$. Combining (68)-(70), we have

$$\dot{V}(\overline{w}) \leq (m/\lambda_n - 2)\|w\|^2 + 2m\|w\||w_n| - \left(\lambda_n - Lc^2\right) w_n^2 \tag{70}$$

Since $(\lambda_0, \cdots, \lambda_n) \in \Omega_\Lambda$, we can see $\lambda_n > \max\{2n+2, m^2 + Lc^2\}$. If $m \leq 1$, then $\lambda_n > 2n+2 > m$; if $m \geq 1$, then $\lambda_n > m^2 + Lc^2 > m$. Therefore, $\lambda_n > m$ always holds whenever $(\lambda_0, \cdots, \lambda_n) \in \Omega_\Lambda$.

By the inequality $\lambda_n > m$, we have $2 - \frac{m}{\lambda_n} \geq 1$, therefore

$$\dot{V}(\overline{w}(t)) \leq -\|w(t)\|^2 + 2m\|w(t)\||w_n(t)| - (\lambda_n - Lc^2) w_n^2(t).$$

Since $\lambda_n > m^2 + Lc^2$, we conclude that $\dot{V}(\overline{w}(t)) \leq -\alpha \|\overline{w}(t)\|^2$ for some $\alpha > 0$, i.e., $\dot{V}(\overline{w}(t)) \leq -2\alpha V(\overline{w}(t))$. Therefore, by the comparison theorem, we have $V(\overline{w}(t)) \leq e^{-2\alpha t} V(\overline{w}(0))$ for $t \in [0, T)$.

Finally, we estimate the upper bound of $\left\|\overline{Y}(t)\right\|$ as follows:

$$\begin{aligned}
\left\|\overline{Y}(t)\right\| &= \|P\overline{w}(t)\| \leq \|P\|\|\overline{w}(t)\| = \|P\|\sqrt{2V(\overline{w}(t))} \\
&\leq \|P\|\sqrt{2e^{-2\alpha t} V(\overline{w}(0))} = e^{-\alpha t}\|P\|\|\overline{w}(0)\| \\
&\leq e^{-\alpha t}\|P\|\|P^{-1}\| \left\|\overline{Y}(0)\right\| \leq c e^{-\alpha t} \left\|\overline{Y}(0)\right\|.
\end{aligned}$$

This completes the proof of Lemma 2.□

Proof of lemma 3. It suffices to show that if the parameters $(k_0, \cdots, k_n) \in \Omega_{L_0, \underline{b}, b_0, c}$, then for any $T_0 < T$ we have

$$\left\|\overline{Y}(t)\right\| \leq c e^{-\alpha t} \left\|\overline{Y}(0)\right\|, \forall t \in [0, T_0).$$

Denote $a \triangleq \sup_{0 \leq t \leq T_0} \left\|\overline{Y}(t)\right\|$, $L' \triangleq \sup_{0 \leq \rho \leq a} \frac{\tau_2(\rho)}{\rho}$, $b' \triangleq \tau_1(a)$.

It is easy to verify that $|a_t| \leq \tau_2(\left\|\overline{Y}(t)\right\|) = \frac{\tau_2(\left\|\overline{Y}(t)\right\|)}{\left\|\overline{Y}(t)\right\|} \left\|\overline{Y}(t)\right\| \leq L' \left\|\overline{Y}(t)\right\|$ and $\underline{b} \leq b_t \leq \tau_1(\left\|\overline{Y}(t)\right\|) \leq \tau_1(a) = b'$ for $t \in [0, T_0)$.

Therefore, by Lemma 2, we have $\left\|\overline{Y}(t)\right\| \leq c e^{-\alpha t} \left\|\overline{Y}(0)\right\|$ for $t \in [0, T_0)$ whenever the parameters $(k_0, \cdots, k_n) \in \Omega_{L', \underline{b}, b', c}$. As a consequence, $a = \sup_{0 \leq t \leq T_0} \left\|\overline{Y}(t)\right\| \leq c \left\|\overline{Y}(0)\right\| \leq cR$ and $b' = \tau_1(a) \leq \tau_1(cR)$, which implies $L' \leq L_0, b' \leq b_0$.

The final thing we need to prove is $\Omega_{L_0, \underline{b}, b_0, c} \subset \Omega_{L', \underline{b}, b', c}$.

From (27)-(28), we know that

$$\Omega_{L, \underline{b}, \overline{b}, c} \triangleq \left\{ \begin{bmatrix} k_0 \\ \vdots \\ k_n \end{bmatrix} \middle| \begin{bmatrix} k_0 \\ \vdots \\ k_n \end{bmatrix} = \frac{1}{\underline{b}} \begin{bmatrix} \prod_{i=0}^{n} \lambda_i \\ \vdots \\ \sum_{i=0}^{n} \lambda_i \end{bmatrix}, \lambda \in \Omega_1 \cap \Omega_2 \right\}$$

where

$$\Omega_2 = \left\{ \lambda \in \mathbb{R}^{n+1} \middle| \lambda_n > \left(Lc^2 + \left(\overline{b} - \underline{b}\right) c/\underline{b}\right)^2 + Lc^2 \right\}. \tag{71}$$

From (21), it is easy to see that Ω_1 does not depend on $L, \underline{b}, \overline{b}$ and c. By (71), we know that Ω_2 depends on $L, \underline{b}, \overline{b}, c$, i.e., $\Omega_2 = \Omega_2(L, \underline{b}, \overline{b}, c)$. It is easy to see that if $\underline{b}$ and c are fixed, then Ω_2 gets smaller for larger $L, \overline{b}$, i.e., $\Omega_2(L_0, \underline{b}, b_0, c) \subset \Omega_2(L', \underline{b}, b', c)$.

Therefore, we have $\Omega_{L_0,\underline{b},b_0,c} \subset \Omega_{L',\underline{b},b',c}$. This means that, if the parameters $(k_0, \cdots, k_n) \in \Omega_{L_0,\underline{b},b_0,c}$, then for $T_0 < T$ we have

$$\left\|\overline{Y}(t)\right\| \leq ce^{-\alpha t} \left\|\overline{Y}(0)\right\|, \quad \forall t \in [0, T_0).$$

Since T_0 is arbitrary, we complete the proof of Lemma 3.□

Proof of lemma 4. Suppose that $(k_0, \cdots, k_n) \in \Omega_{L,\underline{b},\overline{b},c}$ and denote $\overline{Y}(t) = P\overline{w}(t)$, then similar to the proof of Lemma 2, we have

$$\begin{cases} \dot{w}_0 &= -\lambda_0 w_0 + d_0 \left(\Delta_t + b_t \sum_{i=2}^n k_i \epsilon^{n-i} \xi_i\right) \\ &\vdots \\ \dot{w}_n &= -\lambda_n w_n + d_n \left(\Delta_t + b_t \sum_{i=2}^n k_i \epsilon^{n-i} \xi_i\right) \\ \dot{\xi} &= \frac{1}{\epsilon} B\xi + E_t \end{cases} \tag{72}$$

where $\Delta_t \triangleq a_t + \frac{\underline{b} - b_t}{\underline{b}} \sum_{i=0}^n \lambda_i w_i$ and

$$E_t \triangleq \left(0, \cdots, a_t - b_t \left(\sum_{i=0}^n k_i y_i - \sum_{i=2}^n k_i \epsilon^{n-i} \xi_i\right)\right)^T.$$

Now, we introduce a quadratic function as follows:

$$V_0(\overline{w}(t), \xi(t)) = \frac{1}{2}\|\overline{w}(t)\|^2 + \xi^T(t) Q \xi(t),$$

where Q is the unique positive definite matrix satisfying $B^T Q + QB = -I$.

Then it is not difficult to compute the time derivative of V_0,

$$\dot{V}_0(\overline{w}(t), \xi(t)) = \dot{V}(\overline{w}(t)) + b_t \sum_{i=0}^n d_i w_i \sum_{i=2}^n k_i \epsilon^{n-i} \xi_i - \|\xi\|^2/\epsilon + 2\xi^T Q E_t \tag{73}$$

where $\dot{V}(\overline{w}(t))$ is computed in (67).

From the proof of Lemma 2, we conclude that if $(k_0, \cdots, k_n) \in \Omega_{L,\underline{b},\overline{b},c}$, then there exists a constant $\alpha > 0$, such that $\dot{V}(\overline{w}(t)) \leq -\alpha \|\overline{w}(t)\|^2$.

Without loss of generality, assume that $\epsilon < 1$, then we have

$$\left|\sum_{i=2}^n k_i \epsilon^{n-i} \xi_i\right| \leq \sqrt{n} \max\{k_2, \cdots, k_n\} \|\xi\|, \quad \left|\sum_{i=0}^n \lambda_i w_i\right| \leq \sqrt{n+1} \lambda_n \|\overline{w}\| \tag{74}$$

By Lemma 1, we have $|d_i| \leq \frac{c}{(2n+1)\sqrt{n}}, i = 0, \cdots, n$, and therefore

$$|\sum_{i=0}^{n} d_i w_i| \leq c\frac{|w_0| + \cdots + |w_n|}{(2n+1)\sqrt{n}} \leq c\|\overline{w}\|. \tag{75}$$

Hence by $|a_t| \leq L\left\|\overline{Y}(t)\right\| \leq Lc\left\|\overline{w}(t)\right\|$ and $0 < \underline{b} \leq b_t \leq \overline{b}$, we have from (73)-(75)

$$\begin{aligned}\dot{V}_0(\overline{w}(t), \xi(t)) \leq & -\alpha\|\overline{w}\|^2 - \|\xi\|^2/\epsilon + c_1\|\overline{w}\|\|\xi\| + c_2\|\xi\|^2 \\ = & -\alpha\|\overline{w}\|^2 + c_1\|\overline{w}\|\|\xi\| - (1/\epsilon - c_2)\|\xi\|^2 \end{aligned} \tag{76}$$

where c_1, c_2 are two constants defined by

$$c_1 = \sqrt{n}\overline{b}c\max\{k_2, \cdots, k_n\} + 2\lambda_{\max}(Q)\left(Lc + \overline{b}\sqrt{n+1}\lambda_n/\underline{b}\right)$$

and $c_2 = 2\sqrt{n}\overline{b}\lambda_{\max}(Q)\max\{k_2, \cdots, k_n\}$, which are independent of ϵ.

Since α, c_1, c_2 are independent of ϵ, it is easy to see from (76) that there exists $\epsilon^* > 0$ such that whenever $0 < \epsilon < \epsilon^*$, we have

$$\dot{V}_0(\overline{w}(t), \xi(t)) \leq -2\beta V_0(\overline{w}(t), \xi(t))$$

for some $\beta > 0$. By the comparison theorem, we have

$$V_0(\overline{w}(t), \xi(t)) \leq e^{-2\beta t}V_0(\overline{w}(0), \xi(0)).$$

As a consequence, we have

$$\begin{aligned}\|Y(t)\| \leq & \|\overline{Y}(t)\| = \|P\overline{w}(t)\| \leq \|P\|\|\overline{w}(t)\| \\ \leq & \|P\|\sqrt{2V_0(\overline{w}(t), \xi(t))} \leq \|P\|e^{-\beta t}\sqrt{2V_0(\overline{w}(0), \xi(0))} \\ = & \|P\|e^{-\beta t}\left(\|\overline{w}(0)\| + \sqrt{2\lambda_{\max}(Q)}\|\xi(0)\|\right) \\ \leq & \|P\|e^{-\beta t}(\left\|P^{-1}\right\|\left\|\overline{Y}(0)\right\| + \sqrt{2\lambda_{\max}(Q)}\|\xi(0)\|) \\ \leq & ce^{-\beta t}(\|Y(0)\| + |y_0(0)| + \sqrt{2\lambda_{\max}(Q)}\|\xi(0)\|).\end{aligned}$$

Moreover, we have

$$\begin{aligned}&\sqrt{\lambda_{\min}(Q)}\|\xi(t)\| \leq \sqrt{V_0(\overline{w}(t), \xi(t))} \leq e^{-\beta t}\sqrt{V_0(\overline{w}(0), \xi(0))} \\ \leq & e^{-\beta t}\left(\|\overline{w}(0)\| + \sqrt{\lambda_{\max}(Q)}\|\xi(0)\|\right) \leq e^{-\beta t}\left(\left\|P^{-1}\right\|\left\|\overline{Y}(0)\right\| + \sqrt{\lambda_{\max}(Q)}\|\xi(0)\|\right) \\ \leq & e^{-\beta t}\left(c\left\|\overline{Y}(0)\right\| + \sqrt{\lambda_{\max}(Q)}\|\xi(0)\|\right) \Box\end{aligned}$$

6.2 Appendix B

Proof of Example 2. Obviously, (17) has uniform relative degree 2. By simple calculations, we have $\Phi(x) = (x_1, f_1(x_1) + x_2)^T$, $F(x) = f_1'(x_1)(f_1(x_1) + x_2) +$

$f_2(x_1, x_2)$ and $G(x) = 1$ for any $x \in \mathbb{R}^2$, which implies that Assumption (A) is satisfied.

Next, the Jacobian matrix $J_\Phi(x) = \begin{bmatrix} 1 & 0 \\ f_1'(x_1) & 1 \end{bmatrix}$ is nonsingular and it is easy to see that $\lim_{\|x\|\to\infty} \|\Phi(x)\| = \infty$. Therefore, by Theorem A1, we conclude that Assumption (C') is satisfied.

Let y^* be any given setpoint. Then $x^* \stackrel{\triangle}{=} \Phi^{-1}(y^*, 0) = (y^*, -f_1(y^*))$. Now we proceed to estimate the upper bound of $|F(x)G(x^*) - F(x^*)G(x)|$,

$$
\begin{aligned}
& |F(x)G(x^*) - F(x^*)G(x)| \\
=& |F(x) - F(x^*)| = |F(x_1, x_2) - F(y^*, -f_1(y^*))| \\
=& |f_1'(x_1)(f_1(x_1) + x_2) + f_2(x_1, x_2) - f_2(y^*, -f_1(y^*))| \\
\leq& |f_1'(x_1)(f_1(x_1) + x_2)| + |f_2(x_1, x_2) - f_2(y^*, x_2)| + |f_2(y^*, x_2) - f_2(y^*, -f_1(y^*))| \\
\leq& L|f_1(x_1) + x_2| + L|x_1 - y^*| + L|x_2 + f_1(y^*)| \\
\leq& L|f_1(x_1) + x_2| + L|x_1 - y^*| + L(|f_1(x_1) + x_2| + |f_1(y^*) - f_1(x_1)|) \\
\leq& 2L|f_1(x_1) + x_2| + (L + L^2)|x_1 - y^*| \\
\leq& \sqrt{4L^2 + (L + L^2)^2}\sqrt{(f_1(x_1) + x_2)^2 + (x_1 - y^*)^2} \\
=& \sqrt{4L^2 + (L + L^2)^2}\|\Phi(x) - \Phi(x^*)\|
\end{aligned}
$$

Therefore, by Theorem 2 and Remark 5, the classical PID controller $u(t) = k_p e(t) + k_i \int_0^t e(s)ds + k_d \frac{de(t)}{dt}$ can globally stabilize the system and make the regulation error $e(t)$ converge to 0 exponentially for any initial state $x(0) \in \mathbb{R}^2$ as long as $(k_i, k_p, k_d) \in \Omega_{\tilde{L},1,1,c}$, where $\tilde{L} = \sqrt{4L^2 + (L + L^2)^2}$.$\square$

Proof of Example 3. By some simple calculations, it is easy to get $\Phi(x) = (x_1, f_1(x_1, x_2))$, $F(x) = \frac{\partial f_1}{\partial x_1}(x)f_1(x) + \frac{\partial f_1}{\partial x_2}(x)f_2(x)$, $G(x) = \frac{\partial f_1}{\partial x_2}(x)g(x) \geq \underline{b}b_1 > 0$ and therefore Assumption (A) is satisfied.

Let y^* be any given setpoint. Since $\frac{\partial f_1}{\partial x_2}(x) \geq \underline{b} > 0$ for any $x \in \mathbb{R}^2$, then there exists a unique $x_2^* \in \mathbb{R}$ such that $f_1(y^*, x_2^*) = 0$. Denote $x^* \stackrel{\triangle}{=} \Phi^{-1}(y^*, 0) = (y^*, x_2^*)$, $\overline{x}_1 \stackrel{\triangle}{=} x_1 - y^*$ and $\overline{x}_2 \stackrel{\triangle}{=} x_2 - x_2^*$. Then we can easily obtain the following equalities:

$$\|x - x^*\|^2 = (x_1 - y^*)^2 + (x_2 - x_2^*)^2 = \overline{x}_1^2 + \overline{x}_2^2 \tag{77}$$

and $\|\Phi(x) - \Phi(x^*)\|^2 = \overline{x}_1^2 + (f_1(x_1, x_2) - f_1(y^*, x_2^*))^2$. By the mean value theorem, we have

$$\|\Phi(x) - \Phi(x^*)\|^2 = \overline{x}_1^2 + (\theta_1\overline{x}_1 + \theta_2\overline{x}_2)^2, \tag{78}$$

where $|\theta_1| \leq L$, $0 < \underline{b} \leq \theta_2 \leq L$.

Now we proceed to prove $\|H(x) - H(x^*)\| \leq L_0\|\Phi(x) - z^*\|$ for some $L_0 > 0$. First, we will show that

$$\alpha\|x - x^*\| \leq \|\Phi(x) - \Phi(x^*)\|, \forall x \in \mathbb{R}^2 \tag{79}$$

for some $\alpha > 0$.

Without loss of generality, assume that $x - x^* \neq 0$. Then there exists some $r > 0$ and $\theta \in [0, 2\pi]$ such that $\overline{x}_1 = r\cos\theta$ and $\overline{x}_2 = r\sin\theta$. From (77)-(78), we have

$$\|\Phi(x) - \Phi(x^*)\|^2/\|x - x^*\|^2 = \cos^2\theta + (\theta_1\cos\theta + \theta_2\sin\theta)^2.$$

Denote $\alpha = \inf\sqrt{\cos^2\theta + (\theta_1\cos\theta + \theta_2\sin\theta)^2}$, where the infimum is taken for all $\theta \in [0, 2\pi], |\theta_1| \leq L, 0 < \underline{b} \leq \theta_2 \leq L$. It is easy to obtain that $\alpha > 0$, i.e., (79) is satisfied.

Next, we will prove that

$$\|H(x) - H(x^*)\| \leq \beta\|x - x^*\| \tag{80}$$

for some $\beta > 0$.

We only give a sketch proof due to space limitation. First, note that $f_1(x^*) = 0$ and $|f_1(0)| \leq M$, it is not difficult to get the upper bound of $\|x^*\|$ by the assumption $\left\|\frac{\partial f_1}{\partial x}\right\| \leq L$ and $\frac{\partial f_1}{\partial x_2}(x) \geq \underline{b} > 0$. Then by $|f_2(0)| \leq M$ and the upper bound of $\|x^*\|$, it is not difficult to find M_0 such that $|f_2(x^*)| \leq M_0$.

Recall that $f_1(x^*) = 0$, then we have

$$\begin{aligned}
&|F(x) - F(x^*)| \\
=&\left|\frac{\partial f_1}{\partial x_1}(x)f_1(x)\right| + \left|\frac{\partial f_1}{\partial x_2}(x)f_2(x) - \frac{\partial f_1}{\partial x_2}(x^*)f_2(x^*)\right| \\
\leq&\left|\frac{\partial f_1}{\partial x_1}(x)f_1(x)\right| + \left|\frac{\partial f_1}{\partial x_2}(x)(f_2(x) - f_2(x^*))\right| + \left|\left(\frac{\partial f_1}{\partial x_2}(x) - \frac{\partial f_1}{\partial x_2}(x^*)\right)f_2(x^*)\right| \\
\leq&L^2\|x - x^*\| + \left|\frac{\partial f_1}{\partial x_2}(x)(f_2(x) - f_2(x^*))\right| + \left|\left(\frac{\partial f_1}{\partial x_2}(x) - \frac{\partial f_1}{\partial x_2}(x^*)\right)f_2(x^*)\right| \\
\leq&\left(2L^2 + M_0L\right)\|x - x^*\|
\end{aligned}$$

Similarly, we can obtain

$$\begin{aligned}
|G(x) - G(x^*)| &= \left|\frac{\partial f_1}{\partial x_2}(x)g(x) - \frac{\partial f_1}{\partial x_2}(x^*)g(x^*)\right| \\
&= \left|\left(\frac{\partial f_1}{\partial x_2}(x) - \frac{\partial f_1}{\partial x_2}(x^*)\right)g(x)\right| + \left|\frac{\partial f_1}{\partial x_2}(x^*)(g(x) - g(x^*))\right| \\
&\leq \left(L^2 + b_2L\right)\|x - x^*\|.
\end{aligned}$$

Therefore, we have

$$\|H(x) - H(x^*)\| \leq L_0\|\Phi(x) - \Phi(x^*)\| \tag{81}$$

for some $L_0 > 0$. Furthermore, we have

$$\begin{aligned}
|F(x)| \leq&|F(x^*)| + |F(x) - F(x^*)| \leq \left|\frac{\partial f_1}{\partial x_2}(x^*)f_2(x^*)\right| + \|H(x) - H(x^*)\| \\
\leq&LM_0 + L_0\|\Phi(x) - \Phi(x^*)\| \leq L_0\|\Phi(x)\| + LM_0 + L_0|y^*|.
\end{aligned} \tag{82}$$

By (81)-(82) and $\underline{b}b_1 \leq G(x) = \frac{\partial f_1}{\partial x_2}(x)g(x) \leq Lb_2$, we conclude that Assumption (B1) is satisfied.

Finally, by the fact $\frac{\partial f_1}{\partial x_2}(x) \geq \underline{b} > 0$, it is easy to see that $\lim_{\|x\|\to\infty} \|\Phi(x)\| = \infty$. By Theorem A1, we conclude that Assumption (C') is also satisfied. Therefore, by Theorem 2, the classical PID controller $u(t) = k_p e(t) + k_i \int_0^t e(s)ds + k_d \frac{de(t)}{dt}$ can globally stabilize the system and make the regulation error $e(t)$ converge to 0 exponentially for any initial state $x(0) \in \mathbb{R}^2$. $\square$

Proof of Remark 4. We will give an example to show that the super-linear growth rate (14) in Assumption (C) cannot be weakened to (15) for any $\eta > 0$.

We first define a function f as follows:

$$f(x) = \begin{cases} 2 - (\log x)^{-\eta}, & x \geq e \\ -f(-x), & x \leq -e \end{cases}, \tag{83}$$

where e is the natural logarithm. We can extend f as a smooth function defined on $\mathbb{R}$ with $f(0) = 0$ and $f'(x) > 0$, when $-e < x < e$. Let us consider the following nonlinear uncertain plant with PID controller:

$$\begin{cases} \dot{x}_1 &= \epsilon f(x_2) \\ \dot{x}_2 &= \frac{1+u}{\epsilon f'(x_2)} \\ y &= x_1 \\ u(t) &= k_p e(t) + k_i \int_0^t e(s)ds + k_d \frac{de}{dt}(t) \end{cases} \tag{84}$$

where ϵ is an unknown constant with $0 < \epsilon \leq 1$. Let $y^* = 0$ be the setpoint. Then we can verify that both Assumptions (A) and (B) are satisfied and (15) is satisfied for some positive numbers N_1 and N_2. However, we will show that for any $R > 0$ and for any given PID parameters, there always exists initial state $x(0) \in \mathbb{R}^2$ satisfying $\|x(0)\| \leq R$, such that the solution of the closed-loop system (84) with the initial state $x(0)$ has finite escape time for all $\epsilon < \min\{\frac{1}{4(|k_p|+|k_i|+|k_d|)}, \frac{1}{4}\}$.

First, by some simple calculations, we can obtain $\Phi(x) = (x_1, \epsilon f(x_2))$, $F(x) = 1, G(x) = 1$ for any $x \in \mathbb{R}^2$.

Define an increasing function $\tau_1(r) = r + 2$. It is not difficult to see that $\sup_{x_2 \in \mathbb{R}} |f(x_2)| = 2$.

It is easy to obtain

$$\|\Phi(x)\| \leq |x_1| + |\epsilon f(x_2)| \leq \|x\| + 2 = \tau_1(\|x\|)$$

and

$$\|H(x)\| \leq \sqrt{2} < \tau_1(\|x\|)$$

for any $x \in \mathbb{R}^2$. For $y^* = 0$, then $x^* = (0, 0)$. We have

$$\|H(x) - H(x^*)\| = 0 \leq \tau_2(r) = r.$$

Therefore, Assumptions (A) and (B) are satisfied.

By simple calculations, we can obtain

$$J_{\Phi}^{-1}(x) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\epsilon f'(x_2)} \end{bmatrix}.$$

By the definition of f, it is easy to see that

$$f'(x_2) = \eta \left(|x_2| \log^{1+\eta} |x_2|\right)^{-1}, \forall |x_2| \geq e$$

Therefore, we can obtain

$$\left\| J_{\Phi}^{-1}(x) \right\| \leq N_1 \|x\| \log^{1+\eta} \|x\| + N_2, \forall x \in \mathbb{R}^2$$

for some constants N_1 and N_2(possibly depend on ϵ).

Let $R > 0$ and the parameter triple $(k_p, k_i, k_d) \in \mathbb{R}^3$ are given arbitrarily. Let $[0, T)$ be the maximal existence interval of (84) with initial state $x(0) = (0, 0)$. We proceed to prove that the closed-loop equation (84) will have finite escape time for ϵ sufficiently small, i.e., $T < \infty$.

Denote $T_0 = \min\{T, 1\}$. Let us first prove that $u(t) = O(\epsilon)$ for $t \in [0, T_0)$.

Now suppose that $t \in [0, T_0)$, then

$$|e(t)| = |x_1(t)| = \left| x_1(0) + \int_0^t \dot{x}_1(s) ds \right| = \left| \int_0^t \epsilon f(x_2(s)) ds \right| \leq \int_0^t 2\epsilon ds \leq 2\epsilon. \tag{85}$$

By (85), we have

$$\left| \int_0^t e(s) ds \right| \leq \int_0^t |e(s)| ds \leq 2\epsilon t \leq 2\epsilon \tag{86}$$

Recall that $|f(x)| \leq 2$, we can obtain

$$\left| \dot{e}(t) \right| = \left| \dot{x}_1(t) \right| = |\epsilon f(x_2(t))| \leq 2\epsilon. \tag{87}$$

From (85)-(87), we have

$$|u(t)| = \left| k_p e(t) + k_i \int_0^t e(s) ds + k_d \frac{de}{dt}(t) \right| \leq 2(|k_p| + |k_i| + |k_d|)\epsilon. \tag{88}$$

Next, we proceed to prove that $T < \infty$ whenever $\epsilon < \min\{\frac{1}{4(|k_p|+|k_i|+|k_d|)}, \frac{1}{4}\}$.

From $\dot{x}_2(t) = \frac{1+u(t)}{\epsilon f'(x_2(t))}$ and (88), we conclude that if $\epsilon < \frac{1}{4(|k_p|+|k_i|+|k_d|)}$, then we have $\dot{x}_2 \geq \frac{1}{2\epsilon f'(x_2)}$ for $t \in [0, T_0)$, i.e., $f'(x_2) dx_2 \geq \frac{1}{2\epsilon} dt, \ \forall t \in [0, T_0)$.

By the comparison lemma in differential equations, the following inequality will be satisfied

$$f(x_2(t)) - f(x_2(0)) = f(x_2(t)) \geq \frac{t}{2\epsilon}, \quad t \in [0, T_0). \tag{89}$$

Notice that $|f(x)| \leq 2$, therefore by (89), we know that $T_0 \leq 4\epsilon$.

Since $\epsilon < \frac{1}{4}$, we obtain $T_0 \leq 4\epsilon < 1$, which implies $T < 1 < \infty$, i.e., the maximal existence interval $[0, T)$ of the closed-loop system (84) with initial state $(0, 0)$ is finite.□

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