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Convergence of Self-Tuning Regulators Under Conditional Heteroscedastic Noises with Unknown High-Frequency Gain*

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Abstract In the classical theory of self-tuning regulators, it always requires that the conditional variances of the systems noises are bounded. However, such a requirement may not be satisfied when modeling many practical systems, and one significant example is the well-known ARCH (autoregressive conditional heteroscedasticity) model in econometrics. The aim of this paper is to consider self-tuning regulators of linear stochastic systems with both unknown parameters and conditional heteroscedastic noises, where the adaptive controller will be designed based on both the weighted least-squares algorithm and the certainty equivalence principle. The authors will show that under some natural conditions on the system structure and the noises with unbounded conditional variances, the closed-loop adaptive control system will be globally stable and the tracking error will be asymptotically optimal. Thus, this paper provides a significant extension of the classical theory on self-tuning regulators with expanded applicability.

Keywords ARCH model, conditional heteroscedasticity, convergence, self-tuning regulator, weighted least-squares algorithm.

1 Introduction

In the mathematical modeling of practical complex dynamical systems, varies structural and/or disturbance uncertainties are bound to exist, and control theory is a scientific discipline that tries to control or regulate the behaviors of such systems. In the history of automatic control, varies control methods have been developed to deal with internal and external uncertainties. Roughly speaking, there are basically three ways to deal with uncertainties, i.e., learning (or systems identification or estimation), feedback (and feedforward), and the combination of both. The online combination of learning and feedback is usually called adaptive control whose purpose is to achieve the desired control objective in the presence of uncertainties[1].

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A concrete and widely used way for the online combination of learning and feedback is the so called "certainty equivalence principle", where a real-time estimation algorithm is used to estimate the unknown system or controller structure (typically parameterized by finite dimensional unknown parameters), and the resulting online estimates are then regarded as the "true" parameters and directly used in the controller design. Although such a principle is simple, natural and feasible, except for a few special cases, it may well not be necessary, even may not be stable and optimal, which is just reason why we need to establish a theory for stability and optimality in this paper.

A well-known and widely used estimation algorithm is the recursive least-squares. Indeed, the combination of the least-squares parameter estimation algorithm with a dead-beat controller was made by Kalman[2] for discrete-time linear deterministic systems. It was not until the work Åström and Wittenmark[3] that the least-squares estimator was combined with the minimum variance controller for discrete-time linear stochastic systems, that such an adaptive controller was named as self-tuning regulators (STR), and that the first remarkable step towards a convergence analysis of STR was made. Since the STR is very flexible with respect to the underlying design method and it is easy to implement with microprocessors, it has received considerable attention. The convergence theory of the least-squares based STR was an open problem in the 70s and 80s in the last century, which had attracted widespread research efforts from many scientists (see, e.g., Goodwin-Ramadge-Caines[4], Lai and Wei[5], Kumar[6]), and it was finally resolved by Guo and Chen[7] and Guo[8] where a sophisticated combinations of nonlinear analysis, martingale theory and stochastic Lyapunov functions was introduced and successfully used.

It is worth mentioning that in almost all the existing work of stochastic adaptive control theory, the conditional variance sequence of the noises is assumed to be bounded almost surely. However, this assumption may not be true in many interesting situations. For example, in the well-known ARCH and GARCH (generalized autoregressive conditional heteroscedasticity) models in econometrics, the conditional variance sequence of the noises is unbounded[9, 10]. Thus, it is necessary to establish a theory of adaptive control for linear stochastic systems with noises having unbounded conditional variances.

The purpose of this paper is to consider self-tuning control problems of linear stochastic systems with both unknown coefficients (including the high-frequency gain) and conditional heteroscedastic noises. Firstly, we propose some natural conditions on the system structure and the noises, and introduce a suitable sequence of time-varying weights into the least-squares estimation algorithm. Then, we use the "certainty equivalence principle"[11] to design the adaptive controller, modify the high-frequency gain slightly, and finally provide a convergence theory of self-tuning regulators. To be specific, we will prove that under some natural conditions on the system structure and the conditional heteroscedastic noises, the closed-loop adaptive control system is globally stable and the tracking error is asymptotically optimal.

Different from the author's previous work[12], this paper focuses on the case where the high-frequency gain is unknown, since this is the case that we meet commonly in practice[13, 14]. To ensure that the closed-loop control system still has the nice properties mentioned above, some suitable modifications are introduced in the weighted least square (WLS) estimation of the high-frequency gain, and this modification is inspired by [15].

The rest of the paper is organized as follows. We introduce the problem formulation in Section 2. The stability and optimality of the closed-loop systems controlled by STR are presented and proved in Section 3. Finally, some concluding remarks are given in Section 4.

2 Problem Formulation

2.1 ARCH Model and Conditional Heteroscedastic Noises

Traditional time series analysis assumes that the fluctuation amplitude (variance) of time series variables is fixed, but this assumption is not realistic. To deal with conditional heteroscedastic noises, Engle proposed the ARCH model in 1982[16]. This model may effectively capture inflation and stock market returns[17, 18].

In the ARCH model, the conditional heteroscedastic noise is defined as a martingale difference sequence $\{w_n, \mathcal{F}_n\}$ with time-varying conditional varices $\{h_n\}$, i.e.,

$$E[w_n^2 \mid \mathcal{F}_{n-1}] = h_n,$$

where $\{w_n\}$ is defined as the product of a white Gaussian noise $\{e_n\}$ and a time-dependent series $\{h_n\}$,

$$w_n = h_n e_n,$$

and where the series $\{h_n\}$ is modelled as[19]

$$h_n^2 = \lambda_0 + \lambda_1 w_{n-1}^2 + \cdots + \lambda_q w_{n-q}^2 = \lambda_0 + \sum_{i=1}^{q} \lambda_i w_{n-i}^2,$$

with $\lambda_0 > 0,\ q > 0$ and $\lambda_i \geq 0, i > 0$.

2.2 Weighted Least-Squares Self-Tuning Regulator

Consider the single-input single-output (SISO) stochastic systems

$$A(z)y_n = B(z)u_{n-1} + w_n, \quad n \geq 0, \tag{1}$$

where $\{y_n\}$, $\{u_n\}$ and $\{w_n\}$ are output, input and noise (without losing generality, suppose $y_n = u_n = w_n = 0, \forall n < 0$), $A(z)$ and $B(z)$ are polynomials in backward shift operator z,

$$A(z) = 1 + a_1 z + \cdots + a_p z^p, \quad p \geq 0,$$
$$B(z) = b_1 + b_2 z + \cdots + b_q z^{q-1}, \quad q \geq 1,$$

where a_i and b_j $(i = 1, 2, \cdots, p, j = 1, 2, \cdots, q)$ are unknown parameters, p and q are two known upper bounds of system order.

In order to write the model into a compact form, we introduce the following unknown parameter vector:

$$\theta = [-a_1 \cdots - a_p, b_1 \cdots b_q]^{\mathrm{T}} \tag{2}$$

and the corresponding regression vector:

$$\varphi_n = [y_n \cdots y_{n-p+1}, u_n \cdots u_{n-q+1}]^{\mathrm{T}}. \tag{3}$$

Then the system (1) can be abbreviated as the following regression model

$$y_{n+1} = \theta^{\mathrm{T}} \varphi_n + w_{n+1}, \quad n \geq 0. \tag{4}$$

Our purpose is to construct a feedback law u_n, based on the known data $\{y_0 y_1 \cdots y_n, u_0 u_1 \cdots u_{n-1}\}$ at any time n, such that the following averaged tracking error asymptotically approaches to its minimum:

$$J_n \stackrel{\text{def}}{=\!=} \frac{1}{n} \sum_{i=1}^{n} (y_i - y_i^*)^2, \tag{5}$$

where $\{y_i^*\}$ is a known reference signal to be tracked.

In order to analyse the above problem theoretically, the following conditions are introduced:

(A1) The noise sequence $\{w_n, \mathcal{F}_n\}$ is a martingale difference sequence ($\{\mathcal{F}_n\}$ is a sequence of nondecreasing sub σ-algebras) satisfying $\sum_{i=1}^{n} w_i^2 = \Theta(n)^\dagger$. Furthermore, suppose that there exists $\beta > 4$ such that

$$E[| w_{n+1} |^\beta | \mathcal{F}_n] = O(\sigma_\beta(n)) \quad \text{a.s.}, \tag{6}$$

where $\sigma_\beta(n)$ is a nondecreasing sequence which satisfies $\sigma_\beta(n) = O(r_n^\varepsilon)$ for some constant $\varepsilon \in (0,1)$, with r_n defined as $r_n \stackrel{\text{def}}{=\!=} 1 + \sum_{i=1}^{n} || \varphi_i ||^2$;

(A2) $B(z) \neq 0, \forall z : | z | \leq 1$ (minimum phase condition);

(A3) $\{y_n^*\}$ is a deterministic bounded signal.

Condition (A2) is necessary to ensure the stability of the closed-loop system for (4)[20].

To construct the STR, we consider the case where the parameter θ is known firstly.

Since $\{w_n\}$ is an unpredictable noise, it is easily to see that the tracking performence (5) will achieve its minimum, if we have

$$y_{n+1}^* = E[y_{n+1} | \mathcal{F}_n], \tag{7}$$

or

$$\theta^{\mathrm{T}} \varphi_n = y_{n+1}^*, \tag{8}$$

by (4) and Condition (A1). Therefore, we can express the optimal control that minimizes the tracking error (5) as

$$u_n = \frac{1}{b_1}(a_1 y_n + a_2 y_{n-1} + \cdots + a_p y_{n-p+1} - b_2 u_{n-1} - \cdots - b_q u_{n-q-1} + y_{n+1}^*). \tag{9}$$

Substituting (9) into the system (1) (or (8) into (4)), then the ideal closed-loop equation can be written as

$$y_n - y_n^* - w_n \equiv 0, \quad \forall n \geq 0. \tag{10}$$

†there exist constants $C > C' > 0$ such that $C' < \frac{1}{n} \sum_{i=1}^{n} w_i^2 < C$ for all large n.

When the parameter θ is not available, the controller (9) is not applicable, and we now introduce a weighted least-squares algorithm (WLS) to estimate the unknown parameter vector θ,

$$\theta_{n+1} = \theta_n + a_n P_n \varphi_n (y_{n+1} - \varphi_n^{\mathrm{T}} \theta_n), \tag{11}$$

$$P_{n+1} = P_n - a_n P_n \varphi_n \varphi_n^{\mathrm{T}} P_n, \tag{12}$$

$$a_n = (\lambda_n^{-1} + \varphi_n^{\mathrm{T}} P_n \varphi_n)^{-1}, \tag{13}$$

where

$$\lambda_n = \sigma_\beta(n)^{-\frac{2}{\beta}}, \tag{14}$$

with $\sigma_\beta(n)$ defined by Condition (A1), and the initial values θ_0 and $P_0 > 0$ can be chosen arbitrarily.

According to the "certainty equivalence principle", replacing θ in (8) by θ_n, we can get a WLS-type STR as bellow:

$$\theta_n^{\mathrm{T}} \varphi_n = y_{n+1}^* \tag{15}$$

or

$$u_n = \frac{1}{b_{1n}}(a_{1n} y_n + \cdots + a_{pn} y_{n-p+1} - b_{2n} u_{n-1} - \cdots - b_{qn} u_{n-q-1} + y_{n+1}^*), \tag{16}$$

where a_{in}, b_{jn} are θ_n's components

$$\theta_n \stackrel{\text{def}}{=\!=} [-a_{1n} \cdots - a_{pn}, b_{1n} \cdots b_{qn}]^{\mathrm{T}}.$$

Noting that in the ideal case where the parameter is known, the closed-loop equation is Equation (10). It is natural to expect that under the adaptive control (15) the closed-loop equation has the form

$$y_n - y_n^* - w_n \approx 0, \quad \forall n.$$

In the current stochastic adaptive case, it is feasible to achieve a weaker expectation, i.e., to expect the averaged tracking error goes to zero asymptotically. In other words, we expect the accumulated tracking error of the closed-loop system

$$R_n \stackrel{\text{def}}{=\!=} \sum_{i=1}^{n} (y_i - y_i^* - w_i)^2 \tag{17}$$

to satisfy

$$R_n = o(n) \quad \text{a.s.}, \tag{18}$$

which indicates that "$y_i - y_i^* - w_i$" approaches to zero on average.

From (A1) and the martingale convergence theorem[15], it is not difficult to show that

$$R_n = o(n) \Leftrightarrow J_n \xrightarrow[n\to\infty]{} l_w^2 \quad \text{a.s.} \tag{19}$$

with J_n defined by (5), and l_w^2 is defined as $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} w_i^2$ (assume this limit exists), which is the minimum value of the tracking performance (5).

The above analysis yields that the necessary and sufficient condition for the STR (15) to be stable and optimal is that (18) holds. Then we only need to discuss whether (18) would hold or not when the WLS-based STR is defined by (15).

2.3 Modification on High-Frequency Gain

From the system (1), we know there holds

$$\lim_{z\to\infty}\frac{B(z^{-1})}{A(z^{-1})}=b_1,$$

where b_1 is usually called as "high-frequency gain".

In the previous work, we have discussed the case where b_1 is known[12], so this in paper we consider the general case where the high-frequency gain b_1 is unknown. The first problem we meet is that u_n may not be well defined as (16), since the set $\{b_{1n}=0\}$ may have a positive probability.

Since we are not going to make additional assumptions in the analysis, a natural and simple method for overcoming this difficulty is to modify θ_n slightly. Denote this modified estimate by $\widehat{\theta}_n$, then the WLS-based STR can be written as

$$\widehat{\theta}_n^{\mathrm{T}}\varphi_n=y_{n-1}^* \tag{20}$$

or

$$\widehat{u}_n=\frac{1}{\widehat{b}_{1n}}(\widehat{a}_{1n}y_n+\cdots+\widehat{a}_{pn}y_{n-p+1}-\widehat{b}_{2n}u_{n-1}-\cdots-\widehat{b}_{qn}u_{n-q-1}+y_{n+1}^*), \tag{21}$$

where $\widehat{a}_{in}$ and $\widehat{b}_{jn}$ are the components of $\widehat{\theta}_n$, which are the (modified) estimates for a_i and b_j respectively.

We try to design a modification of θ_k, such that the closed-loop system has the convergence rate (30) as long as the system (1) satisfies Conditions (A1)–(A3).

Consider the following form of modifications on θ_n

$$\widehat{\theta}_n=\theta_n+P_n^{\frac{1}{2}}e_{i_n}, \tag{22}$$

where θ_n is the WLS estimate defined by (11), P_n is defined by (12), and $\{i_n\}$ is a sequence of integers taking values on $\{0,1,\cdots,d\}$ with $d=p+q$, which is defined by

$$i_n=\arg\max_{0\le i\le d}\mid b_{1n}+e_{p+1}^{\mathrm{T}}P^{\frac{1}{2}}e_i\mid, \tag{23}$$

where $e_0=0$, and $e_i, 1\le i\le d$, is the ith column of the $d\times d$ identity matrix.

In the proof of Theorem 3.1, we will show that the follows three basic properties are satisfied:

(H1) $\|\widehat{\theta}_n\|^2=O(\log r_{n-1})$ a.s.;

(H2) $\sum_{k=1}^n\frac{\varphi_k^{\mathrm{T}}\widetilde{\theta}_n}{\lambda_k^{-1}+\varphi_k^{\mathrm{T}}P_k\varphi_k}=O(\log r_n)$ a.s.;

(H3) $\liminf_{n\to\infty}\sqrt{\log(n+r_{n-1})}\mid\widehat{b}_{1n}\mid\neq 0$ a.s.;

here $\{\varphi_n\}$ and $\{P_n\}$ are defined by (3) and (12), $\widetilde{\theta}_n\triangleq\theta-\widehat{\theta}_n$, and $\widehat{b}_{1n}$ is the estimate for b_{1n} given by $\widehat{\theta}_n$.

2.4 Notations and Estimations

First, we introduce some notations:

$$\alpha_k \stackrel{\text{def}}{=\!=} \frac{(\varphi_k^{\mathrm{T}}\widetilde{\theta}_k)^2}{\lambda_k^{-1}+\varphi_k^{\mathrm{T}}P_k\varphi_k}, \quad \delta_k \stackrel{\text{def}}{=\!=} \mathrm{tr}(P_k-P_{k+1}), \tag{24}$$

$$r_n \stackrel{\text{def}}{=\!=} 1+\sum_{i=1}^{n}\|\varphi_i\|^2, \quad \overline{r}_n \stackrel{\text{def}}{=\!=} 1+\sum_{i=1}^{n}\lambda_i\|\varphi_i\|^2, \quad \widetilde{\theta}_n=\theta-\theta_n. \tag{25}$$

If System (1) satisfies Condition (A1), then[20]

$$\sum_{k=0}^{n}\alpha_k = O(\log \overline{r}_n), \quad \text{a.s.} \tag{26}$$

It is not difficult to obtain $O(\log r_n) = O(\log \overline{r}_n)$ from the definition (25), so we have

$$\sum_{k=0}^{n}\alpha_k = O(\log r_n), \quad \text{a.s.} \tag{27}$$

Suppose $\{d_n\}$ is a nondecreasing positive deterministic sequence, then

$$w_{n+1}^2 = O(d_n), \quad \text{a.s.} \tag{28}$$

When Condition (A1) is satisfied, using Markov inequality and (6), we have

$$\sum_{n=1}^{\infty}P(w_{n+1}^2 \geq \sigma_\beta^{\frac{2}{\beta}}(n)n^\delta \mid \mathcal{F}_n) \leq \sum_{n=1}^{\infty}\frac{E[|w_{n+1}|^\beta|\mathcal{F}_n]}{\sigma_\beta(n)n^{\frac{\delta\beta}{2}}} < \infty, \quad \text{a.s.},$$

where β is given by (6).

Then by means of Borel-Cantelli lemma we can get

$$d_n = \sigma_\beta^{\frac{2}{\beta}}(n)n^\delta, \quad \forall\delta \in \left(\frac{2}{\beta}, \frac{1}{2}\right). \tag{29}$$

3 The Main Result

Theorem 3.1 *Let Conditions* (A1)–(A3) *be satisfied for the system* (1)*. If the control law is defined by* (20)*, with* $\{\widehat{\theta}_k\}$ *defined by* (22) *and* (23)*, then the closed-loop system is globally stable and asymptotically optimal with the following rate of convergence:*

$$R_n = O(n^\gamma d_n), \quad a.s. \quad \forall\gamma > 0, \tag{30}$$

where R_n*,* d_n *are defined by* (17) *and* (29).

To prove this theorem, we need to establish two lemmas first.

Lemma 3.2 *Consider the system* (1)*, if it satisfies Conditions* (A1)–(A3)*, the control law is defined by* (20)*, then there exists a positive random sequence* $\{L_n\}$ *such that*

$$y_n^2 \leq L_n, \quad a.s. \quad \forall n, \tag{31}$$

and $\{L_n\}$ satisfies following "linear time-varying" relation:

$$L_{n+1} \leq (\xi + Cf_n)L_n + t_n, \tag{32}$$

where the constants $\xi \in (0,1), C > 0$, and α_n and

$$f_n = [\alpha_n \delta_n log(n + r_n)]^2 + \alpha_n \delta_n, \tag{33}$$

$$t_n = O([d_n + \lambda_n^{-1}] \log^4(n + r_n)), \tag{34}$$

with α_n and δ_n defined in (24), $\{t_n\}$ is a positive random sequence satisfying d_n defined by (29) and r_n defined by (25).

Lemma 3.3 *Under the same conditions as those of Lemma 3.1, we have the follow estimation*

$$\| \varphi_n \|^2 = O([d_n + \lambda_n^{-1}][n + r_n]^\gamma), \quad a.s. \quad \forall \gamma > 0, \tag{35}$$

where r_n is defined by (25) and d_n is defined by (29).

The detailed proof of Lemma 3.2 and Lemma 3.3 are supplied in the Appendix.

Proof First, we prove that when $\{\widehat{\theta}_k\}$ satisfies Conditions (H1)–(H3), the closed-loop system is globally stable, the tracking error is asymptotically optimal and has the convergence rate (30).

Using (24), (27), (34) and 3.3, we can get

$$\begin{aligned} R_{n+1} &= \sum_{i=0}^{n} (y_{i+1} - y_{i+1}^* - w_{i+1})^2 \\ &= \sum_{i=0}^{n} (\varphi_i^{\mathrm{T}} \widetilde{\theta}_i)^2 \\ &= \sum_{i=0}^{n} \alpha_i (\lambda_i^{-1} + \varphi_i^{\mathrm{T}} P_i \varphi_i) \\ &= O(\lambda_n^{-1} \log r_n) + O\bigg(\sum_{i=0}^{n} \alpha_i \delta_i \| \varphi_i \|^2 \bigg) \\ &= O(\lambda_n^{-1} \log r_n) + O([n + r_n]^\gamma [d_n + \lambda_n^{-1}] \log r_n) \\ &= O([n + r_n]^\gamma [d_n + \lambda_n^{-1}] \log r_n), \quad \forall \gamma > 0. \end{aligned} \tag{36}$$

We just need to prove $r_n = O(n)$.

Using (36), the boundedness of $\{y_i^*\}$ and the property of $\{w_i\}$ in Condition (A1), we obtain

$$\sum_{i=0}^{n+1} y_i^2 = O(n) + O([n + r_n]^\gamma [d_n + \lambda_n^{-1}]), \quad \forall \gamma > 0. \tag{37}$$

Based on the minimum phase Condition (A2) and the above formula we can get the following estimation from the system (1),

$$\sum_{i=0}^{n} u_i^2 = O(n) + O([n + r_n]^\gamma [d_n + \lambda_n^{-1}]), \quad \forall \gamma > 0. \tag{38}$$

By the last two relationships, (14), (27), and Condition (A1) we have

$$\begin{aligned} r_n &= 1 + \sum_{i=1}^{n} \| \varphi_i \|^2 \\ &= O(n) + O\left([\sigma_\beta^{\frac{2}{\beta}}(n)n^\delta + \sigma_\beta^{\frac{2}{\beta}}(n)][n+r_n]^\gamma\right) \\ &= O(n) + O\left(\sigma_\beta^{\frac{2}{\beta}}(n)n^\delta[n+r_n]^\gamma\right) \\ &= O(n) + O\left(r_n^{\frac{2\varepsilon}{\beta}} n^\delta [n+r_n]^\gamma\right) \\ &= O(n) + O\left(r_n^{\frac{2\varepsilon}{\beta}} n^{\delta+\gamma} + r_n^{\frac{2\varepsilon}{\beta}+\gamma} n^\delta\right) \\ &= O(n) + O\left(r_n^{\frac{2\varepsilon}{\beta}+\gamma} n^{\delta+\gamma}\right), \quad \varepsilon \in (0,1), \quad \forall \gamma > 0. \end{aligned}$$

Since $\beta > 4$, $\delta \in (\frac{2}{\beta}, \frac{1}{2})$, we can choose γ small enough to satisfy $\frac{2\varepsilon}{\beta} + 2\gamma + \delta < 1$, such that

$$\begin{aligned} \frac{r_n}{n} &= O(1) + O\left(\left[\frac{r_n}{n}\right]^{\frac{2\varepsilon}{\beta}+\gamma} \cdot \frac{1}{n^{1-\frac{2\varepsilon}{\beta}-2\gamma-\delta}}\right) \\ &= O(1) + o\left(\left[\frac{r_n}{n}\right]^{\frac{2\varepsilon}{\beta}+\gamma}\right), \quad \varepsilon \in (0,1), \quad \forall \gamma > 0. \end{aligned}$$

So we have

$$r_n = O(n), \quad \text{a.s.}, \tag{39}$$

substituting the above estimation into (36), we obtain

$$R_n = O(n^\gamma d_n), \quad \forall \gamma > 0, \tag{40}$$

which implies

$$R_n = o(n), \quad \text{a.s.}$$

Consequently, the closed-loop system has optimality by (19). Furthermore, based on the optimality and Conditions (A1)–(A3) we have for any initial values y_0 and u_0,

$$\sum_{i=0}^{n} (y_i^2 + u_i^2) = O(n), \quad \text{a.s.},$$

which shows the system has global stability.

Next, we need only show that $\{\widehat{\theta}_n\}$ defined by (22) and (23) satisfies the requirements (H1)–(H3).

From [8] we know that θ_n has the estimation

$$\| \theta_n \| = O(\sqrt{\log r_{n-1}}),$$

since both $\{e_{in}\}$ and $\{P_n^{\frac{1}{2}}\}$ are bounded

$$\| \widehat{\theta}_n \|^2 = \| \theta_n + P_n^{\frac{1}{2}} e_{in} \|^2 = O(\log r_{n-1}).$$

So (H1) is satisfied.

Next, from (12) and (13), we have the relation

$$\sum_{k=0}^{n} \alpha_k \lambda_k \varphi_k^{\mathrm{T}} P_k \varphi_k = \sum_{k=0}^{n} \frac{| P_{k+1}^{-1} | - | P_i^{-1} |}{| P_{k+1}^{-1} |} \\ \leq \log | P_{n+1}^{-1} | + \log | P_0^{-1} |, \tag{41}$$

by (27) and this relation, we have

$$\begin{aligned} &\sum_{k=1}^{n} \frac{[\varphi_k^{\mathrm{T}}(\theta - \widehat{\theta}_k)]^2}{\lambda_k^{-1} + \varphi_k^{\mathrm{T}} P_k \varphi_k} \\ =&\sum_{k=1}^{n} \alpha_k [\varphi_k^{\mathrm{T}}(\theta - \theta_k - P_k^{\frac{1}{2}} e_{ik})]^2 \\ =&O\Big(\sum_{k=1}^{n} \alpha_k [\varphi_k^{\mathrm{T}}(\theta - \theta_k)]^2\Big) + O\Big(\sum_{k=0}^{n} \alpha_k \varphi_k^{\mathrm{T}} P_k \varphi_k\Big) \\ =&O(\log r_n). \end{aligned}$$

Hence, (H2) is also satisfied.

To prove (H3) is satisfied, we set $\beta_n = P_n^{\frac{1}{2}}(\theta - \theta_n)$, so we have $\theta = \theta_n + P_n^{\frac{1}{2}}\beta_n$. Since we know that $\| \beta_n \|^2 = O(\log r_{n-1})$[8],

$$\begin{aligned} | b_1 |^2 &= | b_{1n} + e_{p+1} P_n^{\frac{1}{2}} \beta_n |^2 \\ &= \left| \begin{bmatrix} b_{1n} & e_{p+1} P_1^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} 1 \\ \beta_n \end{bmatrix} \right|^2 \\ &= \left\| \begin{bmatrix} b_{1n} & e_{p+1} P_n^{\frac{1}{2}} \end{bmatrix} \right\|^2 (1 + \| \beta_n \|^2) \\ &= O\Big(\left\| \begin{bmatrix} b_{1n} & e_{p+1} P_n^{\frac{1}{2}} \end{bmatrix} \right\|^2 \log(r_{n-1}) \Big). \end{aligned}$$

By Condition (A2) we know that $b_1 \neq 0$, which means there exists a random variable $c > 0$ such that

$$\left\| \begin{bmatrix} b_{1n} & e_{p+1} P_n^{\frac{1}{2}} \end{bmatrix} \right\|^2 \geq \frac{c}{\log r_{n-1}}, \quad \forall n > 0.$$

Denote

$$M = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e_0 & e_1 & \cdots & e_d \end{bmatrix},$$

it is easily to see that M is nonsingular, so

$$\lambda_0 \triangleq \lambda_{\min}(M M^{\mathrm{T}}) > 0,$$

from this, with the definition of $\widehat{\theta}_n$,

$$\begin{aligned}
|\ \widehat{b}_{1n}\ |^2 &= |\ b_{1n} + e_{p+1}^{\mathrm{T}} P_n^{\frac{1}{2}} e_{in}\ |^2 \\
&= \max_{0\le i\le d} \left| \begin{bmatrix} b_{1n} & e_{p+1}^{\mathrm{T}} P_n^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} 1 \\ e_i \end{bmatrix} \right|^2 \\
&\ge (1+d)^{-1} \left\| \begin{bmatrix} b_{1n} & e_{p+1}^{\mathrm{T}} P_n^{\frac{1}{2}} \end{bmatrix} M \right\|^2 \\
&\ge \lambda_0 (1+d)^{-1} \left\| \begin{bmatrix} b_{1n} & e_{p+1}^{\mathrm{T}} P_n^{\frac{1}{2}} \end{bmatrix} \right\|^2 \\
&\ge \frac{\lambda_0 c}{1+d} \frac{1}{\log r_{n-1}}, \quad \forall n.
\end{aligned}$$

So we prove that (H3) is hold. The proof of Theorem 3.1 is complete. ∎

4 Conclusions

In this paper, we have considered self-tuning problems for dynamical systems where the noise process is a conditional heteroscedastic time series, whose conditional variances are unbounded, and where the high-frequency gain of the control systems is assumed to be unknown. By introducing a weighted least-squares algorithm for estimating of the unknown parameters, and by further introducing a suitable modification on the weighted least squares estimates in the design of STR, we are able to show that the closed-loop system is globally stable with an asymptotically optimal tracking error under certain natural conditions. Compared with the authors' previous work on classical self-tuning regulators where the noises are of bounded conditional variances, the main result of this paper appears to be more broadly applicable.

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Appendix

Proof of Lemma 3.1

Proof By (4) and (15),

$$y_{n+1} = \varphi_n^{\mathrm{T}} \widetilde{\theta}_n + y_{n+1}^* + w_{n+1}. \tag{42}$$

From (12) we have $\varphi_n^{\mathrm{T}} P_{n+1} \varphi_n \leq \lambda_n^{-1}$.

By using (24), (27), Condition (A1) and the boundedness of $\{y_n^*\}$, we can get

$$\begin{aligned} y_{n+1}^2 &\leq 2(\varphi_n^{\mathrm{T}} \widetilde{\theta}_n)^2 + O(d_n) \\ &= 2[\alpha_n \lambda_n^{-1} + \varphi_n^{\mathrm{T}} P_{n+1} \varphi_n + \varphi_n^{\mathrm{T}} (P_n - P_{n+1}) \varphi_n] + O(d_n) \\ &\leq 2\alpha_n [2\lambda_n^{-1} + \delta_n \parallel \varphi_n \parallel^2] + O(d_n) \\ &= 2\alpha_n \delta_n \parallel \varphi_n \parallel^2 + O([d_n + \lambda_n^{-1}] \log r_n). \end{aligned} \tag{43}$$

Since System (1) satisfies the minimum phase Condition (A2), we know that there exists $\xi \in (0,1)$ such that

$$u_{n-1}^2 = O\biggl(\sum_{i=0}^{n} \xi^{n-i} y_i^2 \biggr) + O(d_n), \tag{44}$$

then

$$\| \varphi_n \|^2 - u_n^2 = O\bigg(\sum_{i=0}^{n} \xi^{n-i} y_i^2\bigg) + O(d_n). \tag{45}$$

By the properties (H1) and (H3), it follows from (21) that

$$u_n^2 = O\bigg(\log^2(n + r_{n-1})\bigg(\sum_{i=0}^{p-1} y_{n-i}^2 + \sum_{i=1}^{q-1} x_{n-i}^2\bigg) + \log(n + r_{n-1}\bigg), \tag{46}$$

putting (44) into this, it follows that

$$u_n^2 = O\bigg(\log^2(n + r_{n-1})\bigg(\sum_{i=0}^{n} \xi^{n-i} y_i^2 + d_n\bigg)\bigg). \tag{47}$$

Let

$$L_n = \sum_{i=0}^{n} \xi^{n-i} y_i^2,$$

by (45) and (47), we have

$$\| \varphi_n \|^2 = O(\log^2(n + r_{n-1})[L_n + d_n]). \tag{48}$$

From (8) we can get

$$b_1 u_n = \varphi_n^{\mathrm{T}} \widetilde{\theta}_n + y_{n+1}^* + (b_1 u_n - \theta^{\mathrm{T}} \varphi_n), \tag{49}$$

so, by (45) we have

$$\begin{aligned} b_1^2 u_n^2 &\le 3(\varphi_n^{\mathrm{T}} \widetilde{\theta}_n)^2 + O(1 + | b_1 u_n - \theta^{\mathrm{T}} \varphi_n |^2) \\ &= 3(\varphi_n^{\mathrm{T}} \widetilde{\theta}_n)^2 + O(L_n + d_n). \end{aligned} \tag{50}$$

Similarly to the proof of (43), it is known that

$$(\varphi_n^{\mathrm{T}} \widetilde{\theta}_n)^2 \le 2\lambda_n^{-1} \alpha_n + \alpha_n \delta_n \| \varphi_n \|^2, \tag{51}$$

substituting this into (49), we see that

$$u_n^2 = O(\alpha_n \delta_n \| \varphi_n \|^2) + O(L_n + d_n + \lambda_n^{-1} \log r_n),$$

combining this with (45), we get

$$\| \varphi_n \|^2 = O(\alpha_n \delta_n \| \varphi_n \|^2) + O(L_n + d_n + \lambda_n^{-1} \log r_n),$$

putting (48) into this, we have

$$\| \varphi_n \|^2 = O(\alpha_n \delta_n [L_n + d_n] \log^2(n + r_{n-1})) + O(L_n + d_n + \lambda_n^{-1} \log r_n).$$

Finally, substituting this into (43), we can see there exists a constant $C > 0$, with f_n and t_n defined by (33) and (34)

$$L_{n+1} \le \xi L_n + y_{n+1}^2 \le (\xi + C f_n) L_n + t_n,$$

so we finish the proof of Lemma 3.1. ∎

Proof of Lemma 3.2

Proof From (32) we can obtain

$$\begin{aligned} L_{n+1} &\leq \prod_{j=0}^{n}(\xi + Cf_j)L_0 + \sum_{i=0}^{n}\prod_{j=i+1}^{n}(\xi + Cf_j)t_i \\ &= \xi^{n+1}\prod_{j=0}^{n}(1+\xi^{-1}Cf_j)L_0 + \sum_{i=0}^{n}\xi^{n-i}\prod_{j=i+1}^{n}(1+\xi^{-1}Cf_j)t_i. \end{aligned} \tag{52}$$

We now proceed to estimate the products $\prod_{j=i+1}^{n}(1+\xi^{-1}Cf_j)$ in the above formula. By the property (H2), we have for an arbitrary $\gamma > 0$, there exists $\eta > 0$ and i_0 large enough to make

$$\eta\sum_{j=i}^{n}\alpha_j \leq \gamma \log r_n, \quad \forall n \geq i \geq i_0. \tag{53}$$

From (24) we know that

$$\sum_{j=0}^{\infty}\delta_j = \sum_{j=0}^{\infty}\mathrm{tr}(P_j - P_{j+1}) \leq \mathrm{tr}P_0 < \infty, \tag{54}$$

using this, we know that there exists an integer $i_0 > 0$ sufficiently large such that

$$\frac{4}{\eta}\left(\frac{C}{\xi}\right)^{\frac{1}{2}}\sum_{j=i}^{\infty}\delta_j \leq \varepsilon, \quad \forall i \geq i_0. \tag{55}$$

Using the inequality $1 + x \leq \mathrm{e}^x$ and $(1 + xy) \leq (1 + x)(1 + y)$ $\forall x \geq 0, y \geq 0$, we have

$$\begin{aligned} &\prod_{j=i}^{n}(1+\xi^{-1}C\alpha_j\delta_j\log^2(j+r_j)) \\ \leq& \prod_{j=i+1}^{n}\left[1+\left(\frac{1}{2}\eta\alpha_j\right)^2\right]\prod_{j=i}^{n}\left\{1+\xi^{-1}C\left[\frac{2}{\eta}\delta_j\log(j+r_j)\right]^2\right\} \\ \leq& \exp\left(\eta\sum_{j=i+1}^{n}\alpha_j\right)\exp\left\{\frac{4}{\eta}\left(\frac{C}{\xi}\right)^{\frac{1}{2}}\sum_{j=i+1}^{n}\delta_j\log(j+r_j)\right\} \\ \leq& \exp\{\gamma\log(r_n)\}\exp\left\{[\log(n+r_n)]\left[\frac{4}{\eta}\left(\frac{C}{\xi}\right)^{\frac{1}{2}}\sum_{j=i}^{n}\delta_j\right]\right\} \\ \leq& \gamma\log r_n\exp\{\gamma\log(n+r_n)\} \\ =& O([n+r_n]^{2\gamma}), \quad \forall n \geq i \geq i_0. \end{aligned} \tag{56}$$

Furthermore, for any $n \geq i \geq i_0$

$$\begin{aligned} \prod_{j=i}^{n}(1+\xi^{-1}C\alpha_j\delta_j) &\leq \exp\left(\xi\sum_{j=i}^{n}\alpha_j\right)\exp\left\{\frac{C}{\xi\eta}\sum_{j=i}^{n}\delta_j\right\} \\ &= O(r_n)^{\gamma}, \quad \forall n \geq i \geq i_0. \end{aligned} \tag{57}$$

Finally, by the definition of f_j, it follows from the above two formulae that

$$
\begin{aligned}
&\prod_{j=i+1}^{n}(1+\xi^{-1}Cf_j)\\
\leq&\prod_{j=i+1}^{n}\left\{1+\xi^{-1}C\left[\frac{2}{\eta}\delta_j\log(j+r_j)\right]^2\right\}\prod_{j=i+1}^{n}(1+\xi^{-1}C\alpha_j\delta_j)\\
=&O([n+r_n]^{3\gamma}),\quad \forall n\geq i\geq i_0.
\end{aligned}
$$

Substituting this into (52), we have

$$L_{n+1}=O([n+r_n]^{3\gamma}[d_n+\lambda_n^{-1}]\log^4(n+r_n)),\quad \forall\gamma>0.$$

Thus, the arbitrariness of γ can lead to

$$y_{n+1}^2\leq L_{n+1}=O([n+r_n]^{\gamma}[d_n+\lambda_n^{-1}]),\quad \forall\gamma>0.$$

And using (44) we can get

$$u_n^2=O([n+r_n]^{\gamma}[d_n+\lambda_n^{-1}]),\quad \forall\gamma>0,$$

which completes the proof. ∎