On PID Control Theory for Nonaffine Uncertain Stochastic Systems*

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Abstract PID (proportional-integral-derivative) control is recognized to be the most widely and successfully employed control strategy by far. However, there are limited theoretical investigations explaining the rationale why PID can work so well when dealing with nonlinear uncertain systems. This paper continues the previous researches towards establishing a theoretical foundation of PID control, by studying the regulation problem of PID control for nonaffine uncertain nonlinear stochastic systems. To be specific, a three dimensional parameter set will be constructed explicitly based on some prior knowledge on bounds of partial derivatives of both the drift and diffusion terms. It will be shown that the closed-loop control system will achieve exponential stability in the mean square sense under PID control, whenever the controller parameters are chosen from the constructed parameter set. Moreover, similar results can also be obtained for PD (PI) control in some special cases. A numerical example will be provided to illustrate the theoretical results.

Keywords Asymptotically regulation, global stability, nonaffine PID control, stochastic systems, uncertain structure.

1 Introduction

It is well-known that PID (proportional-integral-derivative) control is the most widely employed feedback strategy by far, and has shown its impact on various systems, ranging from process control to flight control (see, e.g., [1–3]). PID controller is believed to be "bread and butter" of control enigneering[4]. The classical PID controller, which has an easy-to-use linear structure consisting of three terms constructed based on the real-time control error, is a typical data-driven control strategy. It has the ability to eliminate steady state offsets through the integral action and to anticipate the near future behavior via the derivative action. Despite of the remarkable progresses in modern control theory over the past 60 years, the classical PID

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controller still plays a dominating role in control practice (see, e.g., [5]). This largely attributes to the fact that the Newton's second law still plays a fundamental role in modeling mechanical systems, and that the PID control has two-sided large scale robustness with respect to system uncertain structure and controller parameter selection[6].

Ever since the emergence of PID controller in the early 20th century, extensively academic and industrial efforts have been made in improving the effectiveness of PID control loops and, as a result, a large amount of tuning methods continue to emerge (see, e.g., [7, 8]). One may naturally believe that PID control is matured in both theory and practice. However, the de facto situation is that poorly tuned PID controlled loops are quite common in practice (see, e.g., [9–11]), and almost all the existing tuning methods are case dependent and heavily rely on experience or experiment or both, which might lead to unsatisfactory performance. Moreover, practical control systems are bound to be nonlinear with uncertainties and disturbances, while most of the existing analytic results on PID are conducted for linear systems, and only a few explicitly for uncertain nonlinear systems (see, e.g., [12–14]). Therefore, it can be said that "PID controllers remain poorly understood"[15], and a comprehensive understanding towards PID may improve its widespread practice and so contribute to better product quality[1].

Thus, it is necessary to make efforts towards establishing the theoretical foundation on PID control for nonlinear uncertain systems. In order to figure out the rationale why PID can make such amazing achievements, nonlinearity and uncertainty must be taken into consideration. Moreover, efforts must be taken on analyzing the limits of PID control in a general framework. In view of these facts, some rigorous mathematical investigations have been made on a class of second order systems described by the celebrated Newton's second law (see, e.g., [16–20]). For instance, [16, 18] show that global stability can be achieved by PID controllers for a class of nonlinear uncertain systems without input channel uncertainty. [19] proves that similar results can be obtained for PID controlled stochastic systems with structural uncertainty consisting of dynamic uncertainty, input channel uncertainty and diffusion uncertainty. Nonetheless, there is still a long way to go to fill up the gap between theory and practice of PID. On the one hand, results in [19] are conducted under restrictive conditions where the input channel must be described by positive definite constant matrices with known lower and upper bounds. On the other hand, numerous practical systems cannot be characterized by affine nonlinear forms since the changing rate of system states may not be linearly related to control inputs, such as glucose control system[21], pendulum control system[22], flight control system[23], etc. Nevertheless, few theoretical results can be found on PID controlled nonaffine nonlinear systems, even for deterministic systems. To the best of authors' knowledge, [20] provides the first rigorous mathematical analysis on global performance of PID controlled MIMO (multi-input multi-output) nonaffine systems without external disturbances.

In this paper, we will consider the global regulation problem for a class of nonaffine stochastic systems, which is an extension of the deterministic nonlinear uncertain systems considered in [20]. The main result of this paper will also significantly weaken the assumptions given in our previous results[19]. To be specific, a three dimensional unbounded open set for the controller parameters could be constructed, from which various PID controllers could be designed to control the uncertain stochastic systems globally with the regulation errors converge exponentially in the mean square sense. The constructed PID parameter set is based on some prior knowledge on the upper bounds of partial derivatives of both the drift and diffusion terms. Moreover, the control input is no longer needed to be linearly related to the derivative of system states, and only the lower bound of the input channel is needed. A design formula of PD controller will be also provided for a class of nonaffine stochastic systems with relative degree two in the cases where more restrictions on the system dynamics are satisfied. Similar results can further be obtained on designing PI controller for a class of nonaffine stochastic systems with relative degree one.

The remainder of this paper is organized as follows. Some backgrounds and the problem formulation are provided in Section 2. Main results are placed in Section 3 together with their mathematical proofs given in Section 4. Section 5 will provide a numerical simulation example. Some concluding remarks will also be given at the end of this paper.

2 Backgrounds and Problem Formulation

2.1 Notations and Definitions

First, we introduce some notations that will be used throughout this paper:

Denote $\mathbb{R}^+$ as the set of all positive real numbers, moreover, denote

$$\mathbb{R}^{n+} = \{(x_1, x_2, \cdots, x_n) | x_i > 0,\ i = 1, 2, \cdots, n\}.$$

Denote $C^k(\mathbb{R}^n, \mathbb{R}^m)$ as the space of functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ with k-times continuous partial derivatives. For a given vector $x \in \mathbb{R}^n$, $\|x\|$ refers to its Euclidean norm. For a given matrix $Q \in \mathbb{R}^{m\times n}$, $\|Q\|$ refers to the corresponding induced norm (i.e., $\|Q\| = \sup_{x\in\mathbb{R}^n, \|x\|=1} \|Qx\|$). For a given matrix $Q \in \mathbb{R}^{n\times n}, \mathrm{tr}(Q)$ denotes its trace. For a given vector-valued function $\Phi(\cdot) \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, the Jacobian matrix of the mapping Φ is defined as follows:

$$\frac{\partial \Phi}{\partial x} = \begin{bmatrix} \frac{\partial \phi_1}{\partial x_1} & \cdots & \frac{\partial \phi_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \phi_n}{\partial x_1} & \cdots & \frac{\partial \phi_n}{\partial x_n} \end{bmatrix}.$$

For a given function $f(\cdot) \in C^2(\mathbb{R}^n, \mathbb{R})$, the Hessian matrix of f is defined by

$$\frac{\partial^2 f}{\partial x^2} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}.$$

Next, consider the following stochastic differential equation (SDE):

$$\begin{aligned} &dx(t) = f(x(t))dt + \sigma(x(t))dw(t), \\ &x(0) = x_0 \in \mathbb{R}^n, \end{aligned} \tag{1}$$

where $x(t)$ is the state, $w(t)$ is a one-dimensional standard Brownian motion, and $f(\cdot) \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, $\sigma(\cdot) \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ are functions satisfying $f(0) = \sigma(0) = 0$.

Definition 2.1 Given a function $V(x,t) \in C^2(\mathbb{R}^n \times \mathbb{R}^+, \mathbb{R})$ associated with the SDE (1). The differential operator L acting on V is defined by

$$LV(x,t) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x) + \frac{1}{2}\mathrm{tr}\Big\{\sigma(x)^\tau \frac{\partial^2 V}{\partial x^2}\sigma(x)\Big\}.$$

2.2 Problem Formulation

Consider the following nonaffine stochastic control systems with relative degree two:

$$\begin{cases} dx_1(t) = x_2(t)dt, \\ dx_2(t) = f(x_1(t), x_2(t), u(t))dt + \sigma(x_1(t), x_2(t))dw(t), \end{cases} \tag{2}$$

where $x_1(t),\ x_2(t) \in \mathbb{R}^n$ are state variables, $u(t) \in \mathbb{R}^n$ is the control input, and where $f(\cdot) \in C^1(\mathbb{R}^{3n}, \mathbb{R}^n)$ and $\sigma(\cdot) \in C^1(\mathbb{R}^{2n}, \mathbb{R}^n)$ are unknown nonlinear functions of their respective variables, and $w(t)$ is a one-dimensional standard Brownian Motion.

Remark 2.2 The above model describes a class of nonlinear systems with uncertain system dynamics and external disturbances, and includes a large amount of practical mechanical systems modeled by the celebrated Newton's second law, e.g., inverted pendulum, helicopter dynamics[23], etc.

In this paper, we focus on the classical PID controller:

$$u(t) = k_i \int_0^t e(s)ds + k_p e(t) + k_d \dot{e}(t), \qquad e(t) = r^* - x_1(t), \tag{3}$$

where $e(t) \in \mathbb{R}^n$ is the control error, $r^* \in \mathbb{R}^n$ is the reference signal, and show that the classical PID controller has the ability to globally stabilize and to regulate the nonlinear uncertain stochastic system (2), under some suitable conditions on the nonlinear functions $f(\cdot)$ and $\sigma(\cdot)$.

3 Main Results

3.1 PID Control

Following a similar framework as in the investigation of PID control for nonaffine deterministic systems in [20], we assume that the uncertain drift function $f(\cdot)$ belongs to the following function class:

Assumption 3.1 (Function Space for Uncertain Drift Function)

$$\mathcal{F}_{L_1,L_2,\underline{b}} = \left\{ f \in C^1\left(\mathbb{R}^{3n}, \mathbb{R}^n\right) \middle| \left\|\frac{\partial f}{\partial x_1}\right\| \le L_1, \left\|\frac{\partial f}{\partial x_2}\right\| \le L_2, \frac{\partial f}{\partial u} \ge \underline{b} I_n \right\},$$

where $L_1, L_2, \underline{b}$ are positive constants.

Remark 3.1 The three constants L_1, L_2 and $\underline{b}$ will be used to describe the system uncertainty quantitatively, which play a key role in designing the PID parameters, as will be shown shortly. Moreover, we remark that the boundedness of the partial derivatives $\frac{\partial f}{\partial x_i}$ $(i = 1, 2)$, appears to be necessary in general to get global control results, since the PID control is a linear output feedback controller, see Proposition 1 in [24].

We further assume that the uncertain diffusion function $\sigma(\cdot)$ belongs to the following function class:

Assumption 3.2 (Function Space for Uncertain Diffusion Function)

$$\mathcal{G}_{N_1,N_2,r^*} = \left\{ \sigma \in C^1 \left(\mathbb{R}^{2n}, \mathbb{R}^n\right) \middle| \left\| \frac{\partial \sigma}{\partial x_1} \right\| \leq N_1, \left\| \frac{\partial \sigma}{\partial x_2} \right\| \leq N_2, \sigma(r^*, 0) = 0 \right\},$$

where N_1 and N_2 are positive constants and $r^* \in \mathbb{R}^n$ is the reference signal.

Remark 3.2 From the definition of $\mathcal{G}_{N_1,N_2,r^*}$, one can see that the diffusion function is required to be vanished at the point $(r^*, 0)$. The following proposition explains the necessity of this requirement.

Proposition 3.3 *Consider the following SDE with PID control:*

$$\begin{cases} dx_1(t) = x_2(t)dt, \\ dx_2(t) = f(x_1(t), x_2(t), u(t))dt + \sigma(x_1(t), x_2(t))dw(t), \\ u(t) = k_i \int_0^t e(s)ds + k_p e(t) + k_d \dot{e}(t), \qquad e(t) = r^* - x_1(t), \end{cases} \tag{4}$$

where $(x_1(t), x_2(t)) \in \mathbb{R}^2$ are the system states, $f(\cdot)$ and $\sigma(\cdot)$ are Lipschitz functions and r^* is a constant setpoint. Suppose that $\sigma(r^*, 0) \neq 0$, then for any given triple (k_p, k_i, k_d) and for all initial state $(x_1(0), x_2(0)) \in \mathbb{R}^2$, the closed-loop system (4) cannot achieve the following control performance

$$\lim_{t\to\infty} \mathbb{E}(x_1(t) - r^*)^2 = 0 \quad \textit{and} \quad \lim_{t\to\infty} \mathbb{E}x_2^2(t) = 0,$$

exponentially, where $\mathbb{E}(\cdot)$ denotes the mathematical expectation of a random variable.

The proof of Proposition 3.3 will be provided in the Appendix.

In this paper, we will show that the control error can converge to zero in mean square sense with PID control for all initial conditions $x_1(0)$, $x_2(0) \in \mathbb{R}^n$, under above assumptions on the nonlinear functions $f(\cdot)$ and $\sigma(\cdot)$. Furthermore, a three dimensional parameter set can be constructed explicitly as long as some prior information about the bounds of partial derivatives of both uncertain functions $f(\cdot)$ and $\sigma(\cdot)$ are available.

We now introduce the PID parameter set to be used and analyzed as follows:

$$\Omega_{pid} = \left\{ (k_p, k_i, k_d) \in \mathbb{R}^{3+} \middle| k_p^2 \underline{b} > 2k_i k_d \underline{b} + k_d N_1^2 + \overline{k},\ k_d^2 \underline{b} > k_p + k_d N_2^2 + \overline{k} \right\}, \tag{5}$$

where $\overline{k} = (L_1 + L_2)(k_p + k_d)$.

Remark 3.4 Ω_{pid} is an open and unbounded set in $\mathbb{R}^3$, which means that the selection of controller parameters is quite flexible. In fact, for any given $k_i > 0$, it can be shown that $(k_p, k_i, k_d) \in \Omega_{pid}$, as long as $k_p = k_d \geq 2k_i + \frac{1}{\underline{b}}[2(L_1 + L_2) + N_1^2 + N_2^2 + 1]$. Moreover, the parameter set Ω_{pid} is a semi-cone, that is, $(\alpha k_p, \alpha k_i, \alpha k_d) \in \Omega_{pid}$, $\forall \alpha \geq 1$ whenever $(k_p, k_i, k_d) \in \Omega_{pid}$.

Theorem 3.5 *Consider the PID controlled stochastic system* (2)–(3) *with unknown functions satisfying Assumption* 3.1 *and Assumption* 3.2*. Then, whenever the PID parameters are chosen from Ω_{pid}, i.e., $(k_p, k_i, k_d) \in \Omega_{pid}$, the closed-loop control system will achieve the following desired performance*

$$\lim_{t\to\infty} \mathbb{E}\|x_1(t) - r^*\|^2 = 0, \quad \lim_{t\to\infty} \mathbb{E}\|x_2(t)\|^2 = 0,$$

exponentially, for any initial states $x_1(0), x_2(0) \in \mathbb{R}^n$.

Remark 3.6 Theorem 3.5 provides a wide range of "safe" controller parameters to guarantee the global stability of the control systems, and demonstrates that the PID control has large scale robustness with respect to the system uncertainties and to the selection of the controller parameters. These results partially explain the rationale behind the widespread successful applications of the PID control. Of course, it is meaningful to further optimize the PID parameters within the constructed parameter set Ω_{pid} to achieve better control performance.

Remark 3.7 First, we remark that if the control system (2) is not subject to random disturbances, i.e., the diffusion term $\sigma(\cdot) \equiv 0$, the PID parameter set Ω_{pid} will be the same as that given in Theorem 3.1 in [20]. Next, it can be verified that the range of parameter set Ω_{pid} will shrink as the positive constants L_1, L_2, N_1, N_2 increase (or $\underline{b}$ decreases). This shows that when the size of the system uncertainty becomes larger, the range of "safe" controller parameters will become smaller.

3.2 PD Control

For the system (2) without random disturbances, i.e., $\sigma(x) = 0$, it has been shown in [20] that the integral term $\int_0^t e(s)ds$ is not necessary for regulation, when the setpoint r^* is an equilibrium of the uncontrolled system, i.e., $f(r^*, 0, 0) = 0$. This fact inspires us to consider the following function space:

Assumption 3.3 (Uncertain Function Space for Drift Function)

$$\mathcal{F}_{L_1,L_2,\underline{b},r^*} = \Big\{ f \in \mathcal{F}_{L_1,L_2,\underline{b}} \mid f(r^*, 0, 0) = 0 \Big\}.$$

Based on the above assumption, we will further investigate the capability of the PD control:

$$u(t) = k_p e(t) + k_d \dot{e}(t), \tag{6}$$

where $e(t) = r^* - y(t)$ is the control error.

Define the following PD parameter set:

$$\Omega_{pd} = \Big\{ (k_p, k_d) \in \mathbb{R}^{2+} \mid k_p^2 \underline{b} > k_d N_1^2 + \overline{k}, \quad k_d^2 \underline{b} > k_p + k_d N_2^2 + \overline{k} \Big\}, \tag{7}$$

where $\overline{k} = (L_1 + L_2)(k_p + k_d)$.

Remark 3.8 Ω_{pd} is an open unbounded set in $\mathbb{R}^3$. To be specific, take $k_p = k_d = k$, it could be verified that $(k_p, k_d) \in \Omega_{pd}$ as long as $k > \frac{1}{\underline{b}}(2L_1 + 2L_2 + N_1^2 + N_2^2 + 1)$.

The following theorem will show that PD control is sufficient when $(r^*, 0)$ is an equilibrium of the uncontrolled system.

Theorem 3.9 *Consider the PD controlled stochastic system* (2)–(6) *with unknown functions satisfying Assumption* 3.2 *and Assumption* 3.3. *Then, whenever the PD parameters are chosen from* Ω_{pd}, *i.e.*, $(k_p, k_d) \in \Omega_{pd}$, *the closed-loop control system will achieve the following desired performance:*

$$\lim_{t\to\infty} \mathbb{E}\|x_1(t) - r^*\|^2 = 0, \quad \lim_{t\to\infty} \mathbb{E}\|x_2(t)\|^2 = 0,$$

exponentially, for any initial states $x_1(0), x_2(0) \in \mathbb{R}^n$.

3.3 PI Control

In this subsection, we will further investigate the capability of PI control.

Consider the following class of stochastic systems:

$$dx(t) = f(x(t), u(t))dt + \sigma(x(t))dw(t), \tag{8}$$

where $x \in \mathbb{R}^n$ is the observed state and $u(t) \in \mathbb{R}^n$ is the control input.

In this subsection, we will adopt the classical PI controller:

$$u(t) = k_i \int_0^t e(s)ds + k_p e(t), \tag{9}$$

where $e(t) = r^* - x(t)$ is the control error, and $r^* \in \mathbb{R}^n$ is the reference signal.

Next, we introduce the following assumption:

Assumption 3.4 (Uncertain Function Spaces) The nonlinear functions f and σ belong to the following function spaces respectively:

$$\mathcal{F}_{L,\underline{b}} = \left\{ f \in C^1(\mathbb{R}^{2n}, \mathbb{R}^n) \;\middle|\; \left\| \frac{\partial f}{\partial x} \right\| \le L, \quad \frac{\partial f}{\partial u} \ge \underline{b} I_n \right\},$$

$$\mathcal{G}_{N,r^*} = \left\{ \sigma \in C^1(\mathbb{R}^n, \mathbb{R}^n) \;\middle|\; \left\| \frac{\partial \sigma}{\partial x} \right\| \le N, \quad \sigma(r^*) = 0 \right\},$$

where L, N and $\underline{b}$ are positive constants.

Define the following PI parameter set:

$$\Omega_{pi} = \left\{ (k_p, k_i) \in \mathbb{R}^{2+} \;\middle|\; 2\underline{b}(2\underline{b}k_p^2 - 2k_i - 2k_pL - k_pN^2) > L^2 \right\}. \tag{10}$$

Remark 3.10 For any fixed $k_i > 0$, it can be verified that $(k_p, k_i) \in \Omega_{pi}$ as long as $k_p \ge \frac{4L+N^2}{2\underline{b}} + \frac{k_i}{L}$, since

$$2\underline{b}(2\underline{b}k_p^2 - 2k_i - 2k_pL - k_pN^2) \ge 2\underline{b}(2k_pL - 2k_i) \ge 8L^2 + 2N^2L > L^2.$$

Consequently, Ω_{pi} is an open unbounded set.

The following theorem will show that PI control is adequate when tackling with first order systems.

Theorem 3.11 *Consider the PI controlled stochastic system* (8)–(9)*, where the unknown functions satisfy Assumption* 3.4*. Then, whenever the PI parameters are chosen from* Ω_{pi}*, i.e.,* $(k_p, k_i) \in \Omega_{pi}$*, the closed-loop control system will achieve the following desired performance:*

$$\lim_{t\to\infty} \mathbb{E}\|x(t) - r^*\|^2 = 0,$$

exponentially, for any initial states $x(0) \in \mathbb{R}^n$*.*

Remark 3.12 Theorem 3.11 (Theorem 3.5) shows that PI control (PID control) has the ability to deal with first order (second order) nonaffine uncertain stochastic systems. We remark that the extended PID discussed in [25] might be adopted to stabilize systems with a general relative degree.

4 Proofs of the Main Results

4.1 Proof of Theorem 3.5

Proof First, suppose that $f \in \mathcal{F}_{L_1,L_2,\underline{b}}$, $\sigma \in \mathcal{G}_{N_1,N_2,r^*}$ and PID parameters $(k_p, k_i, k_d) \in \Omega_{pid}$. Define a map $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ as follows:

$$\Phi(u) \triangleq f(r^*, 0, u), \quad \forall u \in \mathbb{R}^n.$$

Note that $f \in \mathcal{F}_{L_1,L_2,\underline{b}}$, which means $\frac{\partial \Phi(u)}{\partial u} = \frac{\partial f(r^*,0,u)}{\partial u} \geq \underline{b} I_n > 0$. From Theorem A.1, we know that Φ is a global diffeomorphism on $\mathbb{R}^n$, which means the map Φ is both surjective and injective. Thus, Φ has a unique zero point u^*, such that $\Phi(u^*) = 0$.

Next, we introduce some notations. Denote

$$y_0 = \int_0^t e(s)ds - \frac{u^*}{k_i}, \quad y_1 = e(t), \quad y_2 = \dot{e}(t),$$

and

$$g_1(x_1, x_2, u) = -f(r^* - x_1, -x_2, u + u^*), \quad g_2(x_1, x_2) = -\sigma(r^* - x_1, -x_2),$$

then the PID controlled system (2)–(3) goes into

$$\begin{cases} dy_0 = y_1 dt, \\ dy_1 = y_2 dt, \\ dy_2 = g_1(y_1, y_2, k_i y_0 + k_p y_1 + k_d y_2)dt + g_2(y_1, y_2)dw(t). \end{cases} \tag{11}$$

Note that $g_1(0, 0, 0) = -f(r^*, 0, u^*) = 0$ and $g_2(0, 0) = -\sigma(r^*, 0) = 0$, which means $(0, 0, 0) \in \mathbb{R}^{3n}$ is an equilibrium of the system (11). Recall that $f \in \mathcal{F}_{L_1,L_2,\underline{b}}$ and $\sigma \in \mathcal{G}_{N_1,N_2,r^*}$, it is easy to verify that

$$\left\|\frac{\partial g_1}{\partial x_1}\right\| \leq L_1, \quad \left\|\frac{\partial g_1}{\partial x_2}\right\| \leq L_2, \quad \frac{\partial g_1}{\partial u} \leq -\underline{b} I_n < 0, \quad \forall x_1, x_2, u \in \mathbb{R}^n,$$
$$\left\|\frac{\partial g_2}{\partial x_1}\right\| \leq N_1, \quad \left\|\frac{\partial g_2}{\partial x_2}\right\| \leq N_2, \qquad \forall x_1, x_2 \in \mathbb{R}^n.$$

For simplicity, we denote $\widehat{y} \triangleq k_i y_0 + k_p y_1 + k_d y_2$ and we further point out that $g_1(y_1, y_2, \widehat{y})$ can be expressed as follows:

$$\begin{aligned}
&g_1(y_1, y_2, \widehat{y}) \\
=&[g_1(y_1, 0, 0) - g_1(0, 0, 0)] + [g_1(y_1, y_2, 0) - g_1(y_1, 0, 0)] + [g_1(y_1, y_2, \widehat{y}) - g_1(y_1, y_2, 0)] \\
=&\Big[\int_0^1 \frac{\partial g_1(ty_1, 0, 0)}{\partial x_1} dt\Big] y_1 + \Big[\int_0^1 \frac{\partial g_1(y_1, ty_2, 0)}{\partial x_2} dt\Big] y_2 + \Big[\int_0^1 \frac{\partial g_1\big(y_1, y_2, t\widehat{y}\big)}{\partial u} dt\Big] \widehat{y} \\
\triangleq& a_1(y_1) y_1 + b_1(y_1, y_2) y_2 - \gamma(y_0, y_1, y_2)\widehat{y},
\end{aligned}$$

where

$$a_1(y_1) = \int_0^1 \frac{\partial g_1(ty_1, 0, 0)}{\partial x_1} dt, \quad b_1(y_1, y_2) = \int_0^1 \frac{\partial g_1(y_1, ty_2, 0)}{\partial x_2} dt$$

and

$$\gamma(y_0, y_1, y_2) = -\int_0^1 \frac{\partial g_1\big(y_1, y_2, t\widehat{y}\big)}{\partial u} dt.$$

From the above integral expressions, it is not difficult to obtain the following properties:

$$\|a_1(y_1)\| \leq L_1, \quad \|b_1(y_1, y_2)\| \leq L_2, \quad \gamma(y_0, y_1, y_2) \geq \underline{b} I_n > 0.$$

Similarly, $g_2(y_1, y_2)$ can be expressed as follows:

$$g_2(y_1, y_2) = a_2(y_1) y_1 + b_2(y_1, y_2) y_2,$$

where $a_2(y_1) = \int_0^1 \frac{\partial g_2(ty_1, 0)}{\partial x_1} dt$ and $b_2(y_1, y_2) = \int_0^1 \frac{\partial g_2(y_1, ty_2)}{\partial x_2} dt$. As a result, $a_2(y_1)$ and $b_2(y_1, y_2)$ also satisfy the following properties:

$$\|a_2(y_1)\| \leq N_1, \quad \|b_2(y_1, y_2)\| \leq N_2.$$

From the expressions of the functions $g_1(y_1, y_2, \widehat{y})$ and $g_2(y_1, y_2)$, it follows that the PID control system (11) can simply be written as

$$\begin{cases} dy_0 = y_1 dt, \\ dy_1 = y_2 dt, \\ dy_2 = \big[a_1 y_1 + b_1 y_2 - \gamma(k_i y_0 + k_p y_1 + k_d y_2)\big] dt + \big[a_2 y_1 + b_2 y_2\big] dw(t), \end{cases} \tag{12}$$

where $a_1 = a_1(y_1)$, $b_1 = b_1(y_1, y_2)$, $\gamma = \gamma(y_0, y_1, y_2)$, $a_2 = a_2(y_1)$ and $b_2 = b_2(y_1, y_2)$.

Furthermore, the system (12) can be rewritten in the following compact form:

$$dY = A_1(y_0, y_1, y_2) dt + A_2(y_1, y_2) dw(t), \tag{13}$$

where

$$\begin{aligned}
&Y^\tau = [y_0^\tau, y_1^\tau, y_2^\tau], \\
&A_1(y_0, y_1, y_2) = \begin{bmatrix} y_1 \\ y_2 \\ -\gamma k_i y_0 + (-\gamma k_p + a_1) y_1 + (-\gamma k_d + b_1) y_2 \end{bmatrix},
\end{aligned}$$

$$A_2(y_1, y_2) = \begin{bmatrix} 0 \\ 0 \\ a_2 y_1 + b_2 y_2 \end{bmatrix}.$$

By adopting a similar method as that used for deterministic system (see [20]), we construct the following Lyapunov function:

$$V(Y) = Y^\tau P Y,$$

where the matrix P is defined by

$$P = \frac{1}{2}\begin{bmatrix} 2k_i k_p \underline{b} I_n & 2k_i k_d \underline{b} I_n & k_i I_n \\ 2k_i k_d \underline{b} I_n & (2k_p k_d \underline{b} - k_i) I_n & k_p I_n \\ k_i I_n & k_p I_n & k_d I_n \end{bmatrix}, \tag{14}$$

which can be verified to be positive definite, since $k_p^2 - 2k_i k_d > 0$ and $k_d^2 - k_p/\underline{b}$ (see [20] for detailed discussion).

Denote $\underline{\mu}$ and $\overline{\mu}$ as the minimum and the maximum eigenvalues of P respectively, then we have

$$\underline{\mu} Y^\tau Y \le V(Y) \le \overline{\mu} Y^\tau Y. \tag{15}$$

From (13) and Definition 2.1, we know that the differential operator L acting on V can be calculated as follows:

$$\begin{aligned} LV(Y) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial Y^\tau} A_1 + \frac{1}{2}\mathrm{tr}\left[A_2^\tau \frac{\partial^2 V}{\partial Y^2} A_2\right] \\ &= -Y^\tau Q Y, \end{aligned}$$

where Q is a $3n \times 3n$ symmetric matrix defined by

$$Q = \begin{bmatrix} \gamma k_i^2 & \gamma k_p k_i - c_1 & \gamma k_i k_d - c_2 \\ \gamma k_p k_i - c_1^\tau & \gamma k_p^2 - c_3 & \gamma k_p k_d - c_4 \\ \gamma k_i k_d - c_2^\tau & \gamma k_p k_d - c_4^\tau & \gamma k_d^2 - c_5 \end{bmatrix},$$

and

$$\begin{aligned} c_1 &= \frac{k_i}{2} a_1 + k_p k_i \underline{b} I_n, \\ c_2 &= \frac{k_i}{2} b_1 + k_i k_d \underline{b} I_n, \\ c_3 &= k_p \frac{a_1 + a_1^\tau}{2} + 2k_i k_d \underline{b} I_n + k_d \frac{a_2^\tau a_2}{2}, \\ c_4 &= \frac{k_p}{2} b_1 + \frac{k_d}{2} a_1^\tau + k_p k_d \underline{b} I_n + \frac{k_d}{2} a_2^\tau b_2, \\ c_5 &= k_d \frac{b_1 + b_1^\tau}{2} + k_p I_n + k_d \frac{b_2^\tau b_2}{2}. \end{aligned}$$

Next, we will proceed to show the positive definiteness of matrix Q.

First, we will show that Q has the following lower bound:

$$Q \geq Q_0 \triangleq \begin{bmatrix} \underline{b}k_i^2 I_n & \underline{b}k_p k_i I_n - c_1 & \underline{b}k_i k_d I_n - c_2 \\ \underline{b}k_p k_i I_n - c_1^\tau & \underline{b}k_p^2 I_n - c_3 & \underline{b}k_p k_d I_n - c_4 \\ \underline{b}k_i k_d I_n - c_2^\tau & \underline{b}k_p k_d I_n - c_4^\tau & \underline{b}k_d^2 I_n - c_5 \end{bmatrix}, \tag{16}$$

since it can be verified easily that the following matrix R is positive definite:

$$R = \begin{bmatrix} k_i^2 & k_p k_i & k_i k_d \\ k_p k_i & k_p^2 & k_p k_d \\ k_i k_d & k_p k_d & k_d^2 \end{bmatrix} \geq 0,$$

and the following equation holds:

$$Q - Q_0 = R \otimes (\gamma - \underline{b} I_n) \geq 0^\dagger,$$

where $A \otimes B$ refers to the Kronecker product of matrices A and B.

We shall now show the positive definiteness of matrix Q_0.

Let

$$H = \begin{bmatrix} I_n & 0 & 0 \\ -(\underline{b}k_p k_i I_n - c_1^\tau)\frac{1}{\underline{b}k_i^2} & I_n & 0 \\ -(\underline{b}k_i k_d I_n - c_2^\tau)\frac{1}{\underline{b}k_i^2} & 0 & I_n \end{bmatrix},$$

then

$$HQ_0H^\tau = \begin{bmatrix} \underline{b}k_i^2 I_n & \\ & Q_1 \end{bmatrix},$$

where

$$Q_1 = \begin{bmatrix} k_p^2 \underline{b} I_n - \frac{a_1^\tau a_1}{4\underline{b}} - c_3 & \underline{b}k_p k_d I_n - \frac{a_1^\tau b_1}{4\underline{b}} - c_4 \\ \underline{b}k_p k_d I_n - \frac{b_1^\tau a_1}{4\underline{b}} - c_4^\tau & \underline{b}k_d^2 I_n - \frac{b_1^\tau b_1}{4\underline{b}} - c_5 \end{bmatrix}.$$

To prove the matrix Q is positive definite, we only need to show that Q_1 is positive definite since the matrix H is invertible and $\underline{b}k_i^2 > 0$.

Recall the boundedness of a_1, b_1, a_2 and b_2, it follows that for any $x = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in \mathbb{R}^{2n}$, we have

$$\begin{aligned} x^\tau Q_1 x \geq & \left[(k_p^2 - 2k_i k_d)\underline{b} - \frac{L_1^2}{4\underline{b}} - k_p L_1 - k_d \frac{N_1^2}{2}\right] \|x_1\|^2 \\ & - \left(\frac{L_1 L_2}{2\underline{b}} + k_p L_2 + k_d L_1 + k_d N_1 N_2\right) \|x_1\| \|x_2\| \\ & + \left(k_d^2 \underline{b} - k_p - k_d L_2 - k_d \frac{N_2^2}{2}\right) \|x_2\|^2 \end{aligned}$$

† The Kronecker product of two positive semi-definite matrices is also a positive semi-definite matrix.

$$\begin{aligned}
&\geq \left[(k_p^2 - 2k_ik_d)\underline{b} - \frac{L_1^2}{4\underline{b}} - k_pL_1 - k_dN_1^2\right]\|x_1\|^2 \\
&\quad - \left(\frac{L_1L_2}{2\underline{b}} + k_pL_2 + k_dL_1\right)\|x_1\|\|x_2\| \\
&\quad + \left(k_d^2\underline{b} - k_p - k_dL_2 - k_dN_2^2\right)\|x_2\|^2 \\
&\triangleq \begin{bmatrix} \|x_1\| & \|x_2\| \end{bmatrix} Q_2 \begin{bmatrix} \|x_1\| \\ \|x_2\| \end{bmatrix},
\end{aligned}$$

where Q_2 is a symmetric matrix defined by

$$Q_2 = \begin{bmatrix} q_{11} & -q_{12} \\ -q_{21} & q_{22} \end{bmatrix},$$

where

$$\begin{aligned}
q_{11} &= (k_p^2 - 2k_ik_d)\underline{b} - \frac{L_1^2}{4\underline{b}} - k_pL_1 - k_dN_1^2, \\
q_{12} &= q_{21} = \frac{L_1L_2}{4\underline{b}} + \frac{k_pL_2 + k_dL_1}{2}, \\
q_{22} &= k_d^2\underline{b} - k_p - k_dL_2 - k_dN_2^2.
\end{aligned}$$

From the definition of the PID parameter set Ω_{pid} (5), we know that

$$k_p > \frac{\overline{k}}{k_p\underline{b}} > \frac{L_1 + L_2}{\underline{b}}, \quad k_d > \frac{\overline{k}}{k_d\underline{b}} > \frac{L_1 + L_2}{\underline{b}}.$$

It follows that

$$\begin{aligned}
q_{11} &> k_pL_2 + k_d(L_1 + L_2) - \frac{L_1^2}{4\underline{b}} \\
&> \frac{k_pL_2 + k_dL_1}{2} + \frac{k_d(L_1 + L_2)}{2} - \frac{L_1^2}{4\underline{b}} \\
&> \frac{k_pL_2 + k_dL_1}{2} + \frac{(L_1 + L_2)^2}{2\underline{b}} - \frac{L_1^2}{4\underline{b}} \\
&> q_{12},
\end{aligned}$$

and

$$\begin{aligned}
q_{22} &> k_dL_1 + k_p(L_1 + L_2) \\
&> \frac{k_pL_2 + k_dL_1}{2} + \frac{k_p(L_1 + L_2)}{2} \\
&> \frac{k_pL_2 + k_dL_1}{2} + \frac{(L_1 + L_2)^2}{2\underline{b}} \\
&> q_{12}.
\end{aligned}$$

Note that $q_{12} = q_{21} > 0$. It can be deduced directly that the matrix $Q_2 > 0$, which, in turn, gives the positivity of Q_1. As a result, the positive definiteness of Q_0 is verified.

From the definition of Q_0, it is easy to see Q_0 is continuously dependent on the variables a_1, b_1, a_2 and b_2. Moreover, notice that the matrices a_1, b_1, a_2 and b_2 vary on a compact set, namely $\|a_1\| \le L_1$, $\|b_1\| \le L_2$, $\|a_2\| \le N_1$ and $\|b_2\| \le N_2$. From Lemma 3 in [20], there exists $\eta > 0$, such that $Q_0 \ge \eta I_{3n}$ for all $\|a_1\| \le L_1$, $\|b_1\| \le L_2$, $\|a_2\| \le N_1$ and $\|b_2\| \le N_2$. Therefore, the following inequality holds:

$$LV(Y) \le -\eta Y^\tau Y. \tag{17}$$

By using Theorem A.2, the PID control system (2)–(3) will satisfy

$$\lim_{t\to\infty} \mathbb{E}\|x_1 - r^*\|^2 = 0, \quad \lim_{t\to\infty} \mathbb{E}\|x_2\|^2 = 0,$$

exponentially, for any initial values $x_1(0)$, $x_2(0) \in \mathbb{R}^n$. ∎

4.2 Proof of Theorem 3.9

Proof Suppose $f \in \mathcal{F}_{L_1,L_2,\underline{b},r^*}$, $\sigma \in \mathcal{G}_{N_1,N_2,r^*}$ and $(k_p, k_d) \in \Omega_{pd}$. Denote $y_1 = e(t)$ and $y_2 = \dot{e}(t)$, then the PD controlled system (2)–(6) can be rewritten as

$$\begin{cases} dy_1 = y_2 dt, \\ dy_2 = -f(r^* - y_1, -y_2, k_p y_1 + k_d y_2)dt - \sigma(r^* - y_1, -y_2)dw_t. \end{cases} \tag{18}$$

Since $f(r^*, 0, 0) = \sigma(r^*, 0) = 0$, $0 \in \mathbb{R}^{2n}$ is an equilibrium of the system (18).

Denote $g_1(y_1, y_2, u) = -f(r^* - y_1, -y_2, u)$ and $g_2(y_1, y_2) = -\sigma(r^* - y_1, -y_2)$. Similar to the proof of Theorem 3.5, g_1 and g_2 can be decomposed into

$$\begin{aligned} g_1(y_1, y_2, k_p y_1 + k_d y_2) &= a_1(y_1)y_1 + b_1(y_1, y_2)y_2 - \gamma(y_1, y_2)(k_p y_1 + k_d y_2), \\ g_2(y_1, y_2) &= a_2(y_1)y_1 + b_2(y_1, y_2)y_2, \end{aligned}$$

where $\|a_1\| \le L_1$, $\|b_1\| \le L_2$,$0 < \underline{b} I_n \le \gamma$,$\|a_2\| \le N_1$ and $\|b_2\| \le N_2$.

For simplicity, we set

$$\begin{aligned} Y^\tau &= [y_1^\tau, y_2^\tau], \\ A_1(y_1, y_2) &= \begin{bmatrix} y_2 \\ (-\gamma k_p + a_1)y_1 + (-\gamma k_d + b_1)y_2 \end{bmatrix}, \\ A_2(y_1, y_2) &= \begin{bmatrix} 0 \\ a_2 y_1 + b_2 y_2 \end{bmatrix}, \end{aligned}$$

then the system (18) turns into

$$dY = A_1(y_1, y_2)dt + A_2(y_1, y_2)dw_t. \tag{19}$$

Similar to [20], we now consider a quadratic Lyapunov function

$$V(Y) = Y^\tau PY, \tag{20}$$

where

$$P = \frac{1}{2}\begin{bmatrix} 2k_pk_d\underline{b}I_n & k_pI_n \\ k_pI_n & k_dI_n \end{bmatrix}.$$

From the definition of Ω_{pd}, the following facts can be verified:

$$2k_pk_d\underline{b} > 0, \quad 2k_pk_d^2\underline{b} - k_p^2 = k_p(2k_d^2\underline{b} - k_p) > 0.$$

Thus, the positive definiteness of matrix P is true.

By simple manipulations, the differential operator L acting on (20) along the trajectories of (19) is given by

$$\begin{aligned} LV(Y) =& \frac{\partial V}{\partial t} + \frac{\partial V}{\partial Y^\tau}A_1 + \frac{1}{2}\mathrm{tr}\left[A_2^\tau \frac{\partial^2 V}{\partial Y^2}A_2\right] \\ =& -\left[y_1^\tau, y_2^\tau\right] Q \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \end{aligned}$$

where Q is a symmetric matrix expressed by

$$Q = \begin{bmatrix} k_p(\gamma k_p - \frac{a_1+a_1^\tau}{2}) - \frac{a_2^\tau a_2}{2} & k_pk_d(\gamma - \underline{b}I_n) - \frac{k_da_1^\tau + k_pb_1 + k_da_2^\tau b_2}{2} \\ * & k_d(\gamma k_d - \frac{b_1+b_1^\tau}{2}) - k_p - \frac{k_db_2^\tau b_2}{2} \end{bmatrix}, \tag{21}$$

which has the following lower bound:

$$Q \geq Q_0 \triangleq \begin{bmatrix} k_p(\underline{b}I_nk_p - \frac{a_1+a_1^\tau}{2}) - \frac{a_2^\tau a_2}{2} & -\frac{k_da_1^\tau + k_pb_1 + k_da_2^\tau b_2}{2} \\ -\frac{k_da_1 + k_pb_1^\tau + k_db_2^\tau a_2}{2} & k_d(\underline{b}I_nk_d - \frac{b_1+b_1^\tau}{2}) - k_p - \frac{k_db_2^\tau b_2}{2} \end{bmatrix},$$

since

$$Q - Q_0 = \begin{bmatrix} k_p^2 & k_pk_d \\ k_pk_d & k_d^2 \end{bmatrix} \otimes (\gamma - \underline{b}I_n) \geq 0.$$

Similar to the prove of Theorem 3.5, the positive definiteness of matrix Q_0 could be verified by the fact that for all $x = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in \mathbb{R}^{2n}$, the following inequality holds:

$$x^\tau Q_0 x \geq \begin{bmatrix} \|x_1\| & \|x_2\| \end{bmatrix} Q_1 \begin{bmatrix} \|x_1\| \\ \|x_2\| \end{bmatrix},$$

where

$$Q_1 = \begin{bmatrix} k_p^2\underline{b} - k_pL_1 - k_dN_1^2 & -\frac{k_dL_1 + k_pL_2}{2} \\ -\frac{k_dL_1 + k_pL_2}{2} & k_d^2\underline{b} - k_dL_2 - k_p - k_dN_2^2 \end{bmatrix}.$$

From the definition of Ω_{pd}, it is easy to check the positive definiteness of matrix Q_1. Hence

$$x^\tau Q_0 x \geq 0, \quad \forall x \neq 0,$$

which means Q_0 is positive definite.

Denote the minimum eigenvalue of Q_1 as η. Then, the following results can be obtained:

$$LV(Y) = -Y^\tau QY \leq -Y^\tau Q_0 Y \leq -\begin{bmatrix}\|y_1\|, \|y_2\|\end{bmatrix} Q_1 \begin{bmatrix}\|y_1\| \\ \|y_2\|\end{bmatrix} \leq -\eta Y^\tau Y.$$

From Theorem A.2, the original PD control system (2)–(6) will satisfy

$$\lim_{t\to\infty} \mathbb{E}\|x_1 - r^*\|^2 = 0, \quad \lim_{t\to\infty} \mathbb{E}\|x_2\|^2 = 0,$$

exponentially, for any initial values $x_1(0)$, $x_2(0) \in \mathbb{R}^n$. ∎

4.3 Proof of Theorem 3.11

Proof Similar to the proof of Theorem 3.5, $\Phi(u) \triangleq f(r^*, u)$ is a global diffeomorphism. Thus, there exists a unique zero point $u^* \in \mathbb{R}^n$, such that $\Phi(u^*) = 0$.

Substituting the original state vector x with a related new vector $Y^\tau = \begin{bmatrix}y_0^\tau, y_1^\tau\end{bmatrix} \in \mathbb{R}^{2n}$ by a transformation of the form

$$y_0 = \int_0^t e(s)ds - \frac{u^*}{k_i}, \quad y_1 = e(t).$$

Then, the original PI controlled system (8)–(9) is replaced by a new description

$$\begin{cases} dy_0 = y_1 dt, \\ dy_1 = -f(r^* - y_1, k_i y_0 + k_p y_1 + u^*)dt - \sigma(r^* - y_1)dw_t. \end{cases} \tag{22}$$

From the definition of function spaces $\mathcal{F}_{L,\underline{b}}$ and $\mathcal{G}_{M,r^*}$, it can be easily checked that $0 \in \mathbb{R}^{2n}$ is the equilibrium point of the system (22).

For simplicity, set $g_1(y_1, k_i y_0 + k_p y_1) = -f(r^* - y_1, k_i y_0 + k_p y_1 + u^*)$ and $g_2(y_1) = -\sigma(r^* - y_1)$. Then, similar to the proof of Theorem 3.5, these two functions can be decomposed as

$$\begin{aligned} &g_1(y_1, k_i y_0 + k_p y_1) = a(y_1)y_1 - \gamma(y_0, y_1)(k_i y_0 + k_p y_1), \\ &g_2(y_1) = b(y_1)y_1, \end{aligned}$$

with the following properties

$$\|a(y_1)\| \leq L, \quad \gamma(y_0, y_1) \geq \underline{b} I_n > 0, \quad \|b(y_1)\| \leq N.$$

Accordingly, the system (22) can be replaced by a new description

$$dY = A_1(y_0, y_1)dt + A_2(y_1)dw_t, \tag{23}$$

where

$$Y^\tau = \begin{bmatrix}y_0^\tau, y_1^\tau\end{bmatrix}, \quad A_1(y_0, y_1) = \begin{bmatrix} y_1 \\ -\gamma k_i y_0 + (-\gamma k_p + a)y_1 \end{bmatrix}, \quad A_2(y_1) = \begin{bmatrix} 0 \\ b(y_1)y_1 \end{bmatrix}.$$

Similar to [20], we now construct a Lyapunov function as follows

$$V(Y) = Y^\tau PY, \tag{24}$$

where

$$P = \frac{1}{2}\begin{bmatrix} 2k_pk_i\underline{b}I_n & k_iI_n \\ k_iI_n & k_pI_n \end{bmatrix}.$$

From the definition of the parameter set Ω_{pi}, the positive definiteness of P can be checked easily since

$$k_pk_i\underline{b} > 0, \quad 2k_p^2k_i\underline{b} - k_i^2 = k_i(2k_p^2\underline{b} - k_i) > 0.$$

Calculating the differential operator L acting on (24) associated with the system (23)

$$\begin{aligned} LV(Y) =& \frac{\partial V}{\partial t} + \frac{\partial V}{\partial Y^\tau}A_1 + \frac{1}{2}\mathrm{tr}\left[A_2^\tau \frac{\partial^2 V}{\partial Y^2}A_2\right] \\ =& -\gamma k_i^2 y_0^\tau y_0 + y_0^\tau[2k_pk_i(\underline{b}I_n - \gamma) + k_ia]y_1 \\ &+ y_1^\tau\left(-\gamma k_p^2 + k_p\frac{a+a^\tau}{2} + k_iI_n + \frac{1}{2}k_pb^\tau b\right)y_1 \\ =& -Y^\tau QY, \end{aligned}$$

where Q is a symmetric matrix defined by

$$Q = \begin{bmatrix} \gamma k_i^2 & k_pk_i(\gamma - \underline{b}I_n) - \frac{k_ia}{2} \\ * & \gamma k_p^2 - k_iI_n - k_p\frac{a+a^\tau}{2} - \frac{k_pb^\tau b}{2} \end{bmatrix}.$$

Similar to the proof of Theorem 3.9, there exists a positive definite matrix Q_0 such that $Q \geq Q_0$, which is expressed by

$$Q_0 = \begin{bmatrix} \underline{b}k_i^2I_n & -\frac{k_ia}{2} \\ -\frac{k_ia^\tau}{2} & \underline{b}k_p^2I_n - k_iI_n - k_p\frac{a+a^\tau}{2} - \frac{k_pb^\tau b}{2} \end{bmatrix}.$$

The inequality $Q \geq Q_0$ holds since

$$Q - Q_0 \geq \begin{bmatrix} k_i & k_pk_i \\ k_pk_i & k_p^2 \end{bmatrix} \otimes (\gamma - \underline{b}I_n) \geq 0.$$

To prove the positive definiteness of matrix Q_0, we shall check whether the following property holds

$$x^\tau Q_0x \geq 0, \quad \forall x \neq 0.$$

For any $x \in \mathbb{R}^{2n}$ ($x^\tau = [x_1^\tau, x_2^\tau]$), we have

$$x^\tau Q_0x \geq \Big[\|x_1\|, \|x_2\|\Big]\, Q_1 \begin{bmatrix} \|x_1\| \\ \|x_2\| \end{bmatrix},$$

where

$$Q_1 = \begin{bmatrix} \underline{b}k_i^2 & -\frac{k_i L}{2} \\ -\frac{k_i L}{2} & \underline{b}k_p^2 - k_i - k_p L - \frac{k_p N^2}{2} \end{bmatrix}.$$

Note that $(k_p, k_i) \in \Omega_{pi}$, the following inequalities hold:

$$\underline{b}k_i^2 > 0,$$
$$4\underline{b}k_i^2\left(\underline{b}k_p^2 - k_i - k_p L - \frac{k_p N^2}{2}\right) - k_i^2 L^2 > 0,$$

which means Q_1 is positive definite.

Denote η as the minimum eigenvalue of Q_1. Then,

$$LV = -Y^\tau QY \le -Y^\tau Q_0 Y \le -\left[\|y_0\|, \|y_1\|\right] Q_1 \begin{bmatrix} \|y_0\| \\ \|y_1\| \end{bmatrix} \le -\eta Y^\tau Y.$$

Hence, by using Theorem A.2, the system (8)–(9) will satisfy $\lim_{t\to\infty}\mathbb{E}\|x - r^*\|^2 = 0$, exponentially, for any initial values $x(0) \in \mathbb{R}^n$. ∎

5 Simulations

Consider the following stochastic nonaffine system:

$$\begin{cases} dx_1 = x_2 dt, \\ dx_2 = f(x_1, x_2, u)dt + \sigma(x_1, x_2)dw(t), \end{cases} \tag{25}$$

where $f(x_1, x_2, u) = a\sin x_1 + bx_2 + cu^3 + u$, $\sigma(x_1, x_2) = d\sin x_2$, and a, b, c, d are unknown parameters with known bounds $|a| \le L_1, |b| \le L_2$, $c > 0$ and $|d| \le N$. Note that

$$\left|\frac{\partial f}{\partial x_1}\right| = |a\cos x_1| \le L_1, \quad \left|\frac{\partial f}{\partial x_2}\right| = |b| \le L_2, \quad \frac{\partial f}{\partial u} = 3cu^2 + 1 \ge 1, \quad \forall x_1, x_2, u \in \mathbb{R}^1,$$
$$\left|\frac{\partial \sigma}{\partial x_1}\right| = 0, \quad \left|\frac{\partial \sigma}{\partial x_2}\right| = |d\cos x_2| \le N, \quad \forall x_1, x_2 \in \mathbb{R}^1, \quad \sigma(r^*, 0) = 0, \quad \forall r^* \in \mathbb{R}^1,$$

we conclude that the uncertain functions satisfy $f \in \mathcal{F}_{L_1,L_2,1}$ and $\sigma \in \mathcal{G}_{0,N,r^*}$. For simplicity, we assume that $L_1 = L_2 = N = 1$, then it can be verified that $(k_p, k_i, k_d) = (8, 1, 8) \in \Omega_{pid}$. First, suppose the PID parameters $(k_p, k_i, k_d) = (8, 1, 8)$ are fixed, $(x_1(0), x_2(0)) = (3, 2)$ is the initial state, and $r^* = 1$ is the setpoint. Figure 1 depicts the curves of the control error $e(t)$ under different system parameters a, b, c and d.

From Figure 1, one can see that the given PID controller has the ability to stabilize and regulate the control system (25), even if the system parameters (a, b, c, d) vary in a wide range.

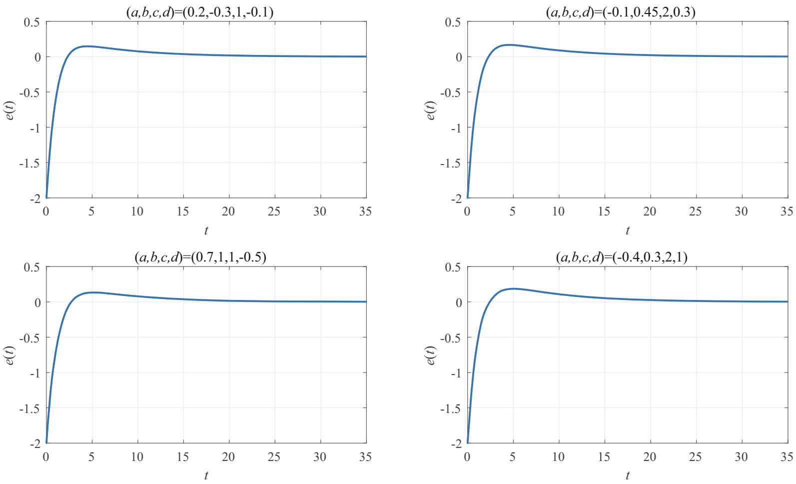

Figure 1 Curves of $e(t)$ under different system parameters (a, b, c, d)

Next, we try to understand how different PID parameters will affect the control performance. Recall $L_1 = L_2 = N = 1$, it can be verified that $(k_p, k_i, k_d) = (k, \frac{k}{7}, k) \in \Omega_{pid}$, $\forall k \geq 7$. In Figure 2, the system parameters are randomly generated ($a \in [-1, 1]$, $b \in [-1, 1]$, $c \in [0, 2]$, and $d \in [-1, 1]$). As we can see from Figure 2, all the PID controlled systems achieve their regulation objectives quite fast. Moreover, from Figure 3, one can also see that large PID gains might increase the amplitude of the controller, and lead to the high-frequency oscillation of the control input.

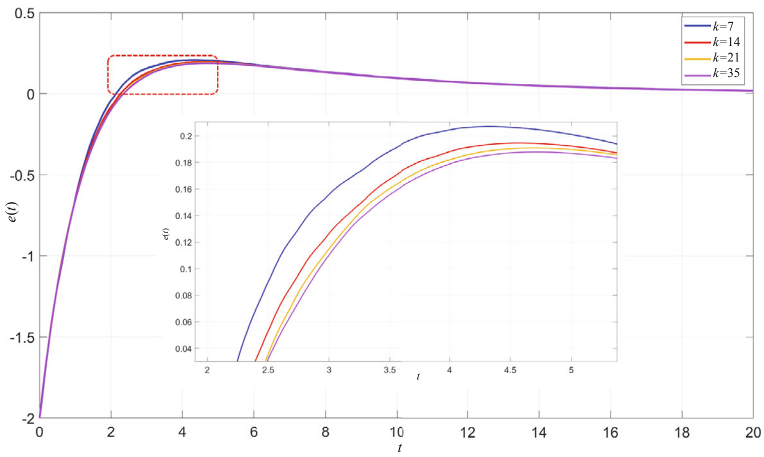

Figure 2 Curves of $e(t)$ under different PID parameters

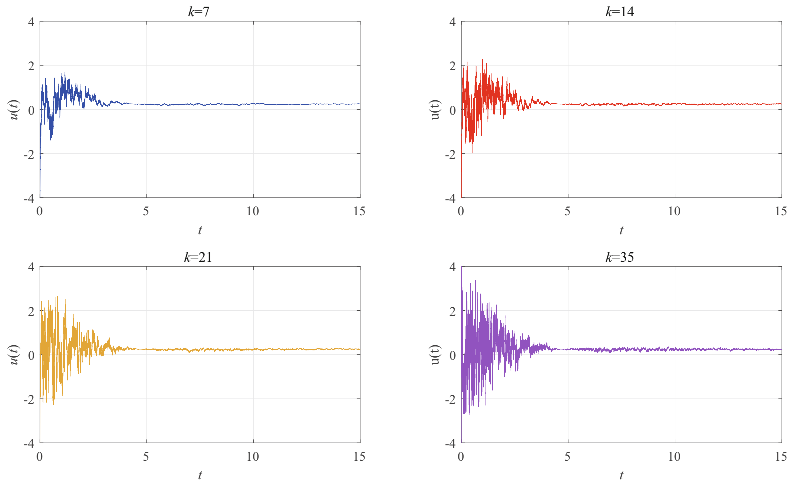

Figure 3 Curves of $u(t)$ under different PID parameters

6 Conclusion

This paper investigates the regulation problem of classical PID control for a class of nonaffine stochastic systems, and provides a rigorous mathematical analysis together with an explicit design formula for the choice of controller parameters. It has been shown that a three dimensional open unbounded parameter set can be constructed based on the bounds of partial derivatives on both the drift and diffusion terms. It has also been shown that whenever the PID parameters are taken from the parameter set, the closed-loop controlled systems will reach the desired setpoint with exponentially convergent regulation error. Similar results have also been provided for PD and PI control. For further investigation, it would be meaningful to further develop some optimal design principles or guidelines in selecting PID parameters within the parameter set provided in the current paper, and to consider more practical situations including time-delay and saturation, etc.

References

[1] Åström K J and Hägglund T, *PID Controllers: Theory, Design, and Tuning*, Instrument society of America Research Triangle Park, NC, 1995.

[2] Åström K J, Hägglund T, and Astrom K J, *Advanced PID Control*, ISA-The Instrumentation, Systems, and Automation Society Research Triangle Park, NC, 2006.

[3] Salih A L, Moghavvemi M, Mohamed H A F, et al., Flight PID controller design for a UAV quadrotor, *Scientific Research and Essays*, 2010, **5**(23): 3660–3667.

[4] Åström K J and Hägglund T, The future of PID control, *Control Engineering Practice*, 2001, **9**(11): 1163–1175.

[5] Samad T, A survey on industry impact and challenges thereof [technical activities], *IEEE Control Systems Magazine*, 2017, **37**(1): 17–18.

[6] Guo L, Feedback and uncertainty: Some basic problems and results, *Annual Reviews in Control*, 2020, **49**: 27–36.

[7] Ziegler J G and Nichols N B, Optimum settings for automatic controllers, *Trans. ASME*, 1942, **64**(11): 759—765.

[8] Chien I L and Fruehauf P S, Consider IMC turning to improve controller performance, *Chemical Engineering Progress*, 1990, **86**(10): 33–41.

[9] Bialkowski W L, Dreams versus reality: A view from both sides of the gap: Manufacturing excellence with come only through engineering excellence, *Pulp & Paper Canada*, 1993, **94**(11): 19–27.

[10] Ender D B, Process control performance: Not as good as you think, *Control Engineering*, 1993, **40**(10): 180–190.

[11] Yu C C, *Autotuning of PID Controllers: A Relay Feedback Approach*, Springer Science & Business Media, Berlin, 2006.

[12] Killingsworth N J and Krstic M, PID tuning using extremum seeking: Online, model-free performance, optimization, *IEEE Control Systems Magazine*, 2006, **26**(1): 70–79.

[13] Romero J G, Donaire A, Ortega R, et al., Global stabilisation of underactuated mechanical systems via PID passivity-based control, *Automatica*, 2018, **96**: 178–185.

[14] Fliess M and Join C, Model-free control, *International Journal of Control*, 2013, **86**(12): 2228–2252.

[15] O'Dwyer A, PI and PID controller tuning rules: An overview and personal perspective, *Proceedings of the IET Irish Signals and Systems Conference*, 2006, 161–166.

[16] Zhao C and Guo L, PID controller design for second order nonlinear uncertain systems, *Science China*, 2017, **60**(2): 1–13.

[17] Cong X R and Guo L, PID control for a class of nonlinear uncertain stochastic systems, *IEEE 56th Annual Conference on Decision and Control (CDC)*, IEEE, 2017, 612–617.

[18] Zhang J K and Guo L, Theory and design of PID controller for nonlinear uncertain systems, *IEEE Control Systems Letters*, 2019, **3**(3): 643–648.

[19] Zhang J K and Guo L, PID control of nonlinear stochastic systems with structural uncertainties, *IFAC-PapersOnLine*, 2020, **53**(2): 2189–2194.

[20] Zhao C and Guo L, Towards a theoretical foundation of PID control for uncertain nonlinear systems, *Automatica*, 2022, **142**: 110360.

[21] Muniyappa R, Lee S, Chen H, et al., Current approaches for assessing insulin sensitivity and resistance in vivo: Advantages, limitations, and appropriate usage, *American Journal of Physiology-Endocrinology and Metabolism*, 2008, **294**(1): E15–E26.

[22] Shiriaev A S, Ludvigsen H, Egeland O, et al, Swinging up of non-affine in control pendulum, *Proceedings of the 1999 American Control Conference (Cat. No. 99CH36251)*, IEEE, 1999, **6**: 4039–4044.

[23] Lee J, Seo J, and Choi J, Output feedback control design using extended high-gain observers and dynamic inversion with projection for a small scaled helicopter, *Automatica*, 2021, **133**: 109883.

[24] Zhao C and Guo L, On the capability of PID control for nonlinear uncertain systems, *IFAC-PapersOnLine*, 2017, **50**(1): 1521–1526.

[25] Zhao C and Guo L, Control of nonlinear uncertain systems by extended PID, *IEEE Transactions*

on Automatic Control, 2021, **66**(8): 3840–3847.

[26] Khasminskii R, *Stochastic Stability of Differential Equations*, Springer Science & Business Media, Berlin, 2011.

Appendix

Theorem A.1 (see [20]) *Let $\Phi(x) \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. Suppose that*

$$\frac{\partial \Phi}{\partial x} \geq R, \quad \forall x \in \mathbb{R}^n,$$

where $R \in \mathbb{R}^{n\times n}$ is positive definite, then Φ is a global diffeomorphism on $\mathbb{R}^n$.

Theorem A.2 (see [26]) *The equilibrium $x = 0$ of the system* (1) *will satisfy*

$$\mathbb{E}\|x(t)\|^2 \leq c\|x_0\|^2 \mathrm{e}^{-\alpha t}, \quad \forall t \geq 0$$

for some positive constants c and α, if there exist positive constants k_1, k_2, k_3, and a function $V(x,t) \in C^2(\mathbb{R}^n \times \mathbb{R}^+, \mathbb{R}^+)$ such that

$$\begin{aligned} &k_1\|x\|^2 \leq V(x,t) \leq k_2\|x\|^2, \\ &LV(x,t) \leq -k_3\|x\|^2. \end{aligned}$$

Proof of Proposition 3.3.

Proof First, without loss of generality, we assume that $r^* = 0$. Suppose that for some k_p, k_i and k_d and for some initial state $(x_1(0), x_2(0))$, the closed-loop equation satisfies

$$\lim_{t\to\infty} \mathbb{E}x_1^2(t) = 0 \ \text{ and } \ \lim_{t\to\infty} \mathbb{E}x_2^2(t) = 0,$$

exponentially, we proceed to show that $\sigma(0,0) = 0$.

Denote $x_0(t) = \int_0^t x_1(s)ds = -\int_0^t e(s)ds$, then $x_0(t) - x_0(s) = \int_s^t x_1(\tau)d\tau$. By Minkowski's integral inequality, it follows that

$$\mathbb{E}(x_0(t) - x_0(s))^2 = \mathbb{E}\left(\int_s^t x_1(\tau)d\tau\right)^2 \leq \left[\int_s^t (\mathbb{E}x_1^2(\tau))^{\frac{1}{2}} d\tau\right]^2, \quad 0 \leq s \leq t. \tag{26}$$

Note that $(\mathbb{E}x_1^2(t))^{\frac{1}{2}} \leq M\mathrm{e}^{-\lambda t}$, for all $t \geq 0$, from (26), we have

$$\mathbb{E}(x_0(t) - x_0(s))^2 \leq \left[\int_s^t M\mathrm{e}^{-\lambda\tau} d\tau\right]^{\frac{1}{2}} \leq \left(\frac{M}{\lambda}\mathrm{e}^{-\lambda s}\right)^{\frac{1}{2}} \to 0, \text{ as } \ s \to \infty,$$

which implies that $\{x_0(t), t \geq 0\}$ is Cauchy in $L^2(\Omega, \mathscr{F}, P)$. Consequently, there exists an L^2-integrable random variable x_∞ such that

$$\lim_{t\to\infty} x_0(t) = x_\infty, \ \text{ in } \ L^2(\Omega, \mathscr{F}, P).$$

Notice that $u(t) = -(k_i x_0(t) + k_p x_1(t) + k_d x_2(t))$, it is easy to see

$$\lim_{t\to\infty} u(t) = -k_i x_\infty \triangleq u_\infty, \ \text{ in } \ L^2(\Omega, \mathscr{F}, P).$$

Recall $dx_2(t) = f(x_1, x_2, u(t))dt + \sigma(x_1, x_2)dw(t)$, it follows that

$$x_2(t+1) - x_2(t) = X_t + Y_t, \tag{27}$$

where $X_t = \int_t^{t+1} f(x_1(s), x_2(s), u(s))ds$ and $Y_t = \int_t^{t+1} \sigma(x_1(s), x_2(s))dw(s)$. Next, we proceed to show that

$$\mathbb{E}[X_t Y_t] \to 0, \text{ as } t \to \infty. \tag{28}$$

First, by the Lipschitz property of f and σ, and notice the facts that $\mathbb{E}x_1^2(t)$, $\mathbb{E}x_2^2(t)$ and $\mathbb{E}u^2(t)$ are bounded functions of t, it is not difficult to obtain that $\mathbb{E}X_t^2$ and $\mathbb{E}Y_t^2$ are also bounded functions of $t \in [0, \infty)$.

Similarly, note that

$$\lim_{t\to\infty} x_1(t) = 0, \quad \lim_{t\to\infty} x_2(t) = 0, \quad \lim_{t\to\infty} u(t) = u_\infty, \quad \text{in} \quad L^2(\Omega, \mathscr{F}, P),$$

we can obtain

$$\lim_{t\to\infty} \mathbb{E}[X_t - f(0,0,u_\infty)]^2 = 0, \qquad \lim_{t\to\infty} \mathbb{E}[Y_t - \sigma(0,0)(w(t+1) - w(t)]^2 = 0.$$

Thus, to prove (28), it suffices to show

$$\mathbb{E}[f(0,0,u_\infty)(w(t+1) - w(t))] \to 0, \text{ as } t \to \infty. \tag{29}$$

It comes immediately since for any given $T > 0$, we have

$$\mathbb{E}f(0,0,u(T))(w(t+1) - w(t)) \to 0, \quad t \to \infty$$

and the fact that

$$\lim_{T\to\infty} f(0,0,u(T)) = f(0,0,u_\infty), \quad \text{in} \quad L^2(\Omega, \mathscr{F}, P).$$

From (27), it follows that

$$\lim_{t\to\infty} \mathbb{E}(X_t^2 + Y_t^2 + 2X_tY_t) = \lim_{t\to\infty} \mathbb{E}[x_2(t+1) - x_2(t)]^2 = 0,$$

which implies $\lim_{t\to\infty} \mathbb{E}Y_t^2 = 0$. Denote $\sigma^2(x_1, x_2) - \sigma^2(0,0) = g(x_1, x_2)$, by Itô's isometry, we have

$$\begin{aligned} 0 = \lim_{t\to\infty} \mathbb{E}Y_t^2 &= \lim_{t\to\infty} \int_t^{t+1} \mathbb{E}\sigma^2(x_1(s), x_2(s))ds \\ &= \lim_{t\to\infty} \int_t^{t+1} \mathbb{E}\big[\sigma^2(0,0) + g(x_1(s), x_2(s))\big]ds \\ &= \sigma^2(0,0) + \lim_{t\to\infty} \int_t^{t+1} \mathbb{E}g(x_1(s), x_2(s))ds \\ &= 0. \end{aligned}$$

From the Lipschitz condition of σ, there exists $M > 0$ such that $|g(x_1, x_2)| \le M(x_1^2 + x_2^2 + |x_1| + |x_2|)$, and therefore $\lim_{t\to\infty} \int_t^{t+1} \mathbb{E}g(x_1, x_2)ds = 0$, which in turn gives $\sigma(0,0) = 0$. ∎