

Adaptive Tracking Control with Binary-Valued Output Observations

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Abstract—This paper considers real-time control and learning problems for finite-dimensional linear systems under binary-valued and randomly disturbed output observations, which arise in various fields such as the information industry and control engineering. This has long been regarded as an open problem because the exact values of the traditional regression vectors used in the construction of adaptive algorithms are unavailable, as one only has binary-valued output information. To overcome this difficulty, we consider the adaptive estimation problem of the corresponding infinite-impulse-response (IIR) dynamical systems and apply the double array martingale theory that has not been previously used in adaptive control. This enables us to establish global convergence results for both the adaptive prediction regret and the parameter estimation error, without resorting to such stringent data conditions as persistent excitation and bounded system signals that have been used in almost all existing related literature. Based on this, an adaptive control law will be designed that can effectively combine adaptive learning and feedback control. Finally, we are able to show that for any given bounded reference signal, the closed-loop adaptive control system is globally stable and the long-run average output tracking error tends to zero as time goes to infinity. To the best of the authors' knowledge, this appears to be the first adaptive control result for general linear systems with general binary sensors and arbitrarily given bounded reference signals.

Index Terms—Stochastic systems, adaptive control, binary-valued observations, double array martingales, adaptive identification, convergence analysis.

I. INTRODUCTION

Binary output observation data, which refers to system output observations that can only take the values 0 or 1, is prevalent in practical systems. One example comes from the signal detection problem in wireless communication [1], [2], where signals are limited to 1-bit via a low-precision analog-to-digital converter, due to constraints in energy consumption and hardware complexity. Another example is the binary classification problem in machine learning [3], where labels provide only discrete class information, rather than precise or continuous values. Furthermore, binary observations are also

widely encountered in neural systems [4], [5], and can be produced by binary sensors in engineering applications, such as gas content sensors in the gas and oil industry [6], and shift-by-wire switch sensors in automotive applications [7].

Given their importance, extensive theoretical research has been devoted to estimation problems under such binary or quantized observations for various classes of systems, including linear systems [8], block-oriented nonlinear systems [15], and hybrid systems [18]. Various parameter identification methods have been proposed for estimating unknown parameters using binary/quantized observation information, including the empirical measure approaches [8], maximum likelihood algorithms [9], set-membership methods [10], stochastic approximation-type algorithms [11], [13], and stochastic Newton algorithms [14], [16]. Meanwhile, for the control problems under binary/quantized observations, numerous studies devoted to problems such as finite data-rate stabilization of stochastic linear systems [19] and hybrid linear systems [20], and consensus under quantized communications [21], [22], under the assumption that the model parameters are known *a priori*. However, in many practical control scenarios, the models of systems are uncertain, one may require simultaneous parameter learning and real-time control. This basic adaptive control problem has rarely been explored in the existing literature, because it turns out that such a problem is highly non-trivial in both controller design and theoretical analysis due to the non-availability of the exact output observations. This is the primary motivation for us to initiate an investigation, by introducing new methods and establishing new results for the adaptive control of a basic class of finite-dimensional uncertain linear systems under binary output observations.

For that purpose, one needs to build on some significant milestones of the existing adaptive control theory. In the area of adaptive control, a natural approach in the design of adaptive control is the certainty equivalence principle, which involves two steps: (1) designing an online estimation algorithm using the observation data to estimate the unknown parameters, and (2) using these online estimated parameters to design the adaptive controller in place of the unknown true parameters. It is well-known that the nonlinear and complex nature of such closed-loop adaptive systems makes the corresponding theoretical analysis quite challenging and thus is regarded as a central issue in adaptive control. Take the basic adaptive control problem of linear stochastic systems as an example, motivated by the need to establish a rigorous theory for the least-squares (LS)-based self-turning regulators proposed by Åström and Wittenmark [23], a great

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deal of effort had been devoted to the convergence study of LS with stochastic feedback signals resulting from stochastic adaptive control ([24]–[32]). Among the many significant contributions in this direction, we mention that Lai and Wei [27] established the asymptotic analysis of LS under the weakest possible excitation condition on the system stochastic signals, by using stochastic Lyapunov function methods and martingale convergence theorems. The convergence rates of either adaptive tracking control [29] or adaptive LQG control [33] were established by resorting to certain kinds of external excitation signals for open-loop stable systems. The complete convergence theories of both the adaptive tracking control and adaptive LQG control were established later in [31], [32], and [34]. These turn out to be foundations for the study of adaptive control under binary-valued observations in the current paper.

The aforementioned work all focuses on adaptive control problems under traditional continuous-valued output observations, which cannot be directly applied to the current binary-valued case. In fact, even the adaptive parameter estimation theory under binary-valued output observations has not been fully established for general linear systems, let alone the investigation of the related adaptive control problems. This is primarily due to the following two reasons. Firstly, the regression vectors traditionally used for the design of adaptive learning algorithms are not available in the current scenario due to the availability of binary-valued observation information only. Consequently, almost all theoretical results on parameter estimation under binary output observations are limited to finite-impulse-response (FIR) systems with no poles in the linear models. Secondly, due to the nonlinearity of the current binary observation function, almost all existing adaptive estimation theories require stringent data conditions, such as independent and stationary conditions, deterministic persistent excitation (PE) conditions, and boundedness signal conditions, which are difficult or even impossible to verify for the closed-loop control system signals. For these reasons, only a few works have been devoted to adaptive control theory under quantized/binary-valued observations. Guo et al. [35] considered the adaptive tracking problem for FIR systems with a scalar unknown parameter. Zhao et al. [36] further investigated adaptive tracking of FIR systems under binary-valued observations for periodic tracking signals with full-rank conditions. In these two scenarios, it is possible to verify the PE and boundedness conditions required for parameter identification. However, such verification is challenging for infinite-impulse-response (IIR) systems and general reference signals. Furthermore, Zhao et al. [37] studied the adaptive regulation problem by using a stochastic approximation-based control algorithm, where the tracking signal is a fixed set-point and the binary observation thresholds are specifically designed and depend on the given fixed set-point. In summary, the adaptive control theory for general linear systems with both general binary sensors and time-varying reference signals remains a challenging research problem.

Partly motivated by the above open problem, we have established an adaptive estimation theory under binary-valued output observations and non-PE data conditions in our recent works [14], [17], but these results cannot be used to directly

solve the above-mentioned basic adaptive control problems, mainly due to the following two reasons: 1) the regression vector contains non-available exact output signals, and 2) the boundedness assumption of the input signals imposed in [14] is hard to verify for closed-loop control systems. To sidestep the difficulty of using the traditional regression vectors, we use the transfer function to transform the current system description into an IIR system description. Then, due to the infinite dimensionality of the unknown parameter in the current IIR system, the standard martingale limit theorems—used for analyzing least-squares algorithms for estimating finite-dimensional parameters—cannot be directly applied. Instead, we adopt the double array martingale theory developed by one of the authors in [38], originally proposed for the adaptive estimation of the $ARX(\infty)$ model, to analyze the convergence properties of martingale difference sequences with growing dimension. By integrating this approach with a time-varying projection operator in the algorithm design, we establish a convergence theory for parameter estimation that does not rely on PE or boundedness assumptions on the system signals, which are required in most of the existing identification literature (see, e.g., [8]–[17], [39]). Furthermore, beyond the existing related work in [14], [17], [38], [39], which focuses solely on adaptive estimation, we develop an adaptive control law by using a switching strategy that depends on a certain level of signal excitation, and show that the resulting closed-loop system signals satisfy the unbounded growth condition required for adaptive estimation. All the above will make it possible for us to establish the stability and optimality of the closed-loop adaptive tracking control systems under binary-valued output observations. To the best of the authors' knowledge, this appears to be the first such result for adaptive control under binary output observations for general tracking signals.

The remainder of this paper is organized as follows. Section II formulates the problem and states the basic assumptions. Section III gives the main results of this paper, including the proposed estimation algorithms, the adaptive control laws, and the main theorems. Section IV provides the proofs of the main results. Numerical examples are provided in Section V. Finally, we conclude the paper with some remarks.

Notations: By $\|\cdot\|$, we denote the Euclidean norm of vectors or matrices. The minimum eigenvalues of a matrix M is denoted by $\lambda_{\min}\{M\}$. Besides, by $\det(M)$ we mean the determinant of the matrix M . Moreover, the sequence $\{x_k, \mathcal{F}_k\}$ is said to be adapted if, for each $k \geq 0$, the random variable x_k is $\mathcal{F}_k$ -measurable, where $\mathcal{F}_k$ is a σ -algebra. Furthermore, for two real sequences $\{x_n, n \geq 1\}$ and $\{y_n, n \geq 1\}$, the notation $x_n = o(y_n), n \rightarrow \infty$ means that $x_n/y_n \rightarrow 0$ as $n \rightarrow \infty$. Also, $x_n = O(y_n), n \rightarrow \infty$, means that there exist positive constants $C > 0$ and $n_0 \in \mathbb{N}$ such that $|x_n|/|y_n| \leq C$ for all $n > n_0$.

II. PROBLEM FORMULATION

Consider the following finite-dimensional linear systems with binary-valued output observations:

$$\begin{cases} A(z)y_{n+1} = B(z)u_n \\ s_{n+1} = I(y_{n+1} + w_{n+1} > c_n) \end{cases}, \quad n \geq 0, \quad (1)$$

where y_n, u_n, s_n and w_n are the system output, input, output observation and noise, respectively (without loss of generality, we assume $y_n = u_n = 0, \forall n < 0$). $A(z)$ and $B(z)$ are polynomials in the backward-shift operator z :

$$\begin{aligned} A(z) &= 1 + a_1 z + \cdots + a_p z^p, \quad p \geq 0, \\ B(z) &= b_1 + b_2 z + \cdots + b_q z^{q-1}, \quad q \geq 1. \end{aligned}$$

Unlike in traditional precise observation scenarios, the system output cannot be obtained; instead, only two values can be observed. $I(\cdot)$ is the indicator function, and $\{c_n, n \geq 1\}$ denotes a given threshold sequence.

The problem considered here is, at any time instant $n \geq 0$, to design a feedback control u_n based only on the past binary-valued output observations $\{s_0, s_1, \dots, s_n\}$ and the known reference signal y_{n+1}^* , such that the limit superior of the averaged tracking error between the unknown system output y_{n+1} and the reference signal y_{n+1}^* , defined as

$$J_{n+1} = \frac{1}{n+1} \sum_{i=0}^n (y_{i+1} - y_{i+1}^*)^2, \quad n \geq 0,$$

is minimized, i.e., $\limsup_{n \rightarrow \infty} J_{n+1}$ achieves its minimum.

To proceed with further discussions, we need the following assumptions.

Assumption 1: Both $A(z)$ and $B(z)$ are stable polynomials, that is $A(z) \neq 0, B(z) \neq 0, \forall |z| \leq 1$.

Assumption 2: The reference signal $\{y_{k+1}^*, \mathcal{F}_k\}$ is a known bounded adapted sequence (where $\{\mathcal{F}_k\}$ is a sequence of nondecreasing σ -algebras).

Assumption 3: The input sequence $\{u_k, \mathcal{F}_k\}$ is an adapted sequence. The threshold $\{c_k, \mathcal{F}_k\}$ is a known bounded adapted sequence, with $\|c_k\| \leq C$, where C is a positive constant.

Assumption 4: The noise w_k is $\mathcal{F}_k$ -measurable. For any $k \geq 1$, the noise w_k is independent of $\mathcal{F}_{k-1}$, and its probability density function, denoted by $f_k(\cdot)$, is known and satisfies

$$\inf_{k \geq 1, x \in \mathfrak{X}} \{f_k(x)\} > 0, \quad (2)$$

for any bounded set $\mathfrak{X} \subset \mathbb{R}$.

Remark 1: The observation thresholds c_n considered in this paper are assumed to be given rather than freely designable. Such settings arise in many practical applications. For example, in the automotive emission control problem, the thresholds c_n are determined by the inherent limitations of the exhaust gas oxygen sensor ([6]); in the one-bit signal detection problem, the thresholds c_n are specified by the quantization function of the given one-bit ADCs ([1], [2]). If the thresholds are designable, it can provide advantages for estimation under binary-valued output observations, particularly, the noise condition in Assumption 4 could potentially be relaxed (see, e.g., [12], [44]), and the step size in Algorithm 1 could also be adjusted to improve convergence performance by suitably choosing the thresholds.

Remark 2: It is easy to verify that condition (2) is satisfied if the noise sequence $\{w_k, k \geq 1\}$ follows an i.i.d. Gaussian distribution, as assumed in previous studies (see, e.g., [9], [13]). Condition (2) ensures that, at time $k+1$, for any given observation threshold c_k and system output y_{k+1} , the output observation s_{k+1} does not take a single value with probability

one. This guarantees the identifiability of the parameter θ . Assumption 4 has been widely used in previous theoretical studies on adaptive identification under binary-valued output observations with given thresholds (see, e.g., [8], [14], [17]).

Since u_n is $\mathcal{F}_n$ -measurable, the best prediction for y_{n+1} given $\mathcal{F}_n$ in the mean square sense is

$$\mathbb{E}[y_{n+1} | \mathcal{F}_n] = \varphi_n^\top \vartheta, \quad (3)$$

where ϑ is the unknown parameter defined by

$$\vartheta = [-a_1, \dots, -a_p, b_1, \dots, b_q]^\top,$$

and φ_n is the corresponding regressor

$$\varphi_n = [y_n, \dots, y_{n-p+1}, u_n, \dots, u_{n-q+1}]^\top. \quad (4)$$

Moreover, since y_{n+1}^* is $\mathcal{F}_n$ -measurable, the control input u_n that satisfies

$$y_{n+1}^* = \mathbb{E}[y_{n+1} | \mathcal{F}_n] = \varphi_n^\top \vartheta. \quad (5)$$

is also $\mathcal{F}_n$ -measurable. Therefore, the minimum value of $\limsup_{n \rightarrow \infty} J_n$ over all adapted control sequences $\{u_n, \mathcal{F}_n, n \geq 1\}$ is 0, and the corresponding optimal adapted control u_n satisfies (5) for all $n \geq 0$. In traditional scenarios where the parameter ϑ is unknown but the outputs $\{y_k\}$ are available, a natural approach is first to provide an estimate $\hat{\vartheta}_k$ for the unknown parameter ϑ at each time instant k based on the past measurements $\{y_i, \varphi_{i-1}, 1 \leq i \leq k\}$, and then design the adaptive control by solving the equation

$$y_{k+1}^* = \varphi_k^\top \hat{\vartheta}_k. \quad (6)$$

However, in the current scenario where the system outputs $\{y_k\}$ are unavailable, since the regression vector $\{\varphi_k\}$ cannot be obtained, it turns out to be impossible to estimate the unknown parameter ϑ using $\{\varphi_k\}$ directly and to design the adaptive control law by solving equation (6) directly. To overcome this difficulty, we consider the transfer function:

$$G(z) = A^{-1}(z)B(z) = \sum_{i=1}^{\infty} G_i z^{i-1}.$$

Under Assumption 1, the following condition holds for any $0 < \rho < 1$:

$$G_1 = b_1, \quad \|G_i\| = O(\rho^i), \quad i > 1. \quad (7)$$

The system can then be transformed into an IIR system as follows:

$$y_{n+1} = \sum_{i=1}^{\infty} G_i u_{n-i+1}. \quad (8)$$

For the system described by (8), we introduce the parameter vector

$$\theta = [G_1, G_2, \dots]^\top \in \mathbb{R}^\infty,$$

and the corresponding regression vector

$$\phi_n = [u_n, u_{n-1}, \dots]^\top \in \mathbb{R}^\infty,$$

so that (1) can be rewritten as

$$\begin{cases} y_{n+1} = \phi_n^\top \theta \\ s_{n+1} = I(y_{n+1} + w_{n+1} > c_n) \end{cases}, \quad n \geq 0. \quad (9)$$

Then, according to (9), the equation (5) that the optimal feedback control satisfies can be rewritten in the following form:

$$y_{n+1}^* = \mathbb{E}[y_{n+1} | \mathcal{F}_n] = \phi_n^\top \theta. \quad (10)$$

Based on (10), the optimal control sequence $\{u_n^0\}$ can then be explicitly expressed as

$$u_n^0 = \frac{1}{G_1} \left(y_{n+1}^* - \sum_{i=2}^{\infty} G_i u_{n-i+1}^0 \right). \quad (11)$$

Since the parameters $\{G_i\}$ in the optimal control u_n^0 are unknown, a natural problem is how to design an estimation algorithm for this infinite-dimensional parameter vector by using binary-valued output observations only, which is needed in the design of an adaptive feedback control law. This issue will be discussed in the subsequent sections.

III. MAIN RESULTS

In this section, we first introduce an adaptive estimation algorithm for the unknown parameter vector θ . Compared to existing algorithms, a notable characteristic of the proposed algorithm is that its asymptotic properties can be established without requiring the PE condition or any boundedness conditions on the regression vectors. Furthermore, based on this estimation algorithm, we design a switching adaptive control and establish its global stability and asymptotic optimality.

A. Adaptive parameter estimation

To estimate the parameter vector in the current large model, we introduce sequences of vectors with progressively increasing dimensions. Specifically, define the dimension sequence $\{p_n, n \geq 1\}$ as

$$p_n = \lfloor (\log n)^a \rfloor, \quad a > 1, \quad (12)$$

where $\lfloor x \rfloor$ denotes the integer part of x . Accordingly, we define a sequence of parameter vectors $\{\theta(n), n \geq 1\}$ with dimensions growing at a rate determined by p_n :

$$\theta(n) = [G_1, G_2, \dots, G_{p_n}]^\top,$$

where $\{G_i, i \geq 1\}$ is defined in (8), and also define the sequence of regression vectors $\{\phi_k(n), 1 \leq k \leq n\}$ as:

$$\phi_k(n) = [u_k, u_{k-1}, \dots, u_{k-p_n+1}]^\top. \quad (13)$$

To constrain the growth rate of parameter estimates, we introduce a projection operator into the algorithm. First, define the function $g(t) : [0, \infty) \rightarrow \mathbb{R}$ as:

$$g(t) = \inf_{k \geq 1} \inf_{\|x\| \leq t+C} f_{k+1}(x), \quad t \geq 0. \quad (14)$$

Under Assumption 4, it follows that the function $g(t)$ is non-increasing and strictly positive for all $t \geq 0$. We then define the inverse function $h(\cdot)$ of $g(\cdot)$ as:

$$h(t) = \inf \{s \geq 0 : g(s) = t\}, \quad 0 < t \leq g(0), \quad (15)$$

which is also non-increasing.

For each $1 \leq k \leq n$, define the projection region as:

$$D_k(n) = \{x \in \mathbb{R}^{p_n} : \|x\|_1 \leq d_k\}, \quad (16)$$

where $\|\cdot\|_1$ denotes the L_1 norm, and the upper bound d_k is given by

$$d_k = \sqrt{h\left(\frac{g(0)}{\alpha_k}\right)}, \quad (17)$$

with $\{\alpha_k, k \geq 1\}$ being any positive, increasing sequence satisfying

$$\alpha_k \rightarrow \infty, \quad \alpha_k^2 = o(\sqrt{k}), \quad a.s., \quad k \rightarrow \infty. \quad (18)$$

The adaptive parameter estimates are given by Algorithm 1.

Algorithm 1

For each $n \geq 1$, the parameter estimate $\theta_n(n)$ is recursively updated as follows.

Inputs: The dimension p_n defined in (12); the initial value $\theta_0(n) \in \mathbb{R}^{p_n}$ and $P_0(n) \in \mathbb{R}^{p_n \times p_n}$ with $P_0(n) > 0$; the observed data $\{(s_{k+1}, c_k, \phi_k(n))\}_{k=0}^{n-1}$, where $\phi_k(n)$ is defined in (13); the constants $\{d_k, 0 \leq k \leq n-1\}$ and C , as specified in (17) and Assumption 3; and the projection region $\{D_k(n), 0 \leq k \leq n-1\}$ defined in (16).

Output: The final parameter estimate $\theta_n(n) \in \mathbb{R}^{p_n}$.

Procedure: For each $n \geq 1$, compute the sequence $\{\theta_k(n)\}_{k=0}^n$ recursively for $0 \leq k \leq n-1$ as follows:

$$\begin{aligned} \theta_{k+1}(n) &= \Pi_{P_{k+1}^{-1}(n)}^{D_k(n)} \{\theta_k(n) + a_k(n) \beta_k P_k(n) \phi_k(n) e_{k+1}(n)\}, \\ e_{k+1}(n) &= s_{k+1} - 1 + F_{k+1}(c_k - \phi_k^\top(n) \theta_k(n)), \\ P_{k+1}(n) &= P_k(n) - \beta_k^2 a_k(n) P_k(n) \phi_k(n) \phi_k^\top(n) P_k(n), \\ a_k(n) &= \frac{1}{1 + \beta_k^2 \phi_k^\top(n) P_k(n) \phi_k(n)}, \\ \beta_k &= \inf_{|x| \leq d_k} \max_{1 \leq i \leq k} |u_i| + C \{f_{k+1}(x)\}, \end{aligned} \quad (19)$$

where $F_{k+1}(\cdot)$ and $f_{k+1}(\cdot)$ are the probability distribution function and probability density function of the noise w_{k+1} , respectively; $\Pi_{P_{k+1}^{-1}(n)}^{D_k(n)}\{\cdot\}$ denotes the projection operator defined via a time-varying Frobenius norm as follows:

$$\Pi_{P_{k+1}^{-1}(n)}^{D_k(n)}\{x\} = \arg \min_{\omega \in D_k(n)} (x - \omega)^\top P_{k+1}^{-1}(n) (x - \omega), \quad \forall x \in \mathbb{R}^{p_n}.$$

Remark 3: Algorithm 1 is inspired by the estimation algorithms for the $ARX(\infty)$ model in [38] and [39]. Its recursive form is similar to the recursive least squares algorithm for linear stochastic models, with the main difference being the design of the update term $e_{k+1}(n)$, which utilizes only binary observation information s_{k+1} . Moreover, our estimation Algorithm 1 can be further simplified computationally by using a similar idea of recursively expanding dimension in [43]. Furthermore, compared to the estimation algorithm with binary-valued observations in [39], the current algorithm differs in two key aspects: Firstly, due to the time-varying design of the projection spaces, our method does not require prior knowledge of the bounded convex set containing the parameter θ , which is not easily obtainable for the current model. Secondly, in designing the scalar gain β_k , it does not rely on the prior assumption of an upper bound of the system

input, which is crucial for establishing adaptive control theory later on.

Based on the adaptive estimates of the unknown parameter, we define the following adaptive predictor for the output at each time step $k \geq 1$:

$$\hat{y}_{k+1} = \phi_k(k)^\top \theta_k(k),$$

where $\theta_k(k)$ is given by Algorithm 1. Here, the difference between the real output y_{k+1} and the adaptive predictor $\hat{y}_{k+1}$ is referred to as regret R_k , which is defined as follows:

$$R_k = [y_{k+1} - \phi_k(k)^\top \theta_k(k)]^2. \quad (20)$$

The asymptotic properties of the regrets are crucial for the theoretical analysis of adaptive control, which will be formally addressed in the following theorem.

Theorem 1: Under Assumptions 1-4, suppose for some $\bar{\delta} > 0$,

$$u_n = O(\min\{d_n, n^{1+\bar{\delta}}\}), \quad a.s., \quad n \rightarrow \infty, \quad (21)$$

where d_n is defined in (17). Then the following properties hold:

- The sample paths of the accumulated regrets have the following asymptotic upper bound as $n \rightarrow \infty$:

$$\sum_{k=0}^n R_k = O(\alpha_n^2 p_n^2 \log n), \quad a.s., \quad (22)$$

where p_n , α_n , and R_n are defined as in (12), (18), and (20), respectively.

- The parameter estimates given by Algorithm 1 have the following asymptotic upper bound as $n \rightarrow \infty$:

$$\|\theta(n) - \theta_n(n)\|^2 = O\left(\frac{\alpha_n^2 p_n \log n}{\lambda_{\min}(n)}\right), \quad a.s., \quad (23)$$

where $\lambda_{\min}(n)$ is defined by

$$\lambda_{\min}(n) = \lambda_{\min} \left\{ P_0^{-1} + \sum_{k=0}^n \phi_k(n) \phi_k^\top(n) \right\}. \quad (24)$$

The proof is provided in Appendix A.

The results of the accumulated regrets in (22) do not require any excitation condition, but a certain growth rate condition (21) on the system signals is required, which will be verified for the closed-loop adaptive systems in the proof of Theorem 2 in the next section.

Remark 4: If $A(z)$ and $B(z)$ have no common factor and $[a_p, b_q] \neq 0$, it is well known that the linear equations connecting $\{a_i, b_j, 1 \leq i \leq p, 1 \leq j \leq q\}$ with $\{G_i, i \geq 1\}$ are as follows:

$$b_i = \sum_{j=0}^{i \wedge p} a_j G_{i-j}, \quad \forall 1 \leq i \leq q,$$

where $\wedge$ represents the minimum value, and

$$[a_1, \dots, a_p]^\top = -(LL^\top)^{-1} L [G_{q+1}, \dots, G_{q+p}]^\top, \quad (25)$$

where

$$L = \begin{bmatrix} G_q & G_{q+1} & \cdots & G_{q+p-1} \\ G_{q-1} & G_q & \cdots & G_{q+p-2} \\ \cdots & \cdots & \cdots & \cdots \\ G_{q-p+1} & G_{q-p+2} & \cdots & G_q \end{bmatrix},$$

and $G_i \triangleq 0$ for $i < 0$ ([42]). Replacing $\{G_i, i \geq 1\}$ in (25) with their estimates $\{\theta_n(n), n \geq 1\}$ generated by Algorithm 1, we can obtain the estimates $\{\vartheta_n, n \geq 1\}$ for the unknown parameter ϑ . From (23), if the following condition holds:

$$\alpha_n^2 p_n \log n = o(\lambda_{\min}(n)), \quad a.s., \quad (26)$$

as $n \rightarrow \infty$, then the estimates ϑ_n are strongly consistent, i.e., $\lim_{n \rightarrow \infty} \|\vartheta - \vartheta_n\| = 0$, *a.s.* The input condition in (26) is weaker than the traditional persistent excitation (PE) condition ([25], [38]), which requires $n = O(\lambda_{\min}(n))$, *a.s.* This relaxed condition enables us to establish the adaptive control results presented in the next section.

B. Adaptive tracking control

1) Main ideas: Although we have relaxed the boundedness conditions on the system inputs in the asymptotic analysis of the parameter estimation algorithms, Theorem 1 still imposes a growth rate constraint on the system inputs. Unfortunately, guaranteeing such a constraint theoretically appears to be quite hard for a control law designed directly using the certainty equivalence principle, i.e.,

$$u_n^0 = \frac{1}{G_{1n}} \left(y_{n+1}^* - \sum_{i=2}^{\infty} G_{in} u_{n-i+1}^0 \right), \quad (27)$$

which prompts us to make some modifications to the controller design.

To motivate the ideas for improving the controller design, we first considered the case where the parameter θ is known. Since $G(z)$ is stable, it is easy to verify that the input sequence $\{u_n, n \geq 0\}$ generated by (11) is a bounded sequence. Furthermore, if the parameter estimation $\theta_n(n)$ is located within a certain “stable neighborhood” of the true parameter θ , it can also be proven that the input sequence remains bounded. See Lemma 2 below for details.

Based on the above analysis, for the case where the parameter is unknown, if the input value u_n given by (27) grows too fast, it indicates that we may not have sufficient signal information to drive the parameter estimates into a certain neighborhood of the true parameter, and in this case, online parameter learning may be called for. On the other hand, according to Theorem 1, if the minimum eigenvalue of the information matrix, i.e., $\lambda_{\min}(n)$, is sufficiently large, then there is adequate signal information to ensure the parameter estimates enter into a “stable neighborhood”, leading to the stability of the adaptive control. These two levels of signal information for estimation inspire a switching adaptive control between adaptive learning and feedback in the initial phase, but it will be proven that after some finite time, the adaptive controller will rarely be switched to the learning phase.

We now proceed to introduce the specific design of adaptive control.

2) Adaptive controller: Firstly, to deal with the potential occurrence of $b_{1n} = 0$ in (27), we make the necessary modification to the estimates produced by Algorithm 1. Let

$$\begin{aligned} \hat{\theta}_n(n) &= \theta_n(n) + (a_n(n) P_n(n))^{\frac{1}{2}} \bar{e}_{i_n} \\ &= [\hat{G}_{1n}, \hat{G}_{2n}, \dots, \hat{G}_{p_n n}]^\top, \end{aligned} \quad (28)$$

where $\{i_n\}$ is a sequence of integers defined by

$$i_n = \arg \max_{0 \leq i \leq p_n} \left| G_{1n} + \bar{e}_1^\top(n) P_n^{\frac{1}{2}}(n) \bar{e}_i(n) \right|,$$

with $\bar{e}_0(n) = 0$, and $\bar{e}_i(n), 1 \leq i \leq p_n$ representing the i th column of the $p_n \times p_n$ identity matrix. Following the proof ideas of Theorem 6.3 in [32], it can be shown that the estimates $\hat{\theta}_n(n)$ still satisfy the same asymptotic properties as the original estimates $\theta_n(n)$ in Theorem 1. Consequently, the modified adaptive control law is given by

$$u_n^s = \frac{1}{\hat{G}_{1n}} \left(y_{n+1}^* - \hat{G}_{2n} u_{n-1} \cdots - \hat{G}_{p_n n} u_{n-p_n+1} \right). \quad (29)$$

To make the adaptive system have some guaranteed level of excitation, we now introduce an arbitrary i.i.d. sequence $\{v_t, t \geq 0\} \subseteq \mathbb{R}$, which is independent of $\{w_t\}$ and satisfies the following properties:

$$\mathbb{E}[v_k] = 0, \quad \mathbb{E}[v_k^2] = 1, \quad |v_k| \leq \bar{v}, \quad (30)$$

where $\bar{v}$ is a constant. Let us partition the time axis by a sequence of stopping times

$$0 = \tau_1 < \sigma_1 < \tau_2 < \sigma_2 < \cdots$$

at which the adaptive controller u_t will be switched, i.e.,

$$u_t = \begin{cases} u_t^s, & \text{if } t \text{ belongs to some } [\tau_k, \sigma_k) \\ v_t, & \text{if } t \text{ belongs to some } [\sigma_k, \tau_{k+1}) \end{cases}, \quad (31)$$

where

$$\sigma_k = \sup \{t > \tau_k : |u_t^s| \leq \min\{d_j, j^b\} + |u_0^s|, \forall j \in [\tau_k, t)\} \quad (32)$$

$$\tau_{k+1} = \inf \left\{ t > \sigma_k : \lambda_{\min}(t) \geq \alpha_t^2 p_t \log^{1+\epsilon} t, \right. \\ \left. \max_{t-p_t \leq j \leq t-1} \{|u_j|, |u_t^s|\} \leq \min\{\sqrt{d_t}, t^{\frac{b}{2}}\} \right\}, \quad (33)$$

where b and ϵ are fixed constants with $b \in (0, \frac{1}{4})$ and $\epsilon > 0$. If the set $\{\cdot\}$ is empty, we define $\sup\{\cdot\}$ as ∞ .

We now make a brief explanation about the switching adaptive control (31). By (31)-(33), it is easy to see that from the random time τ_k , the adaptive control u_t is taken as u_t^s as far as $t < \sigma_k$, where σ_k is the first time when the growth rate of $|u_t^s|$ is greater than $\min\{d_t, t^b\} + |u_0^s|$; then from the random time σ_k , the adaptive control is taken as v_t , until both the growth rate of the minimum eigenvalue $\lambda_{\min}(t)$ is larger than $\alpha_t^2 p_t \log^{1+\epsilon} t$ and the growth rate of the maximum magnitude of the control inputs u_j over the past time interval $[t - p_t, t - 1]$, along with the current ideal control input u_t^s , is less than $\min\{\sqrt{d_t}, t^{\frac{b}{2}}\}$. Moreover, according to (31)-(33), it can easily be seen that for any $t \geq 0$, the growth rate of the adaptive control u_t is constrained within the following bound:

$$|u_t| \leq \min\{d_t, t^b\}, t \geq 0. \quad (34)$$

Remark 5: In the development of the adaptive control theory below, we show that when the control law switches to the learning mode, the system signals will accumulate sufficient excitation to switch back to the control mode within a reasonable period. Moreover, we prove that, up to time n , the total time spent in the learning mode is at most $O(n^\delta p_n)$

as $n \rightarrow \infty$, where $\frac{1}{2} < \delta < 1$. Therefore, the time spent in the learning mode is significantly less than that spent in the control mode, since the ratio of the two tends to 0 as $n \rightarrow \infty$.

Theorem 2: Let Assumptions 1-4 be satisfied. Then the control system (1) under adaptive control law (31)-(33) is globally asymptotically optimal. That is, for any $y_0 \in \mathbb{R}$, $\lim_{n \rightarrow \infty} J_n = 0$, a.s. Moreover, the adaptive tracking error satisfies the following convergence rate as $n \rightarrow \infty$:

$$\sum_{k=1}^n (y_k^* - y_k)^2 = O(n^\delta p_n), \quad a.s. \quad (35)$$

for any $\delta \in (\frac{1}{2} + 2b, 1)$, where p_n and b are defined in (12) and (32), (33), respectively.

Remark 6: Although Theorem 2 implies that the averaged tracking error (or regret in this case) converges to zero, we believe that there still exists room for improving the convergence rate in the current binary-valued observation case. In the traditional full output observation case, the averaged output regrets can achieve a faster convergence rate than that in Theorem 2 if the noise has conditional moments higher than four (see Theorem 1 in [31]), and can even achieve the best possible convergence rate $O(\frac{\log n}{n})$, see [29], [32].

IV. PROOF OF MAIN RESULTS

In this section, we present the proof of Theorem 2. To facilitate the analysis, we first establish two key lemmas.

Define

$$\psi_t(n) = [v_t, v_{t-1}, \dots, v_{t-p_n}]^\top, \quad 1 \leq t \leq n,$$

where the sequence $\{v_t\}$ is defined as in (30). Then, by using the double array martingale theory, we obtain the following lemma concerning the minimum eigenvalue of the information matrix.

Lemma 1: For each $\frac{1}{2} < \eta < 1$, there exists constants $c(\eta) > 0$ and $N(\eta) \geq 1$ such that for each $n \geq N(\eta)$,

$$\lambda_{\min} \left\{ \sum_{t=n-n^\eta}^n \psi_t(n) \psi_t^\top(n) \right\} \geq c(\eta) n^\eta.$$

Given a power series $Q(z) = 1 + \sum_{i=1}^{\infty} q_i z^i$ satisfying

$$\sum_{i=1}^{\infty} |q_i| < \infty, \quad Q(z) \neq 0, \quad \forall |z| \leq 1, \quad (36)$$

The following lemma holds:

Lemma 2: Let the power series $Q(z)$ satisfy condition (36). Then there exists a positive constant Δ , depending only on $Q(z)$, such that for any $\bar{\xi} > 0$, there exist positive constants $c_1(\bar{\xi})$ and $c_2(\bar{\xi})$, depending only on $Q(z)$ and $\bar{\xi}$, for which the following property (referred to as the uniform stability bound) holds: For any given integers n_0 and N , if the sequences $\{\mu_n, n \geq 1\} \subset \mathbb{R}$ and $\{\xi_n, n \geq 1\} \subset \mathbb{R}$ satisfy

- (i). $\max_{n_0 \leq n \leq N} |\xi_n| \leq \bar{\xi}$;
- (ii). There exists a sequence of polynomials $\{Q_n(z), n_0 \leq n \leq N\}$, where each $Q_n(z) = 1 + \sum_{i=1}^{p_n-1} q_{n,i} z^i$, such

that $\max_{n_0 \leq n \leq N} \sum_{i=1}^{\infty} |q_i - q_{n,i}|^2 \leq \Delta$ and $Q_n(z)\mu_n = \xi_n$ for all $n_0 \leq n \leq N$, where z denotes the backwards-shift operator,

then we have

$$|\mu_N| \leq c_1 \max_{n_0 - p_{n_0+1} \leq i \leq n_0 - 1} |\mu_i| + c_2. \quad (37)$$

We now present the proof of Theorem 2.

Proof of Theorems 2. For convenience, we first introduce the following notations:

$$\begin{aligned} U(z) &= 1 + \sum_{i=2}^{\infty} \frac{G_i}{G_1} z^{i-1}, \\ U_n(z) &= 1 + \sum_{i=2}^{p_n} \frac{\hat{G}_{n,i}}{\hat{G}_{n,1}} z^{i-1}, \quad n \geq 1, \\ \zeta_n &= \left[\frac{y_{n+1}^*}{\hat{G}_{n,1}}, 0, \dots \right]^\top, \quad n \geq 1. \end{aligned} \quad (38)$$

To prove Theorem 2, we consider the following three cases:

(1) $\sigma_k < \infty$, $\tau_{k+1} = \infty$ for some $k = k_0$; (2) $\tau_k < \infty$, $\sigma_k = \infty$ for some $k = k_1$; (3) $\tau_k < \infty$, $\sigma_k < \infty$ for all $k \geq 1$.

Case (1). We now prove that Case (1) cannot occur. Assume that $\tau_{k+1} = \infty$ for some $k = k_0$. Then by (31), we have

$$u_i = v_i, \text{ for all } i \geq \sigma_{k_0}. \quad (39)$$

Define $N_1 = \inf\{n \geq 1 : j - j^{3/4} - p_j + 1 \geq \sigma_{k_0}, \forall j \in [n, \infty)\}$. By (39) and the definition of $\phi_t(n)$ in (13), for all $n \geq N_1$ and $t \in [n - n^{3/4}, n]$, it follows that

$$\phi_t(n) = [v_t, v_{t-1}, \dots, v_{t-p_n+1}]^\top. \quad (40)$$

Thus, from (24), (40) and Lemma 1, for all $n \geq \max(N_1, N(\frac{3}{4}))$, we obtain

$$\lambda_{\min}(n) \geq \lambda_{\min} \left\{ \sum_{t=n-n^{3/4}}^n \phi_t(n) \phi_t^\top(n) \right\} \geq c \left(\frac{3}{4} \right) n^{3/4}. \quad (41)$$

Let $N_2 = \inf\{n \geq 1 : \bar{c}j^{3/4} \geq p_j \alpha_j^2 \log j, \forall j \in [n, \infty)\}$. By (24) and (41), we have for all $n \geq \max(N_1, N_2, N(\frac{3}{4}))$,

$$\lambda_{\min}(n) \geq \lambda_{\min} \left\{ \sum_{t=n-n^{3/4}}^n \phi_t(n) \phi_t^\top(n) \right\} \geq p_n \alpha_n^2 \log n.$$

Since $\tau_{k_0+1} = \infty$, by the definition of τ_{k_0+1} and inequality (41), we have for all $n \geq \max(N_1, N_2, N(\frac{3}{4}))$,

$$\max_{n-p_n \leq i \leq n-1} (|u_n^s|, |u_{n-i}|) > \min \left(\sqrt{d_n}, n^{\frac{b}{2}} \right). \quad (42)$$

We now show that inequality (42) cannot hold for all $n \geq \max(N_1, N_2, N(\frac{3}{4}))$.

We first show that the parameter estimate $\hat{\theta}_n(n)$ converges to the true parameter vector $\theta(n)$, that is,

$$\|\hat{\theta}_n(n) - \theta(n)\| = o(1), \quad n \rightarrow \infty. \quad (43)$$

From (23) in Theorem 1 and (41), we obtain for all $n \geq \max(N_1, N_2, N(\frac{3}{4}))$,

$$\|\theta_n(n) - \theta(n)\|^2 = O \left(n^{-3/4} p_n \alpha_n^2 \log n \right), \quad n \rightarrow \infty. \quad (44)$$

Moreover, from (113) in the proof of Theorem 1, we have

$$\alpha_n^{-2} n^{3/4} = O \left(\lambda_{\min} \{P_n^{-1}(n)\} \right), \quad n \rightarrow \infty. \quad (45)$$

Then, by (18) and (45), it follows that

$$\|P_n(n)\| = o(1), \quad n \rightarrow \infty. \quad (46)$$

Hence, from (28) and (46), we obtain

$$\|\hat{\theta}_n(n) - \theta_n(n)\| = O \left(\|P_n(n)\|^{\frac{1}{2}} \right) = o(1), \quad n \rightarrow \infty. \quad (47)$$

Combining (44) and (47) yields (43).

Next, we show that (42) cannot hold for all $n \geq \max(N_1, N_2, N(\frac{3}{4}))$. By Assumption 1, $U(z)$ defined in (38) satisfies condition (36). Moreover, from (43) and Assumption 2, there exists a positive constant ζ_0 such that $|\zeta_n| \leq \zeta_0$ for all $n \geq 1$. Then by Lemma 2, there exists a constant $\Delta_0 > 0$ such that there exist positive constants $c_1(\zeta_0)$ and $c_2(\zeta_0)$, depending only on $U(z)$ and ζ_0 , for which the uniform stability bound property holds. From (43) and (7), we have as $n \rightarrow \infty$,

$$\begin{aligned} \sum_{i=2}^{\infty} \left| \frac{\hat{G}_{n,i}}{\hat{G}_{n,1}} - \frac{G_i}{G_1} \right|^2 &= \sum_{i=2}^{p_n} \left| \frac{\hat{G}_{n,i}}{\hat{G}_{n,1}} - \frac{G_i}{G_1} \right|^2 + \sum_{i=p_n+1}^{\infty} \left| \frac{G_i}{G_1} \right|^2 \\ &= o(1) + O(\rho^{2p_n}) = o(1). \end{aligned} \quad (48)$$

Therefore, there exists $N_3 > 0$ such that for all $n \geq N_3$,

$$\sum_{i=2}^{\infty} \left| \frac{\hat{G}_{n,i}}{\hat{G}_{n,1}} - \frac{G_i}{G_1} \right|^2 \leq \Delta_0. \quad (49)$$

Fixed any $j \geq N_3$, let $n_0 = N = j$ in Lemma 2, and define the modified sequence $\{u_n^{(j)}, n \geq 1\}$ as

$$u_n^{(j)} = \begin{cases} u_n, & n \neq j, \\ u_n^s, & n = j. \end{cases} \quad (50)$$

Then by (29), we have $U_j(z)u_n^{(j)} = \zeta_n$. Together with (49), this implies that sequences $\{u_n^{(j)}, n \geq 1\}$ and $\{\zeta_n, n \geq 1\}$ satisfy the conditions (i) and (ii) in Lemma 2, and thus

$$|u_j^{(j)}| \leq c_1(\zeta_0) \max_{j-p_j+1 \leq i \leq j-1} |u_i| + c_2(\zeta_0).$$

By (50) and (39), for all $n \geq \max(N_1, N_2, N_3, N(\frac{3}{4}))$, we obtain

$$|u_j^s| \leq c_1(\zeta_0) \max_{j-p_j+1 \leq i \leq j-1} |v_i| + c_2(\zeta_0) \leq c_1(\zeta_0) \bar{v} + c_2(\zeta_0), \quad (51)$$

where $\bar{v}$ is the uniform upper bound on $|v_i|$ defined in (30). Since j is arbitrary, (51) holds for all $j \geq \max(N_1, N_2, N_3, N(\frac{3}{4}))$. Furthermore, from (17), we have $\lim_{n \rightarrow \infty} \min \left(\sqrt{d_n}, n^{\frac{b}{2}} \right) = \infty$. Thus, for sufficiently large n ,

we have $\max_{1 \leq i \leq p_n} (|u_n^s|, |u_{n-i}|) \leq \min \left(\sqrt{d_n}, n^{\frac{b}{2}} \right)$, which contradicts (42). Therefore, case (1) cannot occur.

Case (2). If $\sigma_{k_1} = \infty$ for some $k_1 \geq 1$, then for all $t \geq \tau_{k_1}$, we have $u_t = u_t^s$. Therefore, for all $t \geq \tau_{k_1}$, by (29) and (28), it follows that

$$\begin{aligned} y_{t+1}^* &= [u_t^s, u_{t-1}, \dots, u_{t-p_t+1}]^\top \hat{\theta}_t(t) \\ &= \phi_t^\top(t) \left[\theta_t(t) + a_t^{\frac{1}{2}}(t) P_t^{\frac{1}{2}}(t) \bar{e}_{i_t} \right]. \end{aligned} \quad (52)$$

By Assumption 2 and (66), there exists a positive constant $\bar{\zeta}_0$ such that $\sup_{i \geq 1} \max_{n \in [\tau_{k_i}, \sigma_{k_i}]} |\zeta_n| \leq \bar{\zeta}_0$. Then by Lemma 2, there exist constants $c_1(\bar{\zeta}_0) > 0$ and $c_2(\bar{\zeta}_0) > 0$, depending only on $U(z)$ and $\bar{\zeta}_0$, such that the uniform stability bound property holds. Furthermore, by the same reasoning as in (48)-(49), there exists $j_0 > 0$ such that for all $j \geq j_0$,

$$\sup_{n \in [\tau_{k_j}, \sigma_{k_j}]} \sum_{i=2}^{\infty} \left| \frac{\hat{G}_{n,i}}{\hat{G}_{n,1}} - \frac{G_i}{G_1} \right|^2 \leq \Delta_0. \quad (67)$$

For each $j \geq j_0$, let $n_0 = \tau_{k_j}$ and $N = \sigma_{k_j}$ in Lemma 2, and define a modified sequence $\{\bar{u}_n^{(j)}, n \geq 1\}$ by

$$\bar{u}_n^{(j)} = \begin{cases} u_n, & n \notin [\tau_{k_j}, \sigma_{k_j}], \\ u_n^s, & n \in [\tau_{k_j}, \sigma_{k_j}]. \end{cases}$$

By (29), we have $U_n(z)u_n^{(j)} = \xi_n$ for all $\tau_{k_i} \leq n \leq \sigma_{k_i}$. Hence, combining with (67), the sequences $\{\bar{u}_n^{(j)}, n \geq 1\}$ and $\{\zeta_n, n \geq 1\}$ satisfy conditions (i) and (ii) in Lemma 2, and thus

$$|u_{\sigma_{k_i}}^s| = |\bar{u}_{\sigma_{k_i}}^{(j)}| \leq \bar{c}_1(\bar{\zeta}_0) \max_{\tau_{k_i} - p_{\tau_{k_i}} \leq i \leq \tau_{k_i} - 1} |u_i| + \bar{c}_2(\bar{\zeta}_0). \quad (68)$$

By the definition of τ_{k_i} , we have

$$\max_{\tau_{k_i} - p_{\tau_{k_i}} \leq i \leq \tau_{k_i} - 1} |u_i| \leq \min \left\{ \sqrt{d_{\tau_{k_i}}}, \tau_{k_i}^{\frac{b}{2}} \right\}. \quad (69)$$

Therefore, from (68) and (69), we conclude that for sufficiently large k_i ,

$$|u_{\sigma_{k_i}}^s| \leq \min \left\{ d_{\sigma_{k_i}}, \sigma_{k_i}^b \right\} + |u_0^s|. \quad (70)$$

However, by the definition of σ_k , we also have for each $k \geq 1$,

$$|u_{\sigma_k}^s| > \min \left\{ d_{\sigma_k}, \sigma_k^b \right\} + |u_0^s|, \quad (71)$$

which contradicts (70). Hence, we conclude that

$$\sum_{k=1}^{j_n} I(E_k) = O(1), \quad n \rightarrow \infty. \quad (72)$$

Finally, combining (58), (61), and (72), we obtain the desired bound in (57).

Next, we show that for any $\frac{1}{2} < \bar{\eta} < 1$, the following holds:

$$\max_{1 \leq i \leq j_n} |\tau_{i+1} - \sigma_i| = O(n^{\bar{\eta}}), \quad n \rightarrow \infty. \quad (73)$$

Let $s_i = \sigma_i + 3\sigma_i^{\bar{\eta}}$. Since $\sigma_i \rightarrow \infty$ as $i \rightarrow \infty$, we have for sufficiently large i that $s_i - s_i^{\bar{\eta}} - p_{s_i} > \sigma_i$. Define

$$I_0 = \inf \{i \geq 1 : s_j - s_j^{\bar{\eta}} - p_{s_j} + 1 \geq \sigma_j, j \in [i, \infty)\} \quad (74)$$

We now show that $\tau_{i+1} \leq s_i$ for sufficiently large i . Suppose, to the contrary, that this does not hold. Then there exists a subsequence $\{n_i, i \geq 1\}$ with $n_i \rightarrow \infty$ such that

$$\tau_{n_i+1} > s_{n_i}. \quad (75)$$

From (75) and (31), we have $u_t = v_t$ for all $t \in [\sigma_{n_i}, s_{n_i}]$. Then, using (74) and (13), for all $i \geq I_0$ and $s_{n_i} - s_{n_i}^{\bar{\eta}} \leq$

$t \leq s_{n_i}$, we get $\phi_t(s_{n_i}) = [v_t, \dots, v_{t-p_{s_{n_i}}+1}]^\top$. Applying Lemma 1 yields

$$\lambda_{\min} \left\{ \sum_{t=s_{n_i}-s_{n_i}^{\bar{\eta}}}^{s_{n_i}} \phi_t(s_{n_i}) \phi_t^\top(s_{n_i}) \right\} \geq c(\bar{\eta}) s_{n_i}^{\bar{\eta}}. \quad (76)$$

Hence, following a similar analysis to that in (44)-(47), we obtain

$$\|\hat{\theta}_{s_{n_i}}(s_{n_i}) - \theta(s_{n_i})\|^2 = o(1), \quad i \rightarrow \infty. \quad (77)$$

From (77), there exists $\tilde{\zeta} > 0$ such that $|\zeta_{s_{n_i}}| \leq \tilde{\zeta}, i \geq 1$. Then, using the same argument as in (48)-(51), we obtain for sufficiently large i :

$$|u_{s_{n_i}}^s| \leq c_1(\tilde{\zeta})\bar{v} + c_2(\tilde{\zeta}).$$

Consequently, for such sufficiently large i , we have

$$\begin{aligned} \max_{s_{n_i} - p_{s_{n_i}} \leq j \leq s_{n_i} - 1} \{|u_j|, |u_{s_{n_i}}^s|\} &\leq (1 + c_1(\tilde{\zeta}))\bar{v} + c_2(\tilde{\zeta}) \\ &\leq \min \left\{ \sqrt{d_{s_{n_i}}}, \sqrt{s_{n_i}^b} \right\}. \end{aligned} \quad (78)$$

where we have used the fact that $\min \left\{ \sqrt{d_{s_{n_i}}}, \sqrt{s_{n_i}^b} \right\} \rightarrow \infty$ as $i \rightarrow \infty$. However, inequalities (76) and (78) jointly contradict assumption (75). Therefore, for sufficiently large i , we have

$$|\tau_{i+1} - \sigma_i| \leq |s_i - \sigma_i| \leq 3\sigma_i^{\bar{\eta}} \leq 3n^{\bar{\eta}},$$

which establishes (73).

Furthermore, from (8) and (34), we have

$$[y_{t+1}^* - y_{t+1}]^2 = O \left(\max_{1 \leq i \leq t} u_i^2 \right) = O(t^{2b}), \quad t \rightarrow \infty. \quad (79)$$

Finally, combining (57), (73) and (79), we obtain

$$\left(\sum_{t=\sigma_1}^{\tau_2-1} + \dots + \sum_{t=\sigma_{j_n-1}}^{\tau_{j_n}-1} \right) [y_{t+1} - y_{t+1}^*]^2 = O(n^\delta p_n). \quad (80)$$

where $\delta \in (\frac{1}{2} + 2b, 1)$. The desired bound (35) then follows from (56) and (80).

V. NUMERICAL SIMULATION

In this section, we present a simulation example to illustrate the theoretical results obtained in the previous sections.

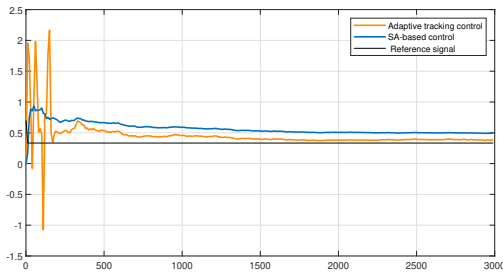
Consider the following stochastic dynamic system under binary-valued output observations:

$$\begin{cases} A(z)y_{k+1} = B(z)u_k \\ s_{k+1} = I(y_{k+1} + w_{k+1} \geq c_k) \end{cases}, \quad k \geq 0, \quad (81)$$

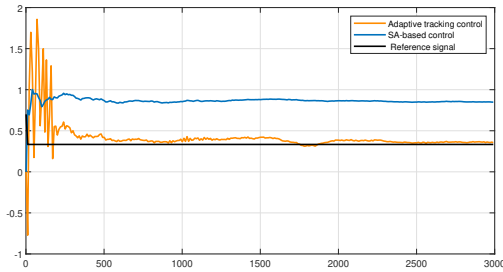
where $A(z) = 1 - 0.1z + 0.5z^2$, $B(z) = 1 + 0.5z - 0.4z^2$. The noise sequence $\{w_k, k \geq 1\}$ is i.i.d. with standard normal distribution $\mathcal{N}(0, 1)$. To estimate the parameter vector θ using Algorithm 1, we set the initial value as $\theta_0(n) = [0, \dots, 0]^\top$ and $P_0(n) = I$ for each $n \geq 1$.

The reference signals $\{y_k^*\}$ are generated by the following logistic map, which is in part a discrete-time demographic model:

$$y_{k+1}^* = r y_k^* (1 - y_k^*), \quad k \geq 0, \quad (82)$$



(a) Specially designed binary observation thresholds



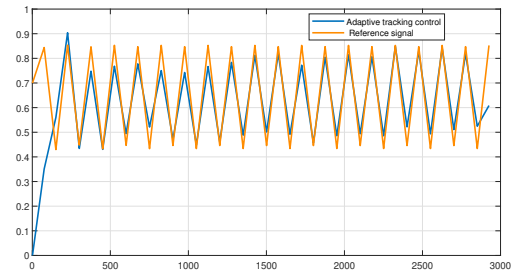
(b) General binary observation thresholds

Fig. 1: $r = 1.5$: Adaptive tracking control for fixed reference signals. a) Specially designed binary observation thresholds b) General binary observation thresholds.

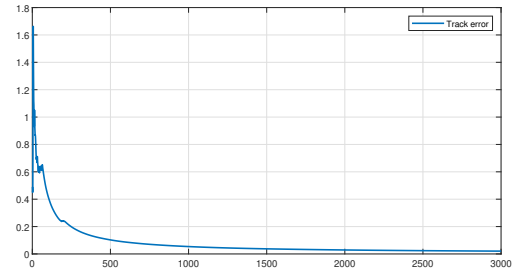
where the initial value is set to $y_0^* = 0.7$. It is well known that when different values of r are selected within the range of $(0, 4)$, the system signal $\{y_k^*, k \geq 0\}$ exhibits different behaviours: 1) When $r \in (0, 3]$, y_k^* converges to a constant value; 2) When r is within $(3, r^*)$ where $r^* = 3.56994 \dots$, y_k^* exhibit a periodic behaviour; 3) When r belongs to the interval $[r^*, 4)$, the maps enter the chaotic region [45]. Below, we validate the tracking performance of the proposed adaptive controller for reference signals generated with r taken from three different intervals.

Case 1: Consider the case where $r = 1.5$; in this scenario, the reference signals generated by (82) quickly converge to the value $\frac{1}{3}$. Fig. 1 compares the tracking performance of our adaptive controller (31)-(33) with the stochastic approximation (SA)-based controller proposed in [37] for this type of reference signal, under both specifically designed binary observation thresholds c_k and general observation thresholds c_k . a) When the binary observation thresholds c_k are specifically designed to match the reference value of $\frac{1}{3}$, both controllers are able to track the reference signal, and our method achieves a faster convergence speed. b) When the binary observation threshold c_k can be set arbitrarily, we may take it as 0.8 for illustration. It can be observed that the system output under the SA-based controller in [37] stabilizes at the observation threshold of 0.8, failing to track the reference signal, while the system output under our adaptive controller ensures that the adaptive tracking error asymptotically converges to zero.

Case 2: Consider the case where $r = 3.44$. In this situation, the reference signals generated by (82) will exhibit periodic oscillations between four values. In the experiment, the observation thresholds are set to $c_k = 0$, and the adaptive



(a) Trajectories of reference signals and system outputs



(b) Tracking error

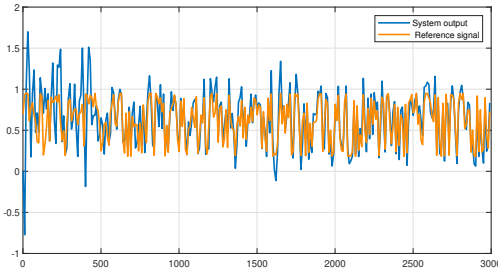
Fig. 2: $r = 3.44$: Adaptive tracking control for periodic reference signals.

controller u_k is provided by (31)-(33). Fig. 2 (a)-(b) presents the tracking results: the upper subfigure shows the trajectories of the system outputs and the reference signals, from which we can see that the system outputs are able to track the variations in the reference signals; The lower subfigure clearly illustrates that the tracking error converges to zero.

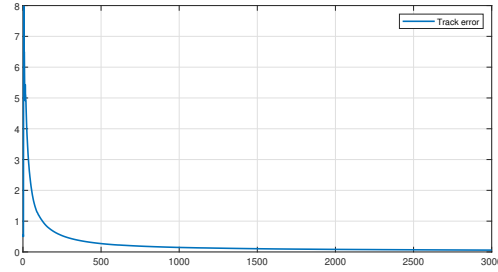
Case 3: Consider the case where $r = 3.8$, which lies within the range where the logistic map may exhibit chaotic dynamics. In this scenario, the reference signal generated from (82) likely exhibits chaotic behavior. Fig. 3 (a)-(b) show the trajectories of the reference signals, the system outputs under the adaptive controller (31)-(33), as well as the tracking error. It can be seen that for these complicated reference signals, the adaptive tracking error can still converge to zero.

VI. CONCLUDING REMARKS

In this paper, we have established an adaptive control theory for finite-dimensional linear systems with any given bounded reference signals under binary-valued output observations. As explained in the paper, one of the difficulties in establishing such a theory lies in the fact that the regression vectors containing the lagged output signals to be used in the construction of the traditional adaptive algorithms are not fully available. Another difficulty is how to sidestep or verify both the boundedness and the excitation conditions required in the convergence analysis of adaptive estimation algorithms in the existing related literature. To overcome these difficulties, we have used both the ideas of large model estimation and the theory of double array martingales in the design and analysis of adaptive algorithms, which makes it possible for us to establish for the first time an adaptive identification theory for



(a) Trajectories of reference signals and system outputs



(b) Tracking error

Fig. 3: $r = 3.8$: Adaptive tracking control for chaotic reference signals.

linear systems with binary-valued output observations under non-PE conditions and possibly unbounded system signals. This further allows us to establish the global stability and asymptotic optimality of the closed-loop adaptive systems. We remark that several extensions of the current work are possible, including more general forms of nonlinear output observations (e.g., multi-level quantized output observations and saturated output observations), and linearly parameterized nonlinear dynamical systems with binary-valued output observations. However, there are many interesting problems that need to be further investigated. At the moment, we do not know whether or not it is possible to establish such a theory without using the IIR-based design ideas and analysis techniques introduced in this paper. It would also be challenging to consider more complicated stochastic dynamical systems, including multi-layer networks and other basic control problems such as linear-quadratic-adaptive control under binary-valued output observations.

APPENDIX

Lemma 3: ([41]). Let $\Sigma_k \in \mathbb{R}^{p \times p}$, $k \geq 0$. For any given $0 < M_1 < 1$ and $0 < M_2 < \infty$, there exists $C(p, M_1, M_2) > 0$, $M > 0$ and $0 < \bar{\rho} < 1$, depending only on p , M_1 , and M_2 , such that if (i) $\max_{n_1 \leq k \leq n_2} \rho(\Sigma_k) \leq M_1$; (ii) $\max_{n_1 \leq k \leq n_2} \|\Sigma_k\| \leq M_2$; (iii) $\sup_{n_1 \leq k \leq n_2-1} \|\Sigma_{k+1} - \Sigma_k\| \leq C(p, M_1, M_2)$, then

$$\left\| \prod_{k=i}^j \Sigma_k \right\| \leq M \bar{\rho}^{j-i}, \forall n_1 \leq j \leq i \leq n_2.$$

Lemma 4: ([38]). Let $\{f_k(n)\}$, $1 \leq k \leq n$ be a $\mathcal{F}_k$ -measurable random vector sequence in $\mathbb{R}^{p_n}$, $p_n \geq 1$, and

$$M_k(n) = I + \sum_{j=1}^k f_j(n) f_j^\top(n), 1 \leq k \leq n. \text{ Then as } n \rightarrow \infty,$$

$$\sum_{k=1}^n f_k^\top(n) M_k^{-1}(n) f_k(n) = O(\log^+ \{\det(M_n(n))\}) + 1 \quad (83)$$

where $\log^+ \{\cdot\}$ denotes the positive part of $\log\{\cdot\}$.

Lemma 5: ([38]). Suppose that $\{w_n, \mathcal{F}_n\}$ is a martingale difference sequence satisfying $\sup_n \mathbb{E}[|w_{n+1}|^2 | \mathcal{F}_n] < \infty$, and

$$\sup_n \mathbb{E}[|w_{n+1}|^4 (\log^+ |w_n|)^{2+\delta}] < \infty, \text{ a.s.}, \quad (84)$$

and $\{x_n, \mathcal{F}_n\}$ is any adapted random sequence satisfying

$$\sum_{i=1}^n \|x\|^2 = O(n^b), \quad \mathbb{E}\|x_n\|^4 \{\log^+(\|x_n\|)\}^{2+\delta} = O(n^{2(b-1)}), \quad (85)$$

where $\delta > 0$, $b \geq 1$ are some constants. Then as $n \rightarrow \infty$:

$$\max_{1 \leq t \leq n} \max_{1 \leq i \leq n} \left\| \sum_{j=1}^i x_{j-t} w_j \right\| = O\left(n^{\frac{b}{2}} \{\log n\}^{\frac{1}{2}}\right) \text{ a.s.}, \forall \eta > 0. \quad (86)$$

Lemma 6: ([40]) Let $\{w_t, \mathcal{F}_t\}$ be a martingale difference sequence satisfying

$$\sup_{t \geq 0} \mathbb{E}[\|w_{t+1}\|^2 | \mathcal{F}_t] < \infty, \quad \|w_t\| = o(\varphi(t)), \text{ a.s.}, \quad (87)$$

where $\varphi(\cdot)$ is a deterministic positive, non-decreasing function and satisfies $\sup_{k \geq 0} \varphi(e^{k+1})/\varphi(e^k) < \infty$. Let $\{f_t(k)\}$, with $t \geq 0$ and $k = 1, 2, \dots$, be a random sequence such that for each fixed $t \geq 0$, $f_t(k)$ is $\mathcal{F}_t$ -measurable for all $k \geq 1$. Then for $p_n = O(\lfloor \log^a n \rfloor)$, $a > 1$, the following property holds:

$$\max_{1 \leq k \leq p_n} \max_{1 \leq i \leq n} \left\| \sum_{t=1}^i f_t(k) w_{t+1} \right\| = O(\bar{r}_n \varphi(n) \log \log n), \text{ a.s.}, n \rightarrow \infty, \quad (88)$$

where

$$\bar{r}_i = \max_{1 \leq k \leq p_n} r_i(k), \quad r_i(k) = \left[\sum_{t=1}^i \|f_t(k)\|^2 + 1 \right]^{\frac{1}{2}}, \quad r_0(k) = 1. \quad (89)$$

A. Proof of Theorem 1, Lemma 1, Lemma 2

Proof of Theorem 1. We first introduce the following notations:

$$\begin{aligned} \psi_k(n) &= F_{k+1}(c_k - \phi_k^\top \theta) - F_{k+1}(c_k - \phi_k^\top(n) \theta_k(n)), \\ \gamma_{k+1} &= s_{k+1} - F_{k+1}(c_k - \phi_k^\top \theta), \\ \delta_k(n) &= \phi_k^\top \theta - \phi_k^\top(n) \theta(n), \\ \tilde{\theta}_k(n) &= \theta(n) - \theta_k(n), \end{aligned} \quad (90)$$

where $F_{k+1}(\cdot)$ denotes the probability distribution function of the noise w_{k+1} . It is clear from (90) that

$$|\psi_k(n)| \leq 1, \quad |\gamma_k| \leq 1 \quad 1 \leq k \leq n. \quad (91)$$

By (9), we have $\mathbb{E}[s_{k+1} | \mathcal{F}_k] = F_{k+1}(c_k - \phi_k^\top \theta)$. Hence, it follows from (90) that $\mathbb{E}[\gamma_{k+1} | \mathcal{F}_k] = 0$, implying that the sequence $\{\gamma_k, \mathcal{F}_k\}$ forms a martingale difference sequence.

Next, we consider the following stochastic Lyapunov function: $V_k(n) = \tilde{\theta}_k^\top(n)P_k^{-1}(n)\tilde{\theta}_k(n)$, $n \geq 1, 1 \leq k \leq n$. By Assumption 4 and the definition of $h(\cdot)$ in (15), we have $\lim_{t \rightarrow 0} h(t) = \infty$. Thus, from the definition of d_k in (17)-(18), it follows that $\lim_{k \rightarrow \infty} d_k = \infty$. Thus, there exists a positive integer N_0 , such that for all $n \geq N_0$,

$$\theta \in D_{N_0}(n), \quad n \geq N_0. \quad (92)$$

Therefore, by (19), (92) and properties of the projection operator (see e.g., Lemma 1 in [14]), we have for each $k \geq N_0$ that

$$V_k(n) \leq \left[\tilde{\theta}_{k-1}(n) - a_{k-1}(n)\beta_{k-1}P_{k-1}(n)\phi_{k-1}(n)e_k(n) \right]^\top \cdot P_k^{-1}(n) \left[\tilde{\theta}_{k-1}^\top(n) - a_{k-1}(n)\beta_{k-1}P_{k-1}(n)\phi_{k-1}(n)e_k(n) \right]. \quad (93)$$

Furthermore, by (19) and the well-known matrix inversion formula, we have

$$P_k^{-1}(n) = P_{k-1}^{-1}(n) + \beta_{k-1}^2 \phi_{k-1}(n)\phi_{k-1}^\top(n), \quad 1 \leq k \leq n, \quad (94)$$

From this, it follows that $1 \leq k \leq n$,

$$\begin{aligned} \tilde{\theta}_{k-1}^\top(n)P_k^{-1}(n)\tilde{\theta}_{k-1}(n) &= \tilde{\theta}_{k-1}^\top(n)P_{k-1}^{-1}(n)\tilde{\theta}_{k-1}(n) \\ &\quad + \beta_{k-1}^2 \tilde{\theta}_{k-1}^\top(n)\phi_{k-1}(n)\phi_{k-1}^\top(n)\tilde{\theta}_{k-1}(n), \end{aligned} \quad (95)$$

and

$$\begin{aligned} &a_{k-1}(n)P_k^{-1}(n)P_{k-1}(n)\phi_{k-1}(n) \\ &= a_{k-1}(n) \left(I + \beta_{k-1}^2 \phi_{k-1}(n)\phi_{k-1}^\top(n)P_{k-1}(n) \right) \phi_{k-1}(n) \\ &= a_{k-1}(n)\phi_{k-1}(n) \left(1 + \beta_{k-1}^2 \phi_{k-1}^\top(n)P_{k-1}(n)\phi_{k-1}(n) \right) \\ &= \phi_{k-1}(n), \end{aligned} \quad (96)$$

where the last step follows from the definition of $a_{k-1}(n)$ in (19). Substituting (95) and (96) into (93), we obtain

$$\begin{aligned} V_k(n) &\leq V_{k-1}(n) + \beta_{k-1}^2 \tilde{\theta}_{k-1}^\top(n)\phi_{k-1}(n)\phi_{k-1}^\top(n)\tilde{\theta}_{k-1}(n) \\ &\quad - 2\beta_{k-1} \tilde{\theta}_{k-1}^\top(n)\phi_{k-1}(n)\psi_{k-1}(n) \\ &\quad + a_{k-1}(n)\beta_{k-1}^2 \phi_{k-1}^\top(n)P_{k-1}(n)\phi_{k-1}(n)\psi_{k-1}^2(n) \\ &\quad + 2a_{k-1}(n)\beta_{k-1}^2 \phi_{k-1}^\top(n)P_{k-1}(n)\phi_{k-1}(n)\gamma_k \\ &\quad - 2\beta_{k-1} \phi_{k-1}^\top(n)\tilde{\theta}_{k-1}(n)\gamma_k \\ &\quad + a_{k-1}(n)\beta_{k-1}^2 \phi_{k-1}^\top(n)P_{k-1}(n)\phi_{k-1}(n)\gamma_k^2. \end{aligned} \quad (97)$$

From the mean-value theorem and the definition of β_k in (19), we have

$$\begin{aligned} \psi_{k-1}(n) &= f_{k+1}(\xi_k) \left(\phi_k^\top \theta - \phi_k^\top(n)\theta_k(n) \right) \\ &\geq \beta_k \left(\phi_k^\top \theta - \phi_k^\top(n)\theta_k(n) \right), \end{aligned} \quad (98)$$

where ξ_k lies between $c_k - \phi_k^\top \theta$ and $c_k - \phi_k^\top(n)\theta_k(n)$, and satisfies $|\xi_k| \leq d_k \max_{1 \leq i \leq k} |u_i| + C$. It then follows that

$$\begin{aligned} &\beta_{k-1} \tilde{\theta}_{k-1}^\top(n)\phi_{k-1}(n)\psi_{k-1}(n) \\ &\geq \beta_{k-1}^2 \tilde{\theta}_{k-1}^\top(n)\phi_{k-1}(n) \left(\tilde{\theta}_{k-1}^\top(n)\phi_{k-1}(n) + \delta_{k-1}(n) \right) \\ &\geq \beta_{k-1}^2 \left(\tilde{\theta}_{k-1}^\top(n)\phi_{k-1}(n) + \delta_{k-1}(n) \right)^2 - \beta_{k-1}\delta_{k-1}(n)\psi_{k-1}(n). \end{aligned} \quad (99)$$

Since $|\psi_{k-1}(n)| \leq 1$, we also obtain

$$\begin{aligned} &a_{k-1}(n)\beta_{k-1}^2 \phi_{k-1}^\top(n)P_{k-1}(n)\phi_{k-1}(n)\psi_{k-1}^2(n) \\ &\leq a_{k-1}(n)\beta_{k-1}^2 \phi_{k-1}^\top(n)P_{k-1}(n)\phi_{k-1}(n). \end{aligned} \quad (100)$$

Substituting (99) and (100) into (97), and summing up both sides of (97) from $k = N_0$ to n , we obtain

$$\begin{aligned} V_n(n) &\leq V_{N_0}(n) + \sum_{k=N_0}^{n-1} \beta_k^2 \left[\tilde{\theta}_k^\top(n)\phi_k(n) \right]^2 \\ &\quad - 2 \sum_{k=N_0}^{n-1} \beta_k^2 \left(\tilde{\theta}_k^\top(n)\phi_k(n) + \delta_k(n) \right)^2 \\ &\quad + 2 \sum_{k=N_0}^{n-1} \beta_k \delta_k(n)\psi_k(n) + \sum_{k=N_0}^{n-1} a_k(n)\beta_k^2 \phi_k^\top(n)P_k(n)\phi_k(n) \\ &\quad - 2 \sum_{k=N_0}^{n-1} \beta_k \phi_k^\top(n)\tilde{\theta}_k(n)\gamma_{k+1} \\ &\quad + 2 \sum_{k=N_0}^{n-1} a_k(n)\beta_k^2 \psi_k(n)\phi_k^\top(n)P_k(n)\phi_k(n)\gamma_{k+1} \\ &\quad + \sum_{k=N_0}^{n-1} a_k(n)\beta_k^2 \phi_k^\top(n)P_k(n)\phi_k(n)\gamma_{k+1}^2. \end{aligned} \quad (101)$$

We next analyze the right-hand side (RHS) of (101) term by term. For the third term of (101), by the definition of $\delta_k(n)$ in (90), the decay property of $\{G_i, i \geq 1\}$ in (7), and the bound on $\{u_i, i \geq 1\}$ in (21), we obtain

$$\begin{aligned} \sum_{k=N_0}^{n-1} \delta_k^2(n) &= O \left(\sum_{k=N_0}^{n-1} \left[\phi_k^\top \theta - \phi_k^\top(n)\theta(n) \right]^2 \right) \\ &= O \left(\sum_{k=N_0}^{n-1} \left(\sum_{i=p_n+1}^{\infty} \|G_i\| \right)^2 \max_{p_n \leq i \leq k} u_{k-i}^2 \right) \\ &= O \left(n^{3+2\bar{\delta}} \rho^{-p_n} \right) = O(1), \quad a.s., n \rightarrow \infty. \end{aligned} \quad (102)$$

Applying Young's inequality, we obtain

$$\begin{aligned} &\sum_{k=N_0}^{n-1} \beta_k^2 \tilde{\theta}_k^\top(n)\phi_k(n)\delta_k(n) \\ &\leq \frac{1}{8} \sum_{k=N_0}^{n-1} \beta_k^2(n) \left[\tilde{\theta}_k^\top(n)\phi_k(n) \right]^2 + 4 \sum_{k=N_0}^{n-1} \beta_k^2(n)\delta_k^2(n), \quad a.s. \end{aligned} \quad (103)$$

Substituting (102) and (103) into the third term of (101), we have

$$\begin{aligned} &- 2 \sum_{k=N_0}^{n-1} \beta_k^2 \left(\tilde{\theta}_k^\top(n)\phi_k(n) + \delta_k(n) \right)^2 \\ &= - \frac{3}{2} \sum_{k=N_0}^{n-1} \beta_k^2 \left[\tilde{\theta}_k^\top(n)\phi_k(n) \right]^2 + O(1), \quad a.s. \end{aligned} \quad (104)$$

For the fourth term in (101), since $|\psi_k(n)| \leq 1$, it follows

from (102) and the definition of β_k in (19) that

$$\sum_{k=N_0}^{n-1} \beta_k \psi_k(n) \delta_k(n) = O \left(\sqrt{n \sum_{k=N_0}^{n-1} \delta_k^2(n)} \right) = O(1), \quad a.s. \quad (105)$$

For the fifth term of (101), we apply (94) and the trace inequality:

$$\begin{aligned} & \log^+ (\det (P_n^{-1}(n))) \\ & \leq p_n \log^+ \left(\text{tr} \left[P_0^{-1}(n) + \sum_{k=0}^{n-1} \beta_k^2 \phi_k(n) \phi_k^\top(n) \right] \right) \\ & = p_n \log^+ \left(\text{tr} [P_0^{-1}(n)] + \sum_{k=0}^{n-1} \beta_k^2 \sum_{i=0}^{p_n+1} u_{k-i}^2 \right) \\ & = O \left(p_n \log (n^{3+3\delta}) \right) = O(p_n \log n), \quad a.s. \end{aligned} \quad (106)$$

Hence, by Lemma 4 and (106), we obtain the bound

$$\begin{aligned} & \sum_{k=N_0}^{n-1} a_k(n) \beta_k^2 \phi_k^\top(n) P_k(n) \phi_k(n) \\ & = O(\log^+ \{\det (P_n^{-1}(n))\} + 1) = O(p_n \log n), \quad a.s. \end{aligned} \quad (107)$$

Furthermore, we analyze the last three terms in (101) which are related to the martingale difference sequence $\{\gamma_n, \mathcal{F}_n\}$, by using the double array martingale limit theory. Since $|\gamma_k| \leq 1$, we have $|\gamma_k| = o(\log \log(k+e))$, *a.s.* Then by Lemma 6, it follows that

$$\begin{aligned} & \sum_{k=N_0}^{n-1} \beta_k \phi_k^\top(n) \tilde{\theta}_k(n) \gamma_{k+1} \\ & = O \left(\left(\sum_{k=N_0}^{n-1} \beta_k^2 [\tilde{\theta}_k^\top(n) \phi_k(n)]^2 \right)^{\frac{1}{2}} (\log \log n)^2 \right), \\ & = o \left(\sum_{k=N_0}^{n-1} \beta_k^2 [\tilde{\theta}_k^\top(n) \phi_k(n)]^2 \right) + o((\log \log n)^5), \quad a.s. \end{aligned} \quad (108)$$

For the seventh term in (101), again by Lemma 6 and (107), we obtain

$$\begin{aligned} & \sum_{k=N_0}^{n-1} a_k(n) \beta_k^2 \psi_k(n) \phi_k^\top(n) P_k(n) \phi_k(n) \gamma_{k+1} \\ & = o \left(\sum_{k=N_0}^{n-1} [a_k(n) \beta_k^2(n) \phi_k^\top(n) P_k(n) \phi_k(n)]^2 \right) + o((\log \log n)^5) \\ & = o(p_n \log n) + o((\log \log n)^5), \quad a.s. \end{aligned} \quad (109)$$

For the eighth term of (101), since $|\gamma_{k+1}| \leq 1$, it follows from (107) that

$$\sum_{k=N_0}^{n-1} a_k(n) \beta_k^2 \phi_k^\top(n) P_k(n) \phi_k(n) \gamma_{k+1}^2 = O(p_n \log n), \quad a.s. \quad (110)$$

Substituting (104), (105), (107)-(110) into (101), we finally obtain

$$\tilde{\theta}_n^\top(n) P_n^{-1}(n) \tilde{\theta}_n(n) + \frac{1}{2} \sum_{k=N_0}^{n-1} \beta_k^2 [\tilde{\theta}_k^\top(n) \phi_k(n)]^2 = O(p_n \log n), \quad a.s. \quad (111)$$

Furthermore, by (21) and the definitions of β_n , $g(\cdot)$, $h(\cdot)$ in (19), (14) and (15), respectively, we have for each $n \geq 1$:

$$\beta_n \geq g \left(d_n \max_{1 \leq i \leq n} |u_i| \right) \geq g(d_n^2) = g(h(g(0)\alpha_n^{-1})) = \frac{g(0)}{\alpha_n}. \quad (112)$$

Then using (94) and (112), we the following lower bound:

$$\begin{aligned} \lambda_{\min} \{P_n^{-1}(n)\} & \geq \lambda_{\min} \left\{ P_0^{-1} + \sum_{t=1}^n \beta_t^2 \phi_t(n) \phi_t^\top(n) \right\} \\ & \geq \left(\frac{g(0)}{\alpha_n} \right)^2 \lambda_{\min} \left\{ P_0^{-1} + \sum_{t=1}^n \phi_t(n) \phi_t^\top(n) \right\} = \left(\frac{g(0)}{\alpha_n} \right)^2 \lambda_{\min}(n), \end{aligned} \quad (113)$$

where $\lambda_{\min}(n)$ is defined in (24). Therefore, we have

$$\begin{aligned} \tilde{\theta}_n^\top(n) P_n^{-1}(n) \tilde{\theta}_n(n) & \geq \lambda_{\min} \{P_n^{-1}(n)\} \|\tilde{\theta}_n(n)\|^2 \\ & \geq g(0)\alpha_n^{-2} \lambda_{\min}(n) \|\tilde{\theta}_n(n)\|^2. \end{aligned} \quad (114)$$

Combining (111) and (114), we arrive at the result (23).

To prove (22), we draw inspiration from the analysis of the $ARX(\infty)$ approximation in [39]. Let $s_i = \lfloor e^{\frac{i}{1+a}} \rfloor$ for each $i \geq 0$. Then for all k such that $\bar{s}_i + 1 \leq k \leq \bar{s}_{i+1}$, we have

$$p_k = p_{\bar{s}_{i+1}}. \quad (115)$$

Since $\{\bar{s}_i, i \geq 0\}$ is an increasing sequence, for each $n \geq 1$, there exists a positive integer $\bar{j}_n$ such that $\bar{s}_{\bar{j}_n} \leq n < \bar{s}_{\bar{j}_n+1}$. Moreover, it can be verified that $\bar{j}_n \leq p_{n+1}$. Then, combining (115) and (111), we obtain

$$\begin{aligned} & \sum_{k=N_0}^{n-1} \beta_k^2 [\tilde{\theta}_k^\top(k) \phi_k(k)]^2 \leq \sum_{i=0}^{\bar{j}_n} \sum_{k=\bar{s}_i+1}^{\bar{s}_{i+1}} \beta_k^2 [\tilde{\theta}_k^\top(k) \phi_k(k)]^2 \\ & = O \left(\sum_{i=0}^{\bar{j}_n} p_{s_{i+1}} \log p_{s_{i+1}} \right) = O(p_n^2 \log n), \quad a.s. \end{aligned} \quad (116)$$

Therefore, by (112) and (116), it follows that

$$\sum_{k=N_0}^{n-1} [\tilde{\theta}_k^\top(k) \phi_k(k)]^2 = O(\alpha_n^2 p_n^2 \log n), \quad a.s. \quad (117)$$

Furthermore, using (102), we have

$$\sum_{k=0}^n \delta_k^2(k) = \sum_{k=0}^n \lambda^{2p_k} k^{2+2\delta} = O(1). \quad (118)$$

Combining this with (117), we finally obtain

$$\sum_{k=0}^n R_k = \sum_{k=0}^n [\tilde{\theta}_k^\top(k) \phi_k(k) + \delta_k(k)]^2 = O(\alpha_n^2 p_n^2 \log n), \quad a.s. \quad (119)$$

Proof of Lemma 1: For each $n \geq 1$, $\frac{1}{2} < \eta < 1$, and any vector $Z_n = [z_1, \dots, z_{p_n}]^\top$ with $\|Z_n\| = 1$, we have

$$\begin{aligned} Z_n^\top \sum_{t=n-n^\eta}^n \psi_t(n) \psi_t^\top(n) Z_n &= \sum_{t=n-n^\eta}^n (z_1 v_t \cdots z_{p_n} v_{t-p_n+1})^2 \\ &= \sum_{t=n-n^\eta}^n (z_1^2 v_t^2 + \cdots + z_{p_n}^2 v_{t-p_n+1}^2) \\ &\quad + 2 \sum_{t=n-n^\eta}^n \sum_{0 \leq s < r \leq p_n} z_{s+1} z_{r+1} v_{t-s} v_{t-r}. \end{aligned} \quad (120)$$

From Lemma 5, we have

$$\begin{aligned} &\max_{1 \leq s \leq p_n} \max_{r < s \leq p_n} \left\| \sum_{t=n-n^\eta}^n v_{t-s} v_{t-r} \right\| \\ &= \max_{1 \leq s \leq p_n} \max_{r < s \leq p_n} \left\| \sum_{t=1}^n v_{t-s} v_{t-r} - \sum_{t=1}^{n-n^\eta} v_{t-s} v_{t-r} \right\| \\ &\leq \max_{1 \leq s \leq p_n} \max_{r < s \leq p_n} \left(\left\| \sum_{t=1}^n v_{t-s} v_{t-r} \right\| + \left\| \sum_{t=1}^{n-n^\eta} v_{t-s} v_{t-r} \right\| \right) \\ &\leq C_1(\eta) \sqrt{n \log \log n}, \end{aligned} \quad (121)$$

where $C_1(\eta)$ does not depend on n . Moreover, we have

$$\sum_{t=n-n^\delta}^n (z_1^2 \mathbb{E}[v_t^2] + \cdots + z_{p_n}^2 \mathbb{E}[v_{t-p_n+1}^2]) \geq n^\eta, \quad (122)$$

and

$$\max_{0 \leq s \leq p_n-1} z_s^2 \left\| \sum_{t=n-n^\eta}^n (v_{t-s}^2 - \mathbb{E}[v_{t-s}^2]) \right\| \leq C_2(\eta) \sqrt{n \log \log n}, \quad (123)$$

where $C_2(\eta)$ does not depend on n . From (120), (121), (122) and (123), we obtain

$$\begin{aligned} &\frac{1}{n^\eta} \lambda_{\min} \left\{ \sum_{t=n-n^\eta}^n \psi_t(n) \psi_t^\top(n) \right\} \\ &\geq 1 - \frac{(C_1(\eta) + C_2(\eta)) p_n \sqrt{n \log \log n}}{n^\eta}, \end{aligned} \quad (124)$$

which establishes Lemma 1.

Proof of Lemma 2: For each positive integer k , define

$$A_k = \begin{bmatrix} -q_1 & \cdots & -q_k & 0 \\ 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}. \quad (125)$$

From condition (36), there exists a positive integer k_0 such that $\rho(A_{k_0}) < 1$ and

$$\sum_{i=k_0+1}^{\infty} |q_i| < \frac{1}{4} - \frac{1}{4} \rho(A_{k_0}). \quad (126)$$

Let $T_1 = \frac{3}{2} \|A_{k_0}\|$, $T_2 = \frac{1}{4} + \frac{3}{4} \rho(A_{k_0})$, and denote $\bar{C} = C(k_0, T_1, T_2)$, where $C(\cdot)$ is defined in Lemma 3. Define

$$\Delta = \min \left\{ \frac{1}{4} \|A_{k_0}\|^2, \left(\frac{1}{12} - \frac{1}{12} \rho(A_{k_0}) \right)^2, \frac{\bar{C}^2}{4} \right\}. \quad (127)$$

Given any $\bar{\xi} > 0$, we now show that the uniform stability bound property holds with constants $c_1(\bar{\xi})$ and $c_2(\bar{\xi})$ depending only on $Q(z)$ and $\bar{\xi}$. Specifically, for any positive integers $n_0 < N$, and for any sequences $\{\mu_n, n \geq 1\} \subset \mathbb{R}$ and $\{\xi_n, n \geq 1\} \subset \mathbb{R}$ satisfying conditions (i) and (ii) in Lemma 2, we aim to verify that inequality (37) holds.

We now define a sequence of matrices $\{A_{n,k_0}, n \geq 1\}$ as follows:

$$A_{n,k_0} = \begin{bmatrix} -q_{n,1} & \cdots & -q_{n,k} & 0 \\ 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}, \quad (128)$$

By condition (i) in Lemma 2, we have

$$\sup_{n_0 \leq n \leq N} \|A_{k_0} - A_{n,k_0}\| \leq \sup_{n_0 \leq n \leq N} \sqrt{\sum_{i=1}^{k_0} |q_i - q_{n,i}|^2} \leq \sqrt{\Delta}, \quad (129)$$

Thus, by the definition of Δ in (127) and (129), it follows that

$$\begin{aligned} \sup_{n_0 \leq n \leq N} \|A_{n,k_0}\| &\leq \|A_{k_0}\| + \Delta \leq \frac{3}{2} \|A_{k_0}\|, \\ \sup_{n_0 \leq n \leq N} \rho(A_{n,k_0}) &\leq \rho(A_{k_0}) + \Delta \leq \frac{1}{4} + \frac{3}{4} \rho(A_{k_0}), \end{aligned} \quad (130)$$

and

$$\sup_{n_0 \leq n \leq N} \sum_{i=k_0+1}^{p_n} |q_{n,i}| \leq \sum_{i=k_0+1}^{\infty} |q_i| + \Delta \leq \frac{1}{3} - \frac{1}{3} \rho(A_{k_0}). \quad (131)$$

Therefore, by (129)-(130), the choice of Δ in (127), and Lemma 3, there exists constants $M > 0$ and $0 < \lambda < 1$, depending only on T_1, T_2 and k_0 , such that

$$\|\Pi_{i=n}^{n+j} A_{i,k_0}\| \leq M \lambda^j, \quad \forall n_0 \leq n \leq n+j \leq N. \quad (132)$$

For each $n \geq 1$, define

$$\bar{\mu}_n = \begin{cases} \mu_n, & n \geq n_0 - p_{n_0} + 1 \\ 0, & n < n_0 - p_{n_0} + 1 \end{cases},$$

and let $x_n = [\bar{\mu}_n, \bar{\mu}_{n-1}, \dots, \bar{\mu}_{n-k+1}]^\top$, $\epsilon_n = [\xi_n, 0, \dots, 0]^\top$. In addition, for each $n \geq 1$ and $i \geq 1$, define

$$\bar{g}_{n,i} = \begin{cases} q_{n,i}, & 1 \leq i \leq p_n \\ 0, & i > p_n \end{cases},$$

and

$$\bar{h}_{n,i} = \begin{bmatrix} -\bar{g}_{n,n-i} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Then by condition (ii) in Lemma 2, we have

$$x_n = A_{n,k} x_{n-1} + \sum_{i=1}^{n-k-1} h_{n,i} x_i + \epsilon_n, \quad n_0 \leq n \leq N. \quad (133)$$

Define the transition matrix

$$\Phi(n+1, i) = A_{n+1,k} \Phi(n, i), \quad \Phi(i, i) = I, \quad \forall N \geq n \geq i \geq n_0. \quad (134)$$

We now prove that for each $N \geq n \geq n_0 + 1$,

$$x_n = \Phi(n, n_0)x_{n_0} + \sum_{i=n_0+1}^n \Phi(n, i)\epsilon_i + \sum_{i=1}^{n-k-1} l_{n,i}x_i. \quad (135)$$

where the coefficients $\{l_{n,i}, 1 \leq i \leq n\}$ are defined as

$$l_{n,i} = \begin{cases} \sum_{j=k+i+1}^n \Phi(n, j)h_{j,i}, & n-k \leq i \leq n-k-1 \\ \sum_{j=n_0+1}^n \Phi(n, j)h_{j,i}, & 1 \leq i \leq n-k-1 \end{cases}. \quad (136)$$

We proceed by induction. For $n = n_0 + 1$, it is easy to verify that

$$l_{n_0+1,i} = h_{n_0+1,i}, \quad 1 \leq i \leq n-k-1, \quad (137)$$

so (135) holds. Assume that (135) holds for some $n \geq n_0 + 1$, we show that it also holds for $n+1$. By definition, we have

$$\begin{aligned} x_{n+1} &= A_{n+1,k}x_n + \sum_{i=1}^{n-k} h_{n+1,i}x_i + \epsilon_{n+1} \\ &= A_{n+1,k}\Phi(n, n_0)x_{n_0} + A_{n+1,k} \sum_{i=n_0+1}^n \Phi(n, i)\epsilon_i \\ &\quad + A_{n+1,k} \sum_{i=1}^{n-k-1} l_{n,i}x_i + \sum_{i=1}^{n-k} h_{n+1,i}x_i + \epsilon_{n+1} \\ &= \Phi(n+1, n_0)x_{n_0} + \sum_{i=n_0+1}^{n+1} \Phi(n+1, i)\epsilon_i + h_{n+1,n-k}x_{n-k} \\ &\quad + \sum_{i=1}^{n-k-1} (A_{n+1,k}l_{n,i} + h_{n+1,i})x_i. \end{aligned} \quad (138)$$

By definition of $l_{n,i}$ in (136), we have $l_{n,n-k-1} = h_{n,n-k-1}$. Besides, for each $n_0 - k \leq i \leq n-k$, we compute:

$$\begin{aligned} A_{n,k}l_{n-1,i} + h_{n,i} &= A_{n,k} \sum_{j=k+i+1}^{n-1} \Phi(n-1, j)h_{j,i} + h_{n,i} \\ &= \sum_{j=k+i+1}^{n-1} \Phi(n, j)h_{j,i} + h_{n,i} = l_{n,i}. \end{aligned} \quad (139)$$

Similarly, for $1 \leq i \leq n_0 - k - 1$, we have

$$A_{n,k}l_{n-1,i} + h_{n,i} = \sum_{j=n_0+1}^n \Phi(n, j)h_{j,i} = l_{n,i}. \quad (140)$$

Therefore, the representation (135) also holds for $n+1$, completing the induction.

According to (132) and (135), for each $N \geq n \geq n_0 + 1$, we have

$$\|x_n\| \leq M\lambda^{n-n_0}\|x_{n_0}\| + \sum_{i=n_0+1}^n \lambda^{n-i}\|\epsilon_i\| + \sum_{i=1}^{n-k-1} \|l_{n,i}\| \|x_i\|. \quad (141)$$

From the definition of $l_{n,i}$ and (131), we obtain

$$\begin{aligned} &\sum_{i=1}^{n-k-1} \|l_{n,i}\| \\ &\leq \sum_{i=1}^{n_0-k-1} \sum_{j=n_0+1}^n \lambda^{n-j} \|h_{j,i}\| + \sum_{i=n_0-k}^{n-k-1} \sum_{j=k+i+1}^n \lambda^{n-j} \|h_{j,i}\| \\ &\leq \sum_{i=1}^{n_0-k-1} \sum_{j=n_0+1}^n \lambda^{n-j} |g_{j,j-i}| + \sum_{i=n_0-k}^{n-k-1} \sum_{j=k+i+1}^n \lambda^{n-j} |g_{j,j-i}| \\ &\leq \sum_{j=n_0+1}^n \sum_{i=1}^{j-k-1} \lambda^{n-j} |g_{j,j-i}| \leq \sum_{j=n_0+1}^n \lambda^{n-j} \left(\sum_{i=k+1}^{\infty} |q_{j,i}| \right) \\ &\leq \frac{1}{1-\lambda} \max_{n_0+1 \leq j \leq n} \left(\sum_{i=k+1}^{\infty} |q_{j,i}| \right) \leq \frac{2}{3}. \end{aligned} \quad (142)$$

Define $y_n = \max_{1 \leq i \leq n} \|x_i\|$. Then, combining (141) and (142), we obtain for each $N \geq n \geq n_0 + 1$,

$$\begin{aligned} y_n &\leq \frac{2}{3}y_{n-1} + M\lambda^{n-n_0}\|x_{n_0}\| + \sum_{i=n_0+1}^n \lambda^{n-i}\|\epsilon_i\| \\ &\leq \left(\frac{2}{3}\right)^{n-n_0} y_{n_0} + \frac{M}{1-\lambda}\|x_{n_0}\| + c_2(\bar{\xi}) \\ &\leq c_1(\bar{\xi}) \max_{n_0-p_{n_0} \leq i \leq n_0} |\mu_i| + c_2(\bar{\xi}). \end{aligned} \quad (143)$$

where $c_1(\bar{\xi})$, $c_2(\bar{\xi})$ depend only on M, λ and $\bar{\xi}$, and thus ultimately only on T_1, T_2, k_0 and $\bar{\xi}$. Therefore, Lemma 2 is established.

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