Control of Nonlinear Uncertain Systems by Extended PID

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Abstract—Since the classical proportional-integral-derivative (PID) controller is the most widely and successfully used ones in industrial processes, it is of vital importance to investigate theoretically the rationale of this ubiquitous controller in dealing with nonlinearity and uncertainty. Recently, we have investigated the capability of the classical PID control for second-order nonlinear uncertain systems, and provided some analytic design methods for the choices of PID parameters, where the system is assumed to be in the form of cascade integrators. In this article, we will consider the natural extension of the classical PID control for high-order affine-nonlinear uncertain systems. In contrast to most of the literature on controller design of nonlinear systems, we do not require such special system structures as normal or triangular forms, thanks to the strong robustness of the extend PID controller. To be specific, we will show that under some suitable conditions on nonlinearity, and uncertainty of the systems, the extended PID controller can semiglobally stabilize the nonlinear uncertain systems, and at the same time the regulation error converges to zero exponentially fast, as long as the control parameters are chosen from an open unbounded parameter manifold constructed in this article.

Index Terms—Affine nonlinear systems, diffeomorphism, normal form, proportional-integral-derivative (PID) control, regulation, system uncertainty.

I. Introduction

Over the past 60 years, remarkable progresses in modern control theory have been made, e.g., numerous advanced control techniques including optimal control, robust control, adaptive control, nonlinear control, intelligent control, etc., have been introduced and investigated. However, the classical PID (proportional-integral-derivative) controller (or its variations), which has nearly 100 years of history, is still the most widely and successfully used one in engineering systems by far (see, e.g., [2] and [26]), which exhibits its lasting vitality.

In fact, a recent survey [26] shows that the PID controller has much higher impact rating than the advanced control technologies and that we still have nothing that compares with PID. However, it has also been reported that most of the practical PID loops are poorly tuned, and there is strong evidence that PID controllers remain poorly understood [25]. Therefore, as pointed out in [1], better understanding of the PID control may considerably improve its widespread practice, and so contribute to better product quality. This is the primary motivation of our theoretical investigation of the PID controller.

As is well-known, the PID controller has been investigated extensively in the literature by numerous control scientists and engineers.

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Except for a few related studies (e.g., [4], [17], and [20]), most existing works focus on linear systems (e.g., [1], [2], [9], and [27]), albeit almost all practical systems are nonlinear with uncertainties. Therefore, to justify the remarkable effectiveness of the PID controllers for real world systems, one has to face with nonlinear uncertain dynamical systems and to understand the rationale and capability of this controller.

Recently, we have given a theoretical investigation for the convergence and design of PID controller for a basic class of nonlinear uncertain systems (see [21], [31], and [32]). For example, in [31], we have shown that for second-order nonlinear uncertain dynamical systems, one can select the three PID parameters to globally stabilize the closed-loop systems and at the same time to make the output of the controlled system converge to any given setpoint, provided that the nonlinear uncertain functions satisfy a Lipschitz condition. Moreover, necessary and sufficient conditions for the selection of the PID parameters have also been discussed and provided in [32]. These results have demonstrated theoretically that the classical PID controller does indeed have large-scale robustness with respect to both the uncertain system structure and the selection of the controller parameters. However, in the work of [31] and [32], we have only considered second-order uncertain nonlinear systems where there is no uncertainty in the control channel.

Actually, in the area of nonlinear control, extensive researches have been conducted on the controller design (e.g., [10], [11], [13], [15], [16], [18], and [22]). For examples, the active disturbance rejection control method (e.g., [12] and [29]), the backstepping approach for pure feedback forms in [22], the extremum seeking methods for nonlinear uncertain systems (see, e.g., [20] and [28]), and many other interesting design methods for certain triangular forms (see, e.g., [10], [12], [14], [15], and [29]), as well as for feedforward nonlinear systems (see, e.g., [24]). We remark that for general affine nonlinear systems, the feedback linearization method may be used, but that needs the full knowledge of the nonlinear functions and usually gives local results (e.g., [6] and [11]). To get global or semiglobal results, global normal forms are usually used directly or assumed to be transformed into it for the investigation of nonlinear systems (see, e.g., [4], [7], [17], and [23]).

In this article, we will consider a general class of single-input and single-output (SISO) affine nonlinear uncertain systems, and investigate the natural extension of the classical PID controller. We will show that for a large class of n-dimensional SISO affine nonlinear uncertain systems, an $(n + 1)$-dimensional parameter manifold can be constructed explicitly, from which the extended PID controller parameters can be arbitrarily chosen to semiglobally stabilize the nonlinear uncertain systems with the regulation error converging to 0 exponentially, even if the system may not be transformed into a global normal form, thanks to the strong robustness of the extended PID controller as will be demonstrated in this article.

The rest of this article is organized as follows. In Section II, we will introduce the problem formulation. The main results are presented in Section III. Section IV contains the proofs of the main theorems. Section V will conclude this article with some remarks.

II. PROBLEM FORMULATION

A. Notations

We first introduce some notations and definitions to be used throughout this article.

Let x be a vector in the n-dimensional Euclidean space $\mathbb{R}^n$, P be an $m \times n$ matrix, and x^T, P^T denotes the transpose of x and P, respectively. Also, let $\|x\|$ denote the Euclidean norm of x, and $\|P\|$ denote the operator norm of the matrix P induced by the Euclidean norm, i.e., $\|P\| = \sup_{x \in \mathbb{R}^n, \|x\|=1} \|Px\|$, which is known to be the largest singular value of P.

Denote $\mathbb{R}_+ = [0, \infty)$.

Let $z(t)$ be a function of time t, then $\dot{z}(t)$ denotes the time derivative of $z(t)$. For simplicity, we oftentimes omit the variable t whenever there is no ambiguity in the sequel.

A map $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is called a global diffeomorphism on $\mathbb{R}^n$ if it is both injective and surjective, and both Φ and its inverse mapping (also denoted by Φ^{-1} for simplicity) are continuously differentiable.

Consider the following SISO affine nonlinear system

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \tag{1}$$

where $f, g : \mathbb{R}^n \to \mathbb{R}^n, h : \mathbb{R}^n \to \mathbb{R}$ are sufficiently smooth unknown nonlinear mappings.

The mappings f, g are called smooth vector fields on $\mathbb{R}^n$. Let the coordinates of x be x_i and the components of f and g be f_i and g_i, respectively, $i = 1, \ldots, n$. Define $L_f h(x) \triangleq \sum_{i=1}^{n} \frac{\partial h}{\partial x_i}(x) f_i(x)$, which is called the Lie derivative of h along with the vector field f. Let us further denote $L_g L_f h(x) \triangleq \sum_{i=1}^{n} \frac{\partial L_f h}{\partial x_i} g_i(x)$, $L_f^k h(x) \triangleq L_f L_f^{k-1} h(x)$, $k \geq 1$, with $L_f^0 h(x) \triangleq h(x)$.

Denote $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ and denote $\overline{\mathbb{R}}^n = \overline{\mathbb{R}} \times \cdots \times \overline{\mathbb{R}}$ is the $n-$ary Cartesian power of $\overline{\mathbb{R}}$. We call x^* is a point at infinity if $x^* \in \overline{\mathbb{R}}^n \backslash \mathbb{R}^n$, i.e., at least one component of x^* is ∞ or $-\infty$.

Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ and x^* be a point at infinity. If $\lim_{x \in \mathbb{R}^n, x \to x^*} \Phi(x)$ exists and finite, then we denote $\Phi(x^*) \triangleq \lim_{x \in \mathbb{R}^n, x \to x^*} \Phi(x)$. For $z \in \mathbb{R}^n$, we denote $\Phi^{-1}(z) \triangleq \{x \in \overline{\mathbb{R}}^n : \Phi(x) = z\}$.

B. Control

Let $y^* \in \mathbb{R}$ be a given setpoint. Our control objective is to design a robust feedback controller $u(t)$ to achieve asymptotic regulation $\lim_{t \to \infty} y(t) = y^*$.

The challenges of this problem attribute at least to the following two facts: First, the feedback linearization method cannot be used because the nonlinear functions are unknown. Second, the traditional design methods such as backstepping cannot be applied because the system is not in a global normal or triangular form.

This article is motivated by our recent theoretical investigation on the classical PID control [32], where it was shown that for a basic class of second-order nonlinear uncertain systems, the classical PID control can globally stabilized the system as long as the upper bounds of the partial derivatives of the system nonlinear function are known as a prior. For dynamical systems with relative degree ≥ 3, the classical PID control cannot achieve global stabilization in general, even for linear time invariant systems [31]. These facts inspires us to consider the following natural extension of PID (called extended PID controller),

defined by

$$u(t) = k_1 e(t) + k_0 \int_0^t e(s)ds + k_2 \dot{e}(t) + \cdots + k_n e^{(n-1)}(t) \tag{2}$$

where $e(t) = y^* - y(t)$ is the regulation error, $\dot{e}(t), \cdots, e^{(n-1)}(t)$ are the time derivatives of $e(t)$ up to the $(n-1)$th order, which are assumed to be available for simplicity.

From the definition (2), we know that the extended PID controller is an output feedback of simple structure and its design does not need the precise model of the plant (1). The control variable $u(t)$ is simply a weighted linear combination of the proportional, integral, and derivative terms of the system regulation error, where the weighting parameters $(k_0, k_1, \ldots, k_n)$ are called extended PID parameters.

We remark that if the nonlinear system is in the normal form of cascade integrators or can be transformed into this form globally, then it may be natural to consider the extended PID control as defined above, and [17] appears to be a pioneer work in this direction. However, for the general nonlinear system (1) to be considered in this article, where the coordinate transformation may not be a global diffeomorphism, a key problem here is: Does the extended PID controller still regulate the nonlinear systems (1) globally or semiglobally? On the other hand, only qualitative design methods for the parameters are given in [17] and the tuning rules are of high gain. Therefore, another key problem is can we provide a concrete design method for the $(n+1)$-extended PID parameters?

We will address this problem in this article by investigating the capability together with the design of the extended PID controller (2) for the uncertain nonlinear system (1). We will rigorously show that the extended PID controller can indeed achieve our control objective, even if the systems may not be transformed into the normal form globally.

C. Assumptions

First, we introduce some notations. Define

$$\Phi(x) \triangleq (h(x), L_f h(x), \ldots, L_f^{n-1} h(x))^T. \tag{3}$$

Let $y^* \in \mathbb{R}$ be the setpoint. Denote

$$z^* \triangleq (y^*, 0, \ldots, 0)^T \in \mathbb{R}^n \tag{4}$$

and define $H : \mathbb{R}^n \to \mathbb{R}^2$ as follows:

$$H(x) \triangleq (F(x), G(x)) \tag{5}$$

where

$$F(x) \triangleq L_f^n h(x), G(x) \triangleq L_g L_f^{n-1} h(x). \tag{6}$$

Assumption A: System (1) has uniform relative degree n, i.e., $L_g L_f^i h(x) = 0, i = 0, \ldots, n-2; G(x) \neq 0, \forall x \in \mathbb{R}^n$. Furthermore, the sign of $G(\cdot)$ is known and $G(x)$ is uniformly bounded away from zero. Without loss of generality, we assume that $G(x) \geq \underline{b} > 0$ for any $x \in \mathbb{R}^n$.

Remark 1: By Assumption A, we know that $J_\Phi(x)$ is invertible for any $x \in \mathbb{R}^n$, where $J_\Phi(x)$ denotes the Jacobian matrix of Φ, (see, e.g., [11]). Under the new coordinates $z = \Phi(x)$, the system (1) transforms into the normal form of cascade integrators

$$\begin{cases} \dot{z}_i = z_{i+1}, i = 1, \ldots, n-1 \\ \dot{z}_n = a(z) + b(z)u \end{cases} \tag{7}$$

locally. However, the system (1) may not be globally transformed in to the normal form (7), unless the n vector fields $(-1)^{i-1} ad_{\tilde{f}}^{i-1} \tilde{g}(x), i =$

$1, \ldots, n$ are complete, where $\tilde{f}(x) = f(x) - \frac{F(x)g(x)}{G(x)}$ and $\tilde{g}(x) = \frac{g(x)}{G(x)}$ (see [11]).

To provide a concrete design method for the extended PID parameters, we need some additional knowledge of the uncertain functions. It turns out that the growth rate of some certain unknown functions need be known as *a prior* in designing the extended PID parameters. Let τ_1, τ_2 be two known increasing functions from $\mathbb{R}^+$ to $\mathbb{R}^+$ with $\limsup_{r \to 0} \frac{\tau_2(r)}{r} < \infty$, we introduce the following Assumption B to measure the size of uncertainty.

Assumption B: The functions Φ and H defined, respectively, by (3) and (5)–(6) satisfy

$$\|\Phi(x)\| \leq \tau_1(\|x\|), \|H(x)\| \leq \tau_1(\|\Phi(x)\|), \forall x \in \mathbb{R}^n$$

and there exists $x^* \in \Phi^{-1}(z^*)$ such that the "gap" of H at x^* is bounded by that of Φ in the sense that

$$\|H(x) - H(x^*)\| \leq \tau_2(\|\Phi(x) - \Phi(x^*)\|), \forall x \in \mathbb{R}^n.$$

Remark 2: We remark that Assumption B is not restrictive. In fact, under Assumption A, and suppose the coordinate transformation Φ is a global diffeomorphism on $\mathbb{R}^n$, then for any setpoint y^*, there always exists some increasing functions τ_1, τ_2 from $\mathbb{R}^+$ to $\mathbb{R}^+$ with $\limsup_{r \to 0} \frac{\tau_2(r)}{r} < \infty$, such that Assumption B is satisfied (see Assumption B0 in [34]).

Remark 3: It will be shown that the constant $\underline{b}$ and the upper bound functions τ_1, τ_2 play a critical role in designing the extended PID parameters. Thus, it is an important task to find out the constant $\underline{b}$ and the two functions τ_1, τ_2 in practice. We remark that for uncertain system in the normal form

$$\begin{cases} \dot{x}_i = x_{i+1}, & i = 1, \ldots, n-1 \\ \dot{x}_n = a(x) + b(x)u, & y = x_1 \end{cases} \tag{8}$$

where $a(x), b(x)$ are both unknown functions. Suppose that

$$b(x) \geq b, |a(x)| + |b(x)| + \|J_a(x)\| + \|J_b(x)\| \leq \rho(\|x\|) \tag{9}$$

for all $x \in \mathbb{R}^n$, where $J_a(x)$ denotes the gradient of $a(x)$, and where $b > 0$ is a known constant and $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is a known increasing function, then for any setpoint $y^* \in \mathbb{R}$, Assumptions A and B are satisfied with $\underline{b} = b$, $\tau_1(r) = \max\{r, \rho(r)\}$ and $\tau_2(r) = \sqrt{2}\rho(r + |y^*|)r$. The proof is given in Appendix B.

Definition 3.1: Denote $\mathbb{S}'$ as the family of nonlinear systems that satisfy both Assumptions A and B, i.e.,

$$\mathbb{S}'(\underline{b}, \tau_1, \tau_2) = \{(f, g, h) : \text{Assumptions A and B hold}\}.$$

A key question is, does the extended PID controller (2) regulate all the nonlinear systems in $\mathbb{S}'(\underline{b}, \tau_1, \tau_2)$ semiglobally? Specifically, for given $R > 0$, does it exist $(k_0, \ldots, k_n) \in \mathbb{R}^{n+1}$, such that the regulation error $e(t)$ converges to 0, for all $(f, g, h) \in \mathbb{S}'(\underline{b}, \tau_1, \tau_2)$ and for all initial states $\|x(0)\| \leq R$?

We point out that, if the uncertain nonlinear system is of the global normal form (8), then the above question is positive (Corollary 1 in Section III). However, for the general affine-nonlinear uncertain system (1), the answer is no! In fact, there exists a triple $(\underline{b}, \tau_1, \tau_2)$, such that any given extended PID controller parameters and for any $R > 0$, there always exist some $(f, g, h) \in \mathbb{S}'(\underline{b}, \tau_1, \tau_2)$ and initial states $\|x(0)\| \leq R$, the solution of the closed-loop system will have finite escape time [34]. Therefore, we need to introduce certain additional assumptions, and it turns out that the following assumption will be sufficient.

Assumption C: There exists two constants $N_1 > 0, N_2 > 0$, such that the inverse of the Jacobian matrix $J_\Phi(x)$ satisfies

$$\left\|J_\Phi^{-1}(x)\right\| \leq N_1 \|x\| \log_+ \|x\| + N_2, \forall x \in \mathbb{R}^n \tag{10}$$

where $\log_+(r) \triangleq \max\{0, \log r\}$ for $r \geq 0$.

Remark 4: First, we point out that the super-linear growth rate of $\|J_\Phi^{-1}(x)\|$ in Assumption C cannot be relaxed essentially in general, (see [34]). Next, we remark that for nonlinear systems already in the normal form (8), Assumption C is automatically satisfied since the coordinate transformation map Φ defined by (3) is identity, i.e., $\Phi(x) = x, \forall x \in \mathbb{R}^n$.

Remark 5: From the proof of Theorem 1 to be given in Section IV, one can see that Assumption C is used only to ensure that the solution of the closed-loop control system exists in $[0, \infty)$ under the extended PID control. Hence, if we can show that the solution of the closed-loop system exists in $[0, \infty)$ by using Assumptions A and B, then Assumption C can be removed from Theorem 1. We already know that such an existence is true when the system (1) can be transformed into the normal form globally, (see [34]). In the case where the system (1) cannot be transformed into the normal form globally, we now give an example to show that Assumptions A and B are satisfied and the solution of the closed-loop system still exists in $[0, \infty)$.

Example 1: Consider the following nonlinear system:

$$\begin{cases} \dot{x}_1 = -\sin x_2 \cos x_2 - \sin x_2 e^{-x_1} u \\ \dot{x}_2 = \sin^2 x_2 - \cos x_2 e^{-x_1} u \\ y = \theta e^{x_1} \cos x_2 \end{cases} \tag{11}$$

where $0 < |\theta| \leq 1$ is an unknown constant. It can be shown that this control system cannot be transformed into the normal form globally, (see [3]). Let $y^* = 0$ be the setpoint, then Assumptions A and B are satisfied with $\underline{b} = 1$, $\tau_1(r) = e^r$, and $\tau_2(r) \equiv 0$. Moreover, it can be shown that the solution of (11) will exist in $[0, \infty)$ for all initial states under PID control

$$u(t) = k_p e(t) + k_i \int_0^t e(s)ds + k_d \dot{e}(t)$$

as long as the cubic equation $s^3 + k_d s^2 + k_p s + k_i = 0$ has three distinct negative roots (see Appendix B).

III. MAIN RESULTS

Definition 3.2: Denote $\mathbb{S}$ as the following class of uncertain systems described by the triple (f, g, h):

$$\mathbb{S}(\underline{b}, \tau_1, \tau_2) = \{(f, g, h) : \text{Assumptions A, B, and C hold}\}.$$

Now, we are in position to present the main results.

Theorem 1: Consider the nonlinear uncertain system (1) with the extended PID controller defined by (2). Then for any given $(\underline{b}, \tau_1, \tau_2)$ and any given $R > 0$, an $(n+1)$-dimensional parameter manifold Ω can be constructed, such that whenever $(k_0, \ldots, k_n) \in \Omega$, the solution of the closed-loop control system will exist in $[0, \infty)$ and the regulation error $e(t)$ will converge to zero exponentially, for any $(f, g, h) \in \mathbb{S}(\underline{b}, \tau_1, \tau_2)$ and any initial state $\|x(0)\| \leq R$.

Remark 6: We emphasize that Ω can be constructed based on the upper bound functions $\tau_1(\cdot), \tau_2(\cdot)$, the constant $\underline{b}$ and R only, and its concrete construction can be found in the proof of Theorem 1. It neither depends on the precise information of the nonlinear functions (f, g, h), nor depends on the initial states. Hence, Theorem 1 demonstrates that the extended PID has large scale robustness with respect to both the system structural uncertainties and the selection of controller parameters.

The following corollary comes immediately from Theorem 1.

Corollary 1: Consider the uncertain nonlinear system (8) with the extended PID controller (2). Suppose (9) is satisfied. Then for any setpoint $y^* \in \mathbb{R}$ and any given $R > 0$, an $(n+1)$-dimensional parameter

manifold Ω (only depend on R, y^*, b and $\rho(\cdot)$) can be constructed, such that whenever $(k_0, \ldots, k_n) \in \Omega$, the regulation error $e(t)$ will converge to zero exponentially, for any unknown functions $a(x)$ and $b(x)$ satisfying (9) and for any initial state $\|x(0)\| \leq R$.

Remark 7: We remark that the semi-global results in Theorem 1 can also be extend to global ones if the upper bound function τ_2 in Assumption B is a linear function, i.e., $\tau_2(r) = Lr$, and $G(x)$ has a known constant upper bound [34]. Moreover, in the case where the derivatives of the regulation error are not available, similar results can also be established (see [34]) by incorporating a differential tracker or a high-gain observer as used in the literature (see, e.g., [8] and [19]). We also remark that the case of zero-dynamics, or the case where the control inputs is nonaffine, would be interesting for further investigation. These cases may be dealt with by using similar techniques as in [12] and [33], where the first paper considered the normal form with zero-dynamics and the second paper considered a class of nonaffine uncertain systems.

IV. PROOFS OF THE MAIN RESULTS

Before proving Theorem 1, we first list some lemmas. Denote $\lambda \triangleq (\lambda_0, \ldots, \lambda_n) \in \mathbb{R}^{n+1}$ and define an open unbounded set $\Omega_1 \subset \mathbb{R}^{n+1}$ as follows:

$$\Omega_1 = \{\lambda | 2 < \lambda_i - 2i < 3, i = 0, \ldots, n-1; \lambda_n > 2n+2\} \quad (12)$$

and for $\lambda \in \Omega_1$, define a matrix $P = P(\lambda)$ as follows (see [5]):

$$P = \begin{bmatrix} (-\lambda_0)^{-n} & \cdots & (-\lambda_n)^{-n} \\ \vdots & & \vdots \\ (-\lambda_0)^{-1} & \cdots & (-\lambda_n)^{-1} \\ 1 & \cdots & 1 \end{bmatrix} \quad (13)$$

and denote $(d_0, \ldots, d_n)^T$ be the last column of P^{-1}, i.e.,

$$(d_0, \ldots, d_n)^T = P^{-1}(0, \ldots, 0, 1)^T \quad (14)$$

Lemma 1. (see [34]): Under the above notations, let us define

$$c_1 \triangleq \sup_{\lambda \in \Omega_1} \|P\|, c_2 \triangleq \sup_{\lambda \in \Omega_1} \|P\| \|P^{-1}\|, c_3 \triangleq \sup_{\lambda \in \Omega_1} \sqrt{n}(2n+1)d_n$$

$$c_4(i) \triangleq \sup_{\lambda \in \Omega_1} |(2n+1)n\lambda_n d_i|, i = 0, \ldots, n-1 \quad (15)$$

and denote $c_0 = \max\{c_1, c_2, c_3, c_4(i), i = 0, \ldots, n-1\}$, then $c_0 < \infty$.

To introduce other lemmas, we now define a parameter manifold first. Let $c \geq c_0$, $L > 0$, and $0 < \underline{b} \leq \overline{b}$ be any given constants. We define the following open unbounded parameter set $\Omega_{L,\underline{b},\overline{b},c} \subset \mathbb{R}^{n+1}$:

$$\Omega_{L,\underline{b},\overline{b},c} \triangleq \left\{ \begin{bmatrix} k_0 \\ \vdots \\ k_n \end{bmatrix} \middle| \begin{bmatrix} k_0 \\ \vdots \\ k_{n-1} \\ k_n \end{bmatrix} = \frac{1}{\underline{b}} \begin{bmatrix} \prod_{i=0}^n \lambda_i \\ \vdots \\ \sum_{i<j} \lambda_i \lambda_j \\ \sum_{i=0}^n \lambda_i \end{bmatrix}, \lambda \in \Omega_\Lambda \right\} \quad (16)$$

where Ω_Λ is defined by

$$\Omega_\Lambda = \left\{ \lambda \in \Omega_1 \middle| \lambda_n > \left(Lc^2 + (\overline{b} - \underline{b}) c/\underline{b}\right)^2 + Lc^2 \right\}. \quad (17)$$

In the following lemmas, the constant T can be a finite positive number $0 < T < \infty$ or an infinity $T = \infty$.

Lemma 2: Let $\overline{Y}(t) = (y_0(t), \ldots, y_n(t))^T$ be a continuously differentiable vector valued function on $[0, T)$. Suppose that there exists two real valued functions a_t and b_t which are both defined on $t \in [0, T)$,

such that the following equalities hold for $t \in [0, T)$:

$$\begin{cases} \dot{y}_i = y_{i+1}, i = 0, \ldots, n-1 \\ \dot{y}_n = a_t - b_t(k_0 y_0 + \cdots + k_n y_n) \end{cases} \quad (18)$$

where $|a_t| \leq L\|\overline{Y}(t)\|$ and $0 < \underline{b} \leq b_t \leq \overline{b}$, for any $t \in [0, T)$. Then for any $(k_0, \ldots, k_n) \in \Omega_{L,\underline{b},\overline{b},c}$, there exists $\alpha > 0$ (only depend on $(k_0, \ldots, k_n)$), such that $\overline{Y}(t)$ satisfies $\|\overline{Y}(t)\| \leq ce^{-\alpha t}\|\overline{Y}(0)\|, \forall t \in [0, T)$.

Lemma 3: Consider the system of equalities (18) again, but where $|a_t| \leq \tau_2(\|\overline{Y}(t)\|)$ and $0 < \underline{b} \leq b_t \leq \tau_1(\|\overline{Y}(t)\|)$, for any $t \in [0, T)$ and where $\tau_1, \tau_2 : \mathbb{R}^+ \to \mathbb{R}^+$ are two increasing functions with $\limsup_{r \to 0} \frac{\tau_2(r)}{r} < \infty$. Then for any $R > 0$, and any $(k_0, \ldots, k_n) \in \Omega_{L_0,\underline{b},b_0,c}$ with $L_0 = \sup_{0 \leq r \leq cR} \frac{\tau_2(r)}{r}$, $b_0 = \tau_1(cR)$, there exists $\alpha > 0$, such that $\overline{Y}(t)$ satisfies $\|\overline{Y}(t)\| \leq ce^{-\alpha t}\|\overline{Y}(0)\|, \forall t \in [0, T)$, provided that $\|\overline{Y}(0)\| \leq R$.

The proofs of the Lemmas are given in Appendix A.

Proof of Theorem 1:

Step 1: First, notice that $e(t) = y^* - y(t) = y^* - h(x(t))$. Therefore, by Assumption A, following a standard calculation (e.g., [11]), we have $e^{(i)}(t) = -L_f^i h(x(t)), i = 1, \ldots, n-1$. Recall $F(x) \triangleq L_f^n h(x), G(x) \triangleq L_g L_f^{n-1} h(x)$, we have:

$$e^{(n)}(t) = -F(x(t)) - G(x(t))u(t). \quad (19)$$

Denote $x_0(t) = \int_0^t [y^* - h(x(s))]ds$, then $x_0(0) = 0$. By the definition for the extended PID in (2), we know that $u(t) = k_0 x_0(t) + k_1(y^* - h(x(t))) - \sum_{i=2}^n k_i L_f^{i-1} h(x(t))$. Combining this with the system (1), we know that the solution of the closed-loop system (1)–(2) with initial state $x(0)$ is equivalent to the solution of the following $(n+1)$th order autonomous differential equation:

$$\begin{cases} \dot{x}_0 = y^* - h(x) \\ \dot{x} = f(x) + g(x)(k_0 x_0 + k_1 y^* - \sum_{j=1}^n k_j L_f^{j-1} h(x)) \end{cases} \quad (20)$$

with initial value $[0, x^T(0)]^T$. Since the functions on the RHS of (20) are smooth, there is a positive constant $T > 0$ (possibly depend on $x(0)$) such that the solution of (1)–(2) exists in $[0, T)$. Denote $y_0(t) \triangleq -\int_0^t e(s)ds - \frac{F(x^*)}{k_0 G(x^*)}$, $y_1(t) \triangleq -e(t)$, $y_i(t) \triangleq -e^{(i-1)}(t)$, $i = 2, \ldots, n$. By (19), we have $\dot{y}_n(t) = -e^{(n)}(t) = F(x(t)) + G(x(t))u(t)$, and $u(t) = -(\sum_{i=0}^n k_i y_i(t) + \frac{F(x^*)}{G(x^*)})$. Therefore, we obtain

$$\begin{cases} \dot{y}_i(t) = y_{i+1}(t), i = 0, \ldots, n-1 \\ \dot{y}_n(t) = a_t - b_t \sum_{i=0}^n k_i y_i(t). \end{cases} \quad (21)$$

where $a_t \triangleq F(x(t)) - \frac{F(x^*)}{G(x^*)}G(x(t))$, $b_t \triangleq G(x(t))$. Denote

$$\overline{Y}(t) \triangleq (y_0(t), \ldots, y_n(t))^T, Y(t) \triangleq (y_1(t), \ldots, y_n(t))^T$$

it is easy to see that $Y(t) = \Phi(x(t)) - z^*$, where z^* is defined in (4).

Step 2: Next, we will apply Lemma 3 to prove that if the initial state $\|x(0)\| \leq R$ and the parameters $(k_0, \ldots, k_n) \in \Omega_{L_0,\underline{b},b_0,c}$, where $L_0 \triangleq \sup_{0 \leq r \leq cR_0} \frac{\underline{b}+\tau_1(|y^*|)}{\underline{b}} \frac{\tau_2(r)}{r}$, $b_0 \triangleq \tau_1(cR_0 + |y^*|)$, and $R_0 \triangleq \tau_1(R) + |y^*| + \tau_1(|y^*|)$, then there exists $\alpha > 0$, such that

$$\|\overline{Y}(t)\| \leq ce^{-\alpha t}\|\overline{Y}(0)\|, t \in [0, T) \quad (22)$$

where $[0, T)$ is the maximal existence interval of the closed-loop system (1)–(2).

By Assumption B, we have $\|H(x^*)\| \leq \tau_1(\|\Phi(x^*)\|) = \tau_1(\|z^*\|) = \tau_1(|y^*|)$. Therefore, $|F(x^*)| \leq \|H(x^*)\| \leq \tau_1(|y^*|)$

and thus we can obtain

$$
\begin{aligned}
|a_t| &= |F(x(t)) - F(x^*)G(x(t))/G(x^*)| \\
&\leq |F(x(t)) - F(x^*)| + |F(x^*)||G(x(t)) - G(x^*)|/G(x^*) \\
&\leq |F(x(t)) - F(x^*)| + \tau_1(|y^*|)|G(x(t)) - G(x^*)|/\underline{b} \\
&\leq (\underline{b} + \tau_1(|y^*|))\|H(x(t)) - H(x^*)\|/\underline{b} \\
&\leq (\underline{b} + \tau_1(|y^*|))\tau_2(\|\Phi(x(t)) - \Phi(x^*)\|)/\underline{b} \\
&= (\underline{b} + \tau_1(|y^*|))\tau_2(\|Y(t)\|)/\underline{b} \leq (\underline{b} + \tau_1(|y^*|))\tau_2\left(\left\|\overline{Y}(t)\right\|\right)/\underline{b}.
\end{aligned}
\tag{23}
$$

On the other hand, by Assumptions A and B, we have

$$
\begin{aligned}
0 < \underline{b} \leq b_t &= G(x(t)) \leq \|H(x(t))\| \leq \tau_1(\|\Phi(x(t))\|) \\
&= \tau_1(\|Y(t) + z^*\|) \leq \tau_1(\|Y(t)\| + |y^*|) \\
&\leq \tau_1(\|\overline{Y}(t)\| + |y^*|).
\end{aligned}
\tag{24}
$$

Since $\|x(0)\| \leq R$, then by Assumption B, we have

$$
\|Y(0)\| = \|\Phi(x(0)) - z^*\| \leq \tau_1(\|x(0)\|) + \|z^*\| \leq \tau_1(R) + |y^*|.
$$

Recall that $y_0(0) = -\frac{F(x^*)}{k_0 G(x^*)}$, we have

$$
\begin{aligned}
\left\|\overline{Y}(0)\right\| &\leq \|Y(0)\| + |y_0(0)| \leq \tau_1(R) + |y^*| + \frac{\tau_1(|y^*|)}{k_0 G(x^*)} \\
&\leq \tau_1(R) + |y^*| + \tau_1(|y^*|)
\end{aligned}
$$

where the last inequality holds since $k_0 = \frac{\prod_{i=0}^n \lambda_i}{\underline{b}} \geq \frac{1}{\underline{b}}$ and $G(x^*) \geq \underline{b}$. By Lemma 3, we know that (22) holds.

Step 3: In this step, we will show that if the initial state $\|x(0)\| \leq R$ and the parameters $(k_0, \ldots, k_n) \in \Omega_{L_0, \underline{b}, b_0, c}$, then the maximal existence interval of the solution of the closed-loop system (1) and (2) is $[0, \infty)$, i.e., $[0, T) = [0, \infty)$.

We use the contradiction argument. Suppose that $T < \infty$ for some initial state $\|x(0)\| \leq R$. Therefore, the maximal existence interval of the solution of (20) with the initial value $[0, x^T(0)]^T \in \mathbb{R}^{n+1}$ is also finite. Then it is well-known from the theory of ordinary differential equations that

$$
\limsup_{0 \leq t < T} \left\|[x_0(t), x^T(t)]\right\| = \infty.
\tag{25}
$$

By (22), we know that $\|\overline{Y}(t)\|$ is bounded on $t \in [0, T)$. Hence, by (23) and (24), the right-hand side of (21) and the boundedness of $\|\overline{Y}(t)\|$, it is not difficult to conclude that $\left\|\dot{\overline{Y}}(t)\right\| \leq N, \forall t \in [0, T)$ for some constant $N > 0$.

On the other hand, denote $z(t) = \Phi(x(t))$, then $\dot{z}(t) = J_\Phi(x(t))\dot{x}(t)$. From $\|\dot{z}(t)\| = \|\dot{Y}(t)\| \leq \|\dot{\overline{Y}}(t)\|$, we obtain

$$
\|\dot{x}(t)\| = \left\|J_\Phi^{-1}(x(t))\dot{z}(t)\right\| \leq N\left\|J_\Phi^{-1}(x(t))\right\|, t \in [0, T).
\tag{26}
$$

By Assumption C, we have

$$
\|\dot{x}(t)\| \leq \alpha_1 \|x(t)\| \log_+ \|x(t)\| + \alpha_2
\tag{27}
$$

for any $t \in [0, T)$, where $\alpha_1 = NN_1$ and $\alpha_2 = NN_2$. Denote $v(t) \triangleq \|x(t)\|$ and $D^+v(t) \triangleq \limsup_{h \to 0^+} \frac{v(t+h) - v(t)}{h}$ be the upper right-hand derivative of $v(t)$. Then it is not difficult to obtain $D^+v(t) \leq \|\dot{x}(t)\|$. Noticing that $v(t) = \|x(t)\|$, from (27), we have $D^+v(t) \leq \|\dot{x}(t)\| \leq \alpha_1 v(t) \log_+ v(t) + \alpha_2, t \in [0, T)$. By the comparison lemma in ordinary differential equations (see, e.g., [18]), we have $\int_{v(0)}^{v(t)} \frac{d\eta}{\alpha_1 \eta \log_+ \eta + \alpha_2} \leq t < T, \forall t \in [0, T)$, which implies $\sup_{0 \leq t < T} v(t) = \sup_{0 \leq t < T} \|x(t)\| < \infty$ since $\int_{v(0)}^\infty \frac{d\eta}{\alpha_1 \eta \log_+ \eta + \alpha_2} =$

∞. From this and the fact $T < \infty$, it is not difficult to see that $\sup_{0 \leq t < T} |x_0(t)| = \sup_{0 \leq t < T} |\int_0^t y^* - h(x(s))ds| < \infty$. Therefore, the solution of (20) with initial state $[0, x^T(0)]$ satisfy $\sup_{0 \leq t < T} \left\|[x_0(t), x^T(t)]\right\| < \infty$, which contradicts to (25). Therefore, if $(k_0, \ldots, k_n) \in \Omega_{L_0, \underline{b}, b_0, c}$, then for any initial state $\|x(0)\| \leq R$, the solution of the closed-loop system will exist in $[0, \infty)$.

Step 4: Since solution of the closed-loop equation exists in $[0, \infty)$, we conclude that (23) and (24) are satisfied in $[0, \infty)$. By using Lemma 3 again, we have $\|\overline{Y}(t)\| \leq ce^{-\alpha t}\|\overline{Y}(0)\|$ for any $t \in [0, \infty)$. Therefore, we have $|e(t)| = |y_1(t)| \leq \|Y(t)\| \leq \|\overline{Y}(t)\| \leq ce^{-\alpha t}\|\overline{Y}(0)\|, \forall t \in [0, \infty)$. This completes the proof of Theorem 1.

V. CONCLUSION

In this article, we have presented a theoretical investigation on the extended PID controller for a general class of SISO affine-nonlinear uncertain dynamical systems, and have established some new results which possess obvious advantages in comparison with the existing literature. First, we have shown that the extended PID controller has the ability to regulate the nonlinear uncertain systems (1) semiglobally, under some fairly general conditions on the nonlinearity and the uncertainty of the systems, which are neither in the conventionally studied normal or triangular forms, nor assumed to be transformed globally into them. Moreover, we have provided a concrete design method for the parameters of the extended PID controller, by constructing an $(n + 1)$-dimensional parameter manifold based on the size of the system uncertainty, improved the existing related qualitative design methods. Furthermore, our main results also demonstrate explicitly that the extended PID controller has large scale robustness with respect to both the system structural uncertainties and the selection of the $(n + 1)$ controller parameters. Of course, many interesting problems still remain open. It would be interesting to consider extended PID control for multi-input–multi-output affine nonlinear uncertain systems, and to generalize our recent results on PID control of coupled multiagent dynamical systems [30]. It would also be interesting to consider more complicated situations such as zero-dynamics, saturation, deadzone, time-delayed inputs, sampled-data PID controllers under a prescribed sampling rate, etc. These belong to further investigation.

APPENDIX

A. Proof of the Lemmas

Proof of Lemma 2: Rewrite (18) as

$$
\begin{cases}
\dot{y}_i = y_{i+1}, i = 0, \ldots, n-1 \\
\dot{y}_n = -\underline{b} \sum_{i=0}^n k_i y_i + a_t + (\underline{b} - b_t) \sum_{i=0}^n k_i y_i
\end{cases}.
\tag{28}
$$

Suppose that $(k_0, \ldots, k_n) \in \Omega_{L, \underline{b}, \overline{b}, c}$ and denote $A \triangleq$

$$
\begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
-\underline{b}k_0 & -\underline{b}k_1 & -\underline{b}k_2 & \cdots & -\underline{b}k_n
\end{bmatrix}.
$$
Then (28) can be rewritten as

$$
\dot{\overline{Y}} = A\overline{Y} + (0, \ldots, 0, a_t + (\underline{b} - b_t)(k_0 y_0 + \cdots + k_n y_n))^T.
\tag{29}
$$

It is easy to see that the characteristic polynomial of A is $\det(sI - A) = s^{n+1} + \sum_{i=0}^n \underline{b}k_i s^i$. By the definition (16) of $\Omega_{L, \underline{b}, \overline{b}, c}$, there exists $(\lambda_0, \ldots, \lambda_n) \in \Omega_\Lambda$ such that $(\underline{b}k_0, \ldots, \underline{b}k_n) = (\prod_{i=0}^n \lambda_i, \ldots, \sum_{i=0}^n \lambda_i)$. Therefore, by Vieta's formulas, we know that $-\lambda_i, i = 0, \ldots, n$ are $(n + 1)$ distinct eigenvalues of A. Hence,

A is similar to J, where J is a diagonal matrix defined by $J \triangleq \mathrm{diag}(-\lambda_0, \ldots, -\lambda_n)$.

It is not difficult to get the relationship $AP = PJ$, where P is defined in (13). To simplify the analysis, we introduce an invertible linear transformation $\overline{Y}(t) = P\overline{w}(t)$, where $\overline{w} = (w_0, \ldots, w_{n-1}, w_n)^T$.

By the relationship $\overline{Y} = P\overline{w}$ (we omit the variable t), we have $y_i = \sum_{k=0}^{n}(-\lambda_k)^{i-n} w_k$. Therefore

$$\sum_{i=0}^{n} k_i y_i = \sum_{k=0}^{n}\left(\sum_{i=0}^{n} k_i(-\lambda_k)^{i-n}\right) w_k$$
$$= \sum_{k=0}^{n} \frac{\sum_{i=0}^{n} k_i(-\lambda_k)^i}{(-\lambda_k)^n} w_k = \frac{1}{\underline{b}}\sum_{k=0}^{n} \lambda_k w_k.$$

The last equality holds since $-\lambda_k$ is the root of the polynomial $s^{n+1} + \sum_{i=0}^{n} \underline{b}k_i s^i$.

By the relationship $A = PJP^{-1}$, then (29) transforms into

$$\dot{\overline{w}} = J\overline{w} + P^{-1}\left(0, \ldots, 0, a_t + (\underline{b} - b_t)\sum_{i=0}^{n} k_i y_i\right)^T. \quad (30)$$

Recall that $(d_0, \ldots, d_n)^T$ is the last column of the matrix P^{-1}, and by $\sum_{i=0}^{n} k_i y_i = \frac{1}{\underline{b}}\sum_{i=0}^{n} \lambda_i w_i$, (30) becomes

$$\dot{w}_k = -\lambda_k w_k + d_k\left(a_t + \frac{\underline{b} - b_t}{\underline{b}}\sum_{i=0}^{n} \lambda_i w_i\right), \quad k = 0, \ldots, n. \quad (31)$$

Now, we consider the following quadratic Lyapunov function: $V(\overline{w}(t)) = \frac{1}{2}\sum_{i=0}^{n} w_i^2(t) = \frac{1}{2}\|\overline{w}(t)\|^2$.

Then it is easy to compute the time derivative of V $\dot{V}(\overline{w}(t)) \triangleq \frac{dV(\overline{w}(t))}{dt}$ as follows:

$$\dot{V}(\overline{w}(t)) = -\sum_{i=0}^{n} \lambda_i w_i^2 + \sum_{i=0}^{n} d_i w_i\left(a_t + \frac{\underline{b} - b_t}{\underline{b}}\sum_{i=0}^{n} \lambda_i w_i\right)$$
$$= \underbrace{-\sum_{i=0}^{n} \lambda_i w_i^2}_{\text{I}} + \underbrace{\left(\sum_{i=0}^{n} d_i w_i\right)a_t}_{\text{II}} + \underbrace{\frac{\underline{b} - b_t}{\underline{b}}\sum_{i=0}^{n} d_i w_i \sum_{i=0}^{n} \lambda_i w_i}_{\text{III}}. \quad (32)$$

Next, we proceed to estimate (32) term by term.

Denote $(w_0, \ldots, w_{n-1})^T \triangleq w$. Since $\lambda_i > 2, i = 0 \cdots, n-1$, the first term

$$\text{I} = -\sum_{i=0}^{n} \lambda_i w_i^2 \leq -2\|w\|^2 - \lambda_n w_n^2. \quad (33)$$

By Lemma 1, we have $\|P\| \leq c$, therefore

$$|a_t| \leq L\|\overline{Y}\| = L\|P\overline{w}\| \leq Lc\|\overline{w}\| \leq Lc(\|w\| + |w_n|).$$

On the other hand, by Lemma 1 and the fact $c \geq c_0$, we also have $|d_i| \leq \frac{c}{(2n+1)n\lambda_n} < \frac{c}{\sqrt{n}\lambda_n}, i = 0, \ldots, n-1$, and $|d_n| \leq \frac{c}{(2n+1)\sqrt{n}} < c$. Therefore, we have

$$|\sum_{i=0}^{n} d_i w_i| \leq c\left(\sum_{i=0}^{n-1}|\frac{w_i}{\sqrt{n}\lambda_n}| + |w_n|\right) \leq c\left(\frac{\|w\|}{\lambda_n} + |w_n|\right).$$

As a consequence, we have the following upper bound for the second term:

$$\text{II} \leq |\left(\sum_{i=0}^{n} d_i w_i\right)a_t| \leq Lc^2(\|w\| + |w_n|)(\|w\|/\lambda_n + |w_n|)$$
$$\leq Lc^2\left(\|w\|^2/\lambda_n + 2\|w\||w_n| + |w_n|^2\right). \quad (34)$$

Finally, we proceed to estimate the third term. Since $d_n = \frac{\lambda_n^n}{\prod_{i=0}^{n-1}(\lambda_n - \lambda_i)} > 0$, it is easy to get

$$\text{III} = \frac{\underline{b} - b_t}{\underline{b}}\sum_{i=0}^{n} d_i w_i \sum_{i=0}^{n} \lambda_i w_i \leq \frac{\underline{b} - b_t}{\underline{b}}\left\{\sum_{i=0}^{n-1} d_i w_i \sum_{i=0}^{n-1} \lambda_i w_i\right.$$
$$\left. + d_n w_n \sum_{i=0}^{n-1} \lambda_i w_i + \left(\sum_{i=0}^{n-1} d_i w_i\right)\lambda_n w_n\right\}.$$

Since we know that $|d_n| \leq \frac{c}{(2n+1)\sqrt{n}}$, $|d_i| \leq \frac{c}{(2n+1)n\lambda_n}$, $i = 0, \ldots, n-1$, and $0 < \lambda_i < 2n+1, i = 0, \ldots, n-1$, and $\sum_{i=0}^{n-1}|w_i| \leq \sqrt{n}\|w\|$, we can easily get the following three inequalities:

$$\left|\sum_{i=0}^{n-1} d_i w_i \sum_{i=0}^{n-1} \lambda_i w_i\right| \leq \frac{c}{n\lambda_n}\left(\sum_{i=0}^{n-1}|w_i|\right)^2 \leq \frac{c}{\lambda_n}\|w\|^2;$$

$$\left|d_n w_n \sum_{i=0}^{n-1} \lambda_i w_i\right| \leq \frac{c(2n+1)\sqrt{n}}{(2n+1)\sqrt{n}}|w_n|\|w\| = c|w_n|\|w\|;$$

$$\left|\left(\sum_{i=0}^{n-1} d_i w_i\right)\lambda_n w_n\right| \leq \frac{c}{(2n+1)\sqrt{n}}\|w\||w_n| \leq c\|w\||w_n|.$$

Therefore, the upper bound of III can be estimated as

$$\text{III} \leq (\overline{b} - \underline{b})\left(c\|w\|^2/\lambda_n + 2c|w_n|\|w\|\right)/\underline{b}. \quad (35)$$

Denote $m = Lc^2 + \frac{(\overline{b}-\underline{b})c}{\underline{b}}$. Combining (33)–(35), we have

$$\dot{V}(\overline{w}) \leq (m/\lambda_n - 2)\|w\|^2 + 2m\|w\||w_n| - \left(\lambda_n - Lc^2\right)w_n^2. \quad (36)$$

Since $(\lambda_0, \ldots, \lambda_n) \in \Omega_\Lambda$, we can see $\lambda_n > \max\{2n+2, m^2 + Lc^2\}$. If $m \leq 1$, then $\lambda_n > 2n+2 > m$; if $m \geq 1$, then $\lambda_n > m^2 + Lc^2 > m$. Therefore, $\lambda_n > m$ always holds whenever $(\lambda_0, \ldots, \lambda_n) \in \Omega_\Lambda$. As a consequence, we have

$$\dot{V}(\overline{w}) \leq -\|w\|^2 + 2m\|w\||w_n| - (\lambda_n - Lc^2)w_n^2. \quad (37)$$

Since $\lambda_n > m^2 + Lc^2$, we conclude that $\dot{V}(\overline{w}(t)) \leq -\alpha\|\overline{w}(t)\|^2$ for some $\alpha > 0$, i.e., $\dot{V}(\overline{w}(t)) \leq -2\alpha V(\overline{w}(t))$. Therefore, by the comparison theorem, we have $V(\overline{w}(t)) \leq e^{-2\alpha t}V(\overline{w}(0))$ for $t \in [0, T)$.

Finally, we estimate the upper bound of $\|\overline{Y}(t)\|$ as follows:

$$\|\overline{Y}(t)\| = \|P\overline{w}(t)\| \leq \|P\|\|\overline{w}(t)\| = \|P\|\sqrt{2V(\overline{w}(t))}$$
$$\leq \|P\|\sqrt{2e^{-2\alpha t}V(\overline{w}(0))} = e^{-\alpha t}\|P\|\|\overline{w}(0)\|$$
$$\leq e^{-\alpha t}\|P\|\|P^{-1}\|\|\overline{Y}(0)\| \leq ce^{-\alpha t}\|\overline{Y}(0)\|. \qquad \square$$

Proof of Lemma 3: It suffices to show that if the parameters $(k_0, \ldots, k_n) \in \Omega_{L_0, \underline{b}, b_0, c}$, then for any $T_0 < T$ we have

$$\|\overline{Y}(t)\| \leq ce^{-\alpha t}\|\overline{Y}(0)\|, \forall t \in [0, T_0).$$

Denote $a \triangleq \sup_{0 \leq t \leq T_0}\|\overline{Y}(t)\|$, $L' \triangleq \sup_{0 \leq r \leq a}\frac{\tau_2(r)}{r}$, $b' \triangleq \tau_1(a)$. It is easy to verify that $|a_t| \leq \tau_2(\|\overline{Y}(t)\|) = \frac{\tau_2(\|\overline{Y}(t)\|)}{\|\overline{Y}(t)\|}\|\overline{Y}(t)\| \leq L'\|\overline{Y}(t)\|$ and $\underline{b} \leq b_t \leq \tau_1(\|\overline{Y}(t)\|) \leq \tau_1(a) = b'$ for $t \in [0, T_0)$.

Therefore, by Lemma 2, we have $\|\overline{Y}(t)\| \leq ce^{-\alpha t}\|\overline{Y}(0)\|$ for $t \in [0, T_0)$ whenever the parameters $(k_0, \ldots, k_n) \in \Omega_{L, '\underline{b}, b, 'c}$. As a consequence, $a = \sup_{0 \leq t \leq T_0}\|\overline{Y}(t)\| \leq c\|\overline{Y}(0)\| \leq cR$ and $b' = \tau_1(a) \leq \tau_1(cR)$, which implies $L' \leq L_0, b' \leq b_0$.

The final thing we need to prove is $\Omega_{L_0, \underline{b}, b_0, c} \subset \Omega_{L, '\underline{b}, b, 'c}$.

From (16)–(17), we know that

$$\Omega_{L,\underline{b},\overline{b},c} \triangleq \left\{ \begin{bmatrix} k_0 \\ \vdots \\ k_n \end{bmatrix} \Bigg| \begin{bmatrix} k_0 \\ \vdots \\ k_n \end{bmatrix} = \frac{1}{\underline{b}} \begin{bmatrix} \prod_{i=0}^{n} \lambda_i \\ \vdots \\ \sum_{i=0}^{n} \lambda_i \end{bmatrix}, \lambda \in \Omega_1 \cap \Omega_2 \right\}$$

where Ω_1 is defined in (12) and

$$\Omega_2 = \left\{ \lambda \in \mathbb{R}^{n+1} \big| \lambda_n > \left(Lc^2 + \left(\overline{b} - \underline{b} \right) c/\underline{b} \right)^2 + Lc^2 \right\}. \quad (38)$$

From (12), it is easy to see that Ω_1 does not depend on $L, \underline{b}, \overline{b}$ and c. By (38), we know that Ω_2 depends on $L, \underline{b}, \overline{b}, c$, i.e., $\Omega_2 = \Omega_2(L, \underline{b}, \overline{b}, c)$. It is easy to see that if $\underline{b}$ and c are fixed, then Ω_2 gets smaller for larger $L, \overline{b}$. Therefore, we have $\Omega_{L_0,\underline{b},b_0,c} \subset \Omega_{L,'\underline{b},b,'c}$. Notice that α can be chosen independent of T_0 and T_0 is arbitrary, we complete the proof of Lemma 3.

B. Proof of Remark 3 and Proof of Example 1

Proof of Remark 3: For nonlinear systems already in the normal form (8), then $\Phi(x) = x, F(x) = a(x), G(x) = b(x), \ x^* = z^*$ and Assumptions A and B reduces to

$$b(x) \geq \underline{b}, \|x\| \leq \tau_1(\|x\|), \|H(x)\| \leq \tau_1(\|x\|), \quad (39)$$

$$\|H(x) - H(z^*)\| \leq \tau_2(\|x - z^*\|), \forall x \in \mathbb{R}^n. \quad (40)$$

Since $H(x) = (a(x), b(x))$, we have $\|H(x)\| \leq |a(x)| + |b(x)|$. From (9), we know that $\|H(x)\| \leq |a(x)| + |b(x)| \leq \rho(\|x\|)$. Therefore, if we denote $\tau_1(r) = \max\{r, \rho(r)\}$, then (39) is satisfied. On the other hand, by the mean value theorem, we have $a(x) - a(z^*) = J_a(\theta(x - z^*) + z^*)(x - z^*)$ for some $0 \leq \theta \leq 1$ and therefore $|a(x) - a(z^*)| \leq \|J_a(\theta(x - z^*) + z^*)\|\|x - z^*\| \leq \rho(\|x - z^*\| + |y^*|)\|x - z^*\|$, the last inequality holds since $0 \leq \theta \leq 1$ and ρ is increasing. Similarly, we have $|b(x) - b(z^*)| \leq \rho(\|x - z^*\| + |y^*|)\|x - z^*\|$. Therefore, we have $\|H(x) - H(z^*)\| \leq \sqrt{2}\max\{|a(x) - a(z^*)|, |b(x) - b(z^*)|\} \leq \sqrt{2}\rho(\|x - z^*\| + |y^*|)\|x - z^*\|$. If we denote $\tau_2(r) = \sqrt{2}\rho(r + |y^*|)r$, then (40) is satisfied. Finally, it is easy to see that $\tau_1(r) = \max\{r, \rho(r)\}$ and $\tau_2(r) = \sqrt{2}\rho(r + |y^*|)r$ are two increasing functions, and $\limsup_{r\to 0} \tau_2(r)/r = \limsup_{r\to 0} \sqrt{2}\rho(r + |y^*|) = \sqrt{2}\rho_+(|y^*|) < \infty$, where $\rho_+(|y^*|)$ denotes the right limit of ρ at $|y^*|$.

Proof of Example 1: By simple calculations, we have

$$\Phi(x) = (\theta e^{x_1} \cos x_2, -\theta e^{x_1} \sin x_2), F(x) = 0, G(x) = 1.$$

Obviously, Φ is not a global diffeomorphism since $\Phi(x_1, x_2 + 2\pi) = \Phi(x_1, x_2)$ for any $(x_1, x_2) \in \mathbb{R}^2$. It is easy to see that $\|\Phi(x)\| = \theta e^{x_1} \leq e^{\|x\|}, \|H(x)\| \leq 1, x^* = (-\infty, 0)$ and $H(x) - H(x^*) = 0$. Therefore, both Assumptions A and B are satisfied with $\underline{b} = 1, \tau_1(r) = e^r$, and $\tau_2(r) \equiv 0$. Denote $y_0(t) = \int_0^t -y(s)ds, \ y_1(t) = -y(t) = -\theta e^{x_1} \cos x_2$, and $y_2(t) = -\dot{y}(t) = \theta e^{x_1} \sin x_2$, then $u = k_i y_0 - k_p \theta e^{x_1} \cos x_2 + k_d \theta e^{x_1} \sin x_2$. Therefore, we have

$$\begin{cases} \dot{y}_0 = -\theta e^{x_1} \cos x_2 \\ \dot{x}_1 = \eta_1(x) - k_i y_0 \sin x_2 e^{-x_1} \\ \dot{x}_2 = \eta_2(x) - k_i y_0 \cos x_2 e^{-x_1} \end{cases} \quad (41)$$

where $\eta_1(x) = (k_p \theta - 1) \sin x_2 \cos x_2 - k_d \theta \sin^2 x_2$ and $\eta_2(x) = \sin x_2^2 + k_p \theta \cos^2 x_2 - k_d \theta \sin x_2 \cos x_2$ are bounded functions on $\mathbb{R}^2$. Suppose that the solution of the closed-loop equation only exists in $[0, T)$ for some initial state $x(0)$ and for some $0 < |\theta| \leq 1$ with $T < \infty$. Then the maximal existence interval of (41) with initial

value $[0, x(0)]$ is also $[0, T)$. Since $\dot{y}_2 = -k_i y_0 - k_p y_1 - k_d y_2$, therefore $[y_0(t), y_1(t), y_2(t)]^T$ in bounded in $[0, T)$. Therefore, from (41), we know $e^{-x_1(t)}$ is unbounded in $[0, T)$. Otherwise the RHS of (41) will be bounded, thus $(y_0(t), x_1(t), x_2(t))$ is bounded in $[0, T)$, which contradicts to that the solution of (41) only exists in a finite interval. Therefore, there exists a sequence $t_n \to T$, such that $e^{x_1(t_n)}$ tends to 0 as $n \to \infty$. Since $\|(y_1(t), y_2(t))\| = \theta e^{x_1(t)}$, therefore we have $\lim_{t_n \to \infty} y_1(t_n) = \lim_{t_n \to \infty} y_2(t_n) = 0$. On the other hand, denote $[y_0(t), y_1(t), y_2(t)]^T = P[w_0(t), w_1(t), w_2(t)]^T$, where $P = \begin{bmatrix} \lambda_1^{-2} & \lambda_2^{-2} & \lambda_3^{-2} \\ \lambda_1^{-1} & \lambda_2^{-1} & \lambda_3^{-1} \\ 1 & 1 & 1 \end{bmatrix}$ and $\lambda_1, \lambda_2, \lambda_3$ are the roots of $s^3 + k_d s^2 + k_p s + k_i = 0$. Then, $[y_0(t), y_1(t), y_2(t)]^T = P[e^{\lambda_1 t} w_0(0), e^{\lambda_2 t} w_1(0), e^{\lambda_3 t} w_2(0)]$. Recall $y_0(0) = 0$ and $\lim_{t_n \to \infty} y_1(t_n) = \lim_{t_n \to \infty} y_2(t_n) = 0$, thus $\begin{bmatrix} \lambda_1^{-2} & \lambda_2^{-2} & \lambda_3^{-2} \\ \lambda_1^{-1} e^{\lambda_1 T} & \lambda_2^{-1} e^{\lambda_2 T} & \lambda_3^{-1} e^{\lambda_3 T} \\ e^{\lambda_1 T} & e^{\lambda_2 T} & e^{\lambda_3 T} \end{bmatrix} \begin{bmatrix} w_0(0) \\ w_1(0) \\ w_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. This implies $[w_0(0), w_1(0), w_2(0)] = [0, 0, 0]$ since the coefficients matrix is invertible. Therefore, $y_1(0) = y_2(0) = 0$. This is impossible since $\|(y_1(0), y_2(0))\| = e^{x_1(0)} > 0$. □

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