A Robust Stochastic Adaptive Controller

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Abstract—**This paper considers the adaptive tracking problem for discrete-time stochastic systems consisting of a modeled part being a stable ARMAX process and unmodeled dynamics dominated by a small constant ϵ multiplied by a quantity independent of ϵ but tending to infinity as the past input, output, and noise grow. The adaptive control law proposed in this paper is switched at stopping times and is disturbed by a sequence of random vectors bounded by an arbitrarily small but fixed constant σ. It is shown that the closed-loop system is globally stable, the estimation error for parameter contained in the modeled part are of order ϵ, and the tracking error differs from the minimum tracking error for systems without unmodeled dynamics by a value of $O(\epsilon^2) + O(\sigma^2)$.**

I. INTRODUCTION

It is of great importance to analyze the influence of unmodeled dynamics contained in a system upon the behavior of adaptive control systems. Issues such as stability, control performance, and accuracy of parameter estimation for the modeled part of the system must be readdressed in the face of unmodeled dynamics, since, in general, a real system can rarely be modeled by an exact linear deterministic or stochastic system. Indeed, in [1]–[3] it is shown that unmodeled dynamics or even small bounded disturbances can cause many adaptive control algorithms to go unstable, without other precautions being taken.

Much attention has therefore been given in recent years to the issue of robust adaptive control, especially in deterministic adaptive control, to determine under what conditions signals in the system remain bounded under violations of standard assumptions (e.g., [4]–[11]). In the case of bounded disturbances, a dead zone is introduced (e.g., [2]) in the adaptive law so that adaptation takes place only when the identification error exceeds a certain threshold. In order to choose the size of the dead zone appropriately, a bound on the disturbance must be known. In [4], the parameter estimator is modified in terms of normalized signals, and the stability of the systems is studied using sector stability and passivity theorems. The σ-modification, i.e., an adaptive control law with the extra leakage term $-\sigma\theta$, $\sigma > 0$, is suggested and analyzed in [7] and [8], and it is shown that the algorithm guarantees boundedness of all signals in the adaptive loop and small residual tracking errors. Another approach guaranteeing robustness is to produce persistency of excitation in order to make the adaptive system (undisturbed) exponentially stable, and then to obtain stability in the presence of bounded disturbances (e.g., [2], [9]). In [9] and [10], an averaging method is used to analyze the local stability of adaptive systems when the strictly positive real (SPR) condition is violated. In [5] and [6], a relative dead zone method is used to deal with the case when the disturbance is internally generated and thus depends on the actual plant input and output signals. Relative dead zone means that the dead zone acts on the suitably normalized, relative identification error. In this method, an *a priori* bound on the unmodeled dynamics is necessary. In summary, we point out that in all the above-mentioned approaches in deterministic robust adaptive control, the proof of stability depends crucially on the *a priori* boundedness of the external noise disturbance.

In the adaptive control of stochastic systems, however, noise is an essential feature of the system, and it need not necessarily be bounded. For example, if the system noise $\{w_n\}$ is a Gaussian white noise sequence with zero mean and variance $\sigma^2 > 0$, it is known that (see, e.g., [12, p. 64]) $\{w_n\}$ is unbounded almost surely. In the stochastic case, it is of interest not only to guarantee boundedness of the system input and output under noise disturbance and unmodeled dynamics, but it is also important to reject the noise optimally, or at least close to optimally. Thus, performance of the adaptive control algorithm, in rejecting the corrupting noise, and tracking the desired reference signal with small tracking error, is also an important goal in robust stochastic adaptive control.

In a recent paper [11], the authors have given a preliminary analysis of the robustness of parameter identification for discrete-time stochastic systems which are viewed as a sum of an ARMAX (autoregressive moving-average with auxiliary (or exogenous) input) process representing the principal part of the system and an additional signal representing unmodeled dynamics. This signal is dominated by a small constant ϵ multiplied by a quantity diverging to infinity as any one of the input, output, and noise of the system increases. This means that the unmodeled dynamics considered in [11] are not negligible when the amplitudes of the input, or output, or noise are not small. Then it is shown that both the estimation error generated by the ELS algorithm for the modeled part and the difference between the tracking error and its minimum value of order ϵ if the system is persistently excited and some other conditions imposed on signals of the closed-loop systems are satisfied.

In this paper we consider the parameter estimation of the modeled part and adaptive tracking problem for the same system as discussed in [11], but with no conditions imposed on signals of the closed-loop systems. However, we must assume that the open-loop system is stable. The main contributions of this paper are the presentation of a robust adaptive controller for stochastic systems and the proof of the following properties.

1) The closed-loop system is stable.
2) The estimation error for the unknown parameters in the modeled part tends to zero as the unmodeled part of the system decays.
3) The tracking error differs from its minimum value corresponding to a system without unmodeled dynamics by a value tending to zero as the unmodeled dynamics vanishes.

The paper is organized as follows. In Section II we give the model of the plant and the estimation algorithm. The robustness of the estimation algorithm is studied in Section III. In Section IV we give the structure of the adaptive control algorithm and analyze the robustness properties of the algorithm with respect to unmodeled dynamics and state our main results. Some concluding remarks are presented in Section V.

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II. Description of the System

Consider the following stochastic system with unmodeled dynamics η_n:

$$A(z^{-1})y_{n+1}=B(z^{-1})u_n+C(z^{-1})w_{n+1}+\eta_n, \qquad n\geq 0,$$

$$y_n=w_n=\eta_n=0,\ u_n=0, \qquad n<0 \tag{1}$$

where

$$A(z^{-1})=I+A_1z^{-1}+\cdots+A_pz^{-p}, \qquad p\geq 0, \tag{2}$$

$$B(z^{-1})=B_1+B_2z^{-1}+\cdots+B_qz^{-q+1}, \qquad q\geq 1 \tag{3}$$

$$C(z^{-1})=I+C_1z^{-1}+\cdots C_rz^{-r}, \qquad r\geq 0 \tag{4}$$

are matrix polynomials in backwards shift operator z^{-1} with unknown matrix coefficient

$$\theta=[-A_1\cdots -A_p,\ B_1\cdots B_qC_1\cdots C_r]^\tau \tag{5}$$

but with known upper bounds for orders p, q, r, and where y_n and u_n are the m-output l-input vectors, respectively. The driven noise $\{w_n\}$ in the modeled part of the system is assumed to be a Martingale difference sequence with respect to a nondecreasing family of σ-algebras $\{\mathfrak{F}_n\}$ with properties

$$\sup_n E(\|w_{n+1}\|^2|\mathfrak{F}_n)<\infty \qquad \text{a.s.}, \tag{6}$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}w_iw_i^\tau=R>0 \qquad \text{a.s.} \tag{7}$$

The unmodeled dynamics η_n is $\mathfrak{F}_n$-measurable and assumed to be dominated by

$$\|\eta_n\|\leq\epsilon\sum_{i=0}^{n}a^{n-i}(\|y_i\|+\|u_i\|+\|w_i\|+1) \tag{8}$$

with $a\in(0,1)$, $\epsilon\geq 0$.

Remark 1: It is shown in [11] that the system described by (1) and (8) contains a variety of systems which are characterized by structured parameter uncertainty and unstructured modeling error; for example, the following two models are special cases of (1) and (8).

Model 1 (System with Structural Uncertainty):

$$[I+\mu_1z^{-1}H_1(z^{-1})]A(z^{-1})y_{n+1}=[I+\mu_2H_2(z^{-1})]B(z^{-1})u_n$$
$$+[I+\mu_3z^{-1}H_3(z^{-1})]C(z^{-1})w_{n+1}+\zeta_n(y,u)$$

where $H_1(z^{-1})$, $H_2(z^{-1})$, and $H_3(z^{-1})$ are matrix polynomials with unknown coefficients and orders, and μ_1, μ_2, and μ_3 are constants. $\zeta_n(y,u)$ is $\mathfrak{F}_n$ measurable unmodeled nonlinear dynamics satisfying

$$\|\zeta_n(y,u)\|\leq\mu_4\sum_{i=0}^{n}a_1^{n-i}(\|y_i\|+\|u_i\|+1)$$

where $\mu_4\geq 0$ and $a_1\in[0,1)$.

Model 2 (System with Slowly Varying Parameters):

$$y_{n+1}+A_{1n}y_n+\cdots+A_{pn}y_{n-p+1}$$
$$=B_{1n}u_n+\cdots+B_{qn}u_{n-q+1}+w_{n+1}+C_{1n}w_n+\cdots+C_{rn}w_{n-r+1}$$

with $\|\theta(n)-\theta\|\leq\mu_5$, $\mu_5\geq 0$, where $\theta(n)$ denotes

$$\theta(n)=[-A_{1n}\cdots -A_{pn}\ B_{1n}\cdots B_{qn}\ C_{1n}\cdots C_{rn}]^\tau.$$

It is also shown that the smaller μ_i ($i=1,\cdots,5$) is in Models 1 and 2, the smaller ϵ is in (8).

For estimating the unknown parameter θ in the modeled part of the system (1), we use the extended least squares (ELS) algorithm

$$\theta_{n+1}=\theta_n+a_nP_n\phi_n(y_{n+1}^\tau-\phi_n^\tau\theta_n), \tag{9}$$

$$P_{n+1}=P_n-a_nP_n\phi_n\phi_n^\tau P_n,\ a_n=(1+\phi_n^\tau P_n\phi_n)^{-1}, \tag{10}$$

$$\phi_n=[y_n^\tau\cdots y_{n-p+1}^\tau,\ u_n^\tau\cdots,\ u_{n-q+1}^\tau,$$
$$y_n^\tau-\phi_{n-1}^\tau\theta_n,\ \cdots,\ y_{n-r+1}^\tau-\phi_{n-r}^\tau\theta_{n-r+1}]^\tau \tag{11}$$

with $P_0=dI$, $d=mp+lq+mr$, and with θ_0 arbitrarily chosen.

III. Robust Parameter Estimation

For the exact model we need the following standard condition.

Assumption 1: $C^{-1}(z^{-1})-1/2I$ is strictly positive real (SPR).

The SPR requirement has been relaxed for the local stability of deterministic adaptive algorithms with the averaging method for persistently excited system in [9] and [10], however, for the global stability, the SPR condition is still needed.

Set

$$\phi_n^0=[y_n^\tau\cdots y_{n-p+1}^\tau,\ u_n^\tau\cdots,\ u_{n-q+1}^\tau,\ w_n^\tau,\cdots,\ w_{n-r+1}^\tau]^\tau \tag{12}$$

$$\phi_n^\xi=\phi_n-\phi_n^0 \tag{13}$$

and denote by $\lambda_{\max}(n)$ ($\lambda_{\max}^0(n)$) and $\lambda_{\min}(n)$ ($\lambda_{\min}^0(n)$) the maximum and minimum eigenvalue of

$$\sum_{i=0}^{n-1}\phi_i\phi_i^\tau+\frac{1}{d}I\left(\sum_{i=0}^{n-1}\phi_i^0\phi_i^{0\tau}+\frac{1}{d}I\right),$$

respectively.

Define condition numbers K and K_0 as follows:

$$K=\limsup_{n\to\infty}\lambda_{\max}(n)/\lambda_{\min}(n), \tag{14}$$

$$K_0=\limsup_{n\to\infty}\lambda_{\max}^0(n)/\lambda_{\min}^0(n). \tag{15}$$

Lemma 1: For the system described by (1)–(8) and ELS algorithm (9)–(11), if Assumption 1 is satisfied, then

i) $\limsup_{n\to\infty}\|\theta_n-\theta\|\leq c_o\sqrt{K}\epsilon$, a.s.

where $c_o>0$ is a constant independent of K and ϵ, and ϵ and K are given in (8) and (14), respectively.

ii) $K\leq 4K_0/[1-c_1(1+2K_0)\epsilon^2]$, a.s.

provided that $K_0<\infty$ and $\epsilon<1/\sqrt{c_1(1+2K_0)}$, where K_0 is defined by (15) and c_1 is a constant independent of K, K_0, and ϵ.

iii) $\sum_{i=0}^{n+1}\|\phi_i^\xi\|^2\leq c_2\epsilon^2r_n$, a.s.

where $c_2>0$ is a constant independent of ϵ, and r_n is defined by

$$r_n=1+\sum_{i=0}^{n}\|\phi_i\|^2. \tag{16}$$

Proof: Conclusions i) and ii) follow from [11, Theorem 1], while conclusion iii) follows from (39) of that paper.

Remark 2: From Lemma 1 we see that the ELS algorithm contains some degree of robustness provided that the condition number $K_0<\infty$, i.e., the persistence of excitation condition is satisfied.

We now show how to design the control law so that this requirement is met.

In stochastic adaptive control, it is well known that if the control law is obtained by the certainty equivalence principle, then the parameter estimate, in general is not strongly consistent due to lack of sufficient excitations (e.g., [14]), even though the adaptive controller is asymptotically optimal (e.g., [13]). In [15], it is suggested to disturb the reference signal by a white noise dither with constant variance to get persistently exciting signals, and hence the consistent parameter estimate in an adaptive tracking system. The controller designed in such a way is termed the continuously disturbed control [15]. Under this kind of controller, the suboptimality of the tracking error only is achieved. A similar treatment was used in [16], where the modified least-squares based algorithm was analyzed. The problem of how to achieve simultaneously the strong consistency of parameter estimates and the minimality of tracking error or quadratic cost is studied in [17], [18] by adding a random dither with diminishing variance to the controller. This method, called the attenuating excitation technique [17]–[19], is based on the fact that strong consistency of estimates can still be established when the standard persistence of excitation condition is not satisfied. A related method adopted in [20] is to occasionally use white-noise probing inputs.

In the presence of unmodeled dynamics, however, some robustness in the estimation algorithm can only be guaranteed for persistently exciting signals as studied in the deterministic case [9], [10] and examined in the stochastic case [11] (see also Lemma 1 and Remark 2). This is why we use random dither with constant covariance rather than diminishing covariance in the sequel.

Let $\{v_n\}$ be a sequence of l-dimensional independent and identically distributed (iid) random vectors independent of $\{w_n\}$ and such that

$$Ev_n=0,\ Ev_nv_n^\tau=\mu I,\ \mu>0,\ \|v_n\|\le\sigma,\ n\ge 0. \tag{17}$$

Without loss of generality, assume

$$\mathfrak{F}_n=\sigma\text{-algebra generated by } \{v_i,\ w_i,\ i\le n\} \tag{18}$$

and set

$$\mathfrak{F}'_n=\sigma\text{-algebra generated by } \{v_{i-1},\ w_i,\ i\le n\}. \tag{19}$$

Let the desired control u_n^0 be $\mathfrak{F}'_n$-measurable and let the input u_n applied to system (1) be defined by

$$u_n=u_n^0+v_n. \tag{20}$$

Define a sequence of auxiliary variables $\{z_n\}$ by

$$A(z^{-1})z_n=B(z^{-1})u_{n-1}+C(z^{-1})w_n \tag{21}$$

with the same $\{u_n\}$, $\{w_n\}$, $A(z^{-1})$, $B(z^{-1})$, and $C(z^{-1})$ as those for system (1).

We note that unlike $\{y_n\}$ the process $\{z_n\}$ is not influenced by the unmodeled dynamics $\{\eta_n\}$, and satisfies an exact ARMAX model. This allows us to apply the existing results to $\{z_n\}$ first, and then to get the desired results for $\{y_n\}$ by the connection between (1) and (21).

Set

$$\psi_n^0=[z_n^\tau,\ \cdots,\ z_{n-p+1}^\tau,\ u_n^\tau,\ \cdots,\ u_{n-q+1}^\tau,\ w_n^\tau,\ \cdots,\ w_{n-r+1}^\tau]^\tau, \tag{22}$$

$$\psi_n^1=[\det A(z^{-1})]\psi_n^0,\ \phi_n^1=[\det A(z^{-1})]\phi_n^0 \tag{23}$$

where ϕ_n^0 is defined by (12).

In the sequel denote by $\lambda_{\min}(X)$ the minimum eigenvalue of a matrix X, and by $a_n = O(b_n)$ we mean that $\|a_n\| \le Cb_n$ for all n and for some constant $C > 0$ which depends neither on n nor on ϵ, where the norm $\|X\|$ for a complex matrix is defined by $\|X\| = \sqrt{\operatorname{tr} XX^*}$.

We need the following identifiability condition for the nominal model.

Assumption 2: $A(z^{-1})$, $B(z^{-1})$, and $C(z^{-1})$ have no common left factor and A_p is of full rank.

Lemma 2: For the auxiliary system defined by (20)–(23), if Assumption 2 is satisfied and if

$$\sum_{i=0}^{n}(\|u_i\|^2+\|z_i\|^2)\le bn, \qquad \text{a.s.}$$

then for sufficiently large n

$$\lambda_{\min}\left(\sum_{i=0}^{n}\psi_i^1\psi_i^{1\tau}\right)\ge\gamma_o n, \qquad \text{a.s.}$$

where $\gamma_o > 0$ and $b > 0$ are constants independent of ϵ.

Proof: The proof is essentially the same as that for [19, eq. (46)] if we note that the ϵ, δ, and α which appeared in [19] are equal to 0, 0, and 1, respectively, in the present case.

In the following theorem the requirements on the condition number in Lemma 1 are transferred to more easily verifiable conditions imposed on the growth rate of the input–output data.

Theorem 1: Consider the system and algorithm defined by (1)–(11), with Assumptions 1 and 2 satisfied. Let the control u_n defined by (20) be applied to (1) and assume that

$$\sum_{i=0}^{n}(\|u_i\|^2+\|y_i\|^2)\le Mn, \qquad n>0. \tag{24}$$

Then the ELS algorithm has a margin of robustness in the sense that there exists a constant $\epsilon_o^* > 0$ such that

$$\limsup_{n\to\infty}\|\theta_n-\theta\|\le C_3\epsilon,\ \epsilon\in[o,\ \epsilon_o^*), \qquad \text{a.s.}$$

where $M > 0$, $C_3 > 0$ are constants independent of ϵ.

Proof: Set

$$\phi_n^\eta=[\eta_n^\tau,\ \eta_{n-1}^\tau\ \cdots\ \eta_{n-p+1}^\tau,\ 0\ \cdots\ 0]^\tau.$$

By (1), (12), (21), and (22) we have

$$mp\left\{\begin{bmatrix} A(z^{-1}) & 0 & 0\\ & \ddots & \\ 0 & A(z^{-1}) & \vdots\\ 0 & \cdots & I_{mr+ql}\end{bmatrix}\right.(\phi_n^0-\psi_n^0)=\phi_n^\eta,$$

hence, by (23)

$$\phi_n^1=\psi_n^1+\zeta_n \tag{25}$$

where

$$\zeta_n=\begin{bmatrix}\operatorname{Adj} A(z^{-1}) & 0 & 0\\ & \ddots & \\ 0 & \operatorname{Adj} A(z^{-1}) & \vdots\\ 0 & \cdots & I_{mr+ql}\end{bmatrix}\phi_n^\eta. \tag{26}$$

From (7), (8), (24), and (26) it is easy to see that

$$\sum_{i=0}^{n-1}\|\zeta_i\|^2\le C_4\epsilon^2 n, \qquad \text{for some } C_4>0. \tag{27}$$

For any $x\in R^d$ from (25), it follows that

$$\|x^\tau\psi_n^1\|^2\le 2\|x^\tau\phi_n^1\|^2+2\|x^\tau\zeta_n\|^2$$

and, hence,

$$\lambda_{\min}\left(\sum_{i=0}^{n-1}\psi_i^1\psi_i^{1\tau}\right)\le 2\lambda_{\min}\left(\sum_{i=0}^{n-1}\phi_i^1\phi_i^{1\tau}\right)+2C_4\epsilon^2 n.$$

From here and Lemma 2 it follows that

$$\lambda_{\min}\left(\sum_{i=0}^{n-1}\phi_i^1\phi_i^{1\tau}\right)\geq\frac{1}{2}(\gamma_o-2C_4\epsilon^2)n. \tag{28}$$

Let

$$\det A(z^{-1})=a_o+a_1z^{-1}+\cdots+a_{mp}z^{-mp}.$$

By (23) and the Schwartz inequality and the fact that $\phi_n^o=0$, for $n<0$, it is easy to see that

$$\lambda_{\min}\left(\sum_{i=0}^{n}\phi_i^1\phi_i^{1\tau}\right)=\inf_{\|x\|=1}\sum_{i=0}^{n}(x^\tau\phi_i^1)^2=\inf_{\|x\|=1}\sum_{i=0}^{n}\left(\sum_{j=0}^{mp}a_jx^\tau\phi_{i-j}^o\right)^2\leq(mp+1)\sum_{j=0}^{mp}a_j^2\lambda_{\min}^0(n).$$

From this and (24)–(28) we see that

$$K_0=\limsup_{n\to\infty}\lambda_{\max}^0(n)/\lambda_{\min}^0(n)\leq\left\{2[(p+q)M+r\operatorname{tr}R](mp+1)\sum_{j=0}^{mp}a_j^2\right\}\Big/(\gamma_o-2C_4\epsilon^2)<\infty \tag{29}$$

provided that $\epsilon<\sqrt{\gamma_o/(2C_4)}$. Take ϵ_1^*: $0<\epsilon_1^*<\sqrt{\gamma_o/(2C_4)}$ and set

$$K_o^*=\left\{2[(p+q)M+r\operatorname{tr}R](mp+1)\sum_{j=0}^{mp}a_j^2\right\}\Big/[\gamma_o-2C_4(\epsilon_1^*)^2].$$

Then Theorem 1 follows by taking ϵ_o^*: $0<\epsilon_o^*<1/\sqrt{C_1}(1+2K_o^*$ and by noting (29) and Lemma 1.

IV. Robust Stochastic Adaptive Controller

Let $\{y_n^*\}$ be a sequence of bounded deterministic reference signals $\|y_n^*\|\leq l_o$. The purpose of this section is to design an adaptive control law so that the closed-loop system is stable and both the estimation error and the tracking error are of order ϵ. We need the following condition for the modeled part of system (1).

Assumption 3: The dimension of the system input equals that of the output, i.e., $l=m$, zeros of $\det A(z^{-1})$ and $\det B(z^{-1})$ lie inside the closed unit disk, and the upper bounds for the following quantities are available:

$$\|\theta\|,\ \operatorname{tr}R,\ \|B_1^{-1}\|,\ \sup_{|z|=1}\|A^{-1}(z)B(z)\|,\ \sup_{|z|=1}\|A^{-1}(z)C(z)\|,\ \sup_{|z|=1}\|B^{-1}(z)A(z)\|$$

and

$$\sup_{|z|=1}\|B^{-1}(z)C(z)\|.$$

These bounds are, respectively, denoted by α, β, γ, K_{AB}, K_{AC}, K_{BA}, and K_{BC}.

We first present the adaptive control law and then explain its form.

Define two sequences of stopping times $\{\tau_k\}$ and $\{\sigma_k\}$ with

$$1=\tau_1<\sigma_1<\tau_2<\sigma_2<\cdots$$

as follows:

$$\sigma_k=\sup\left\{t>\tau_k:\ \sum_{i=\tau_k}^{j-1}\|u_i^1\|^2\leq M_0(j-1)+\|u_{\tau_k}^1\|^2,\ \forall j\in(\tau_k,t]\right\}, \tag{30}$$

$$\tau_{k+1}=\inf\left\{t>\sigma_k:\ 2^k\sum_{i=\tau_k}^{\sigma_k-1}\|u_i^1\|^2\leq t,\ \|u_t^1\|^2\leq M_1t\right\} \tag{31}$$

where u_n^1 is defined from

$$B_{1n}u_n^1=(B_{1n}u_n-\theta_n^\tau\phi_n)+y_{n+1}^* \tag{32}$$

if $\det B_1\neq0$, otherwise $u_n^1=0$, where B_{1n} is the estimate for B_1 given by θ_n, and where M_0 and M_1 are chosen to satisfy

$$M_0\geq[32(\alpha+1)^2\gamma^2(p+q+r)+2][(4K_{AB}^2+1)M_o^1+(4K_{AC}^2+1)\beta+2(\sigma^2+1)], \tag{33}$$

$$M_1\geq32(\alpha+1)^2\gamma^2(p+q+r)[(4K_{AB}^2+1)(\sigma^2+1)+(4K_{AC}^2+1)\beta+2] \tag{34}$$

where

$$M_o^1=2\sigma^2+8K_{BC}^2\beta+1+16K_{BA}^2M_o^2,$$

$$M_o^2=6l_o^2+6(\alpha+1)^2\sigma^2+6\beta+8K_{AB}^2(\sigma^2+1)+4K_{AC}^2\beta+\frac{1}{2}.$$

It is easy to see by (11) that the term $(B_{1n}u_n-\theta_n^\tau\phi_n)$ on the right-hand side of (32) is free of u_n, therefore, u_n^1 can indeed be determined from (32).

The *adaptive control law* is defined by

$$u_n=u_n^0+v_n \tag{35}$$

with v_n satisfying (17) and with u_n^0 given by

$$u_n^0=\begin{cases}u_n^1, & \text{if } n\in[\tau_k,\sigma_k)\\ 0, & \text{if } n\in[\sigma_k,\tau_{k+1}).\end{cases} \tag{36}$$

We now make some explanations of the proposed controller. From Theorem 1 we see that, for the robustness of the parameter estimates, besides conditions on the nominal model structure, there are growth rate requirements for the system input and output when the external excitation is applied to the controller. (It will be seen shortly from Lemma 3 ii) that these requirements are essentially the requirement for the system input under Assumption 3.) However, the standard adaptive tracking controller u_n^1 defined by (32) (e.g., [13]) may not meet this requirement because of the unmodeled dynamics. This is the motivation for introducing stopping times $\{\tau_k\}$ and $\{\sigma_k\}$ and truncating the controller at randomly varying bounds.

From the random time τ_k, by (35) and (36) we see that the adaptive control u_n is defined as u_n^1 excited by v_n as far as $n<\sigma_k$, where σ_k is the first time when the value of $1/j\ \Sigma_{i=\tau_k}^j\|u_i^1\|^2$ is greater, roughly speaking, than M_0. During this time interval the main purpose of the controller is to track the desired signal y_n^*. From the random time σ_k the adaptive control is defined as a pure dither v_n until $n<\tau_{k+1}$, where τ_{k+1} indicates the time when $\|u_n^1\|^2/n$ is less than M_1 (a technical bound) and when another technical condition is satisfied. During the time interval $[\sigma_k,\tau_{k+1})$, the purpose of the controller is to slow down the possible undesirable growth rate of signals. This is possible because of the stability assumption made on the nominal model. Thus, the control law operates by switching on the tracking feedback controller during intervals $[\tau_k, \sigma_k)$ and reverting to simpler dither (a persistently exciting signal) applied to the stable plant in case the local signal growth rate exceeds the *a priori* bound, flagged by σ_k. We seek to prove then that for some k, $\tau_k < \infty$, $\sigma_k = \infty$.

We now present the main results of this paper.

Theorem 2: For the system and algorithm defined by (1)-(11), if Assumptions 1-3 hold, and the control law is defined by (35) and (36), then there exists a constant $\epsilon^* > 0$ such that for any $\epsilon \in [0, \epsilon^*)$ the closed-loop system has the following properties:

$$\text{i) } \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} (\|y_i\|^2 + \|u_i\|^2) < \infty, \quad \text{a.s.} \tag{37}$$

$$\text{ii) } \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \|y_i - y_i^*\|^2 \le \operatorname{tr} R + c\epsilon^2 + \|B_1\|^2\sigma^2, \tag{38}$$

$$\text{iii) } \limsup_{n\to\infty} \|\theta_n - \theta\| \le c'\epsilon \tag{39}$$

where ϵ, R, and σ are given by (8), (7), and (17), respectively, and c and c' are constants independent of ϵ.

We separate the proof into lemmas.

Lemma 3: If Assumptions 1 and 3 are fulfilled, then for small ϵ, the following estimation holds:

$$\text{i) } \sum_{i=0}^{n} \|\eta_i\|^2 = O(\epsilon^2) \left[\sum_{i=0}^{n} (\|y_i\|^2 + \|u_i\|^2) + n \right], \quad \text{a.s.} \tag{40}$$

$$\text{ii) } \sum_{i=0}^{n} \|y_{i+1}\|^2 \le 4K_{AB}^2 \sum_{i=0}^{n} \|u_i\|^2 + 4K_{AC}^2 \beta n + O(\epsilon^2)n, \quad \text{a.s.} \tag{41}$$

$$\text{iii) } \sum_{i=0}^{n} \|u_i\|^2 \le 4K_{BA}^2 \sum_{i=0}^{n} \|y_{i+1}\|^2 + 4K_{BC}^2 \beta n + O(\epsilon^2)n, \quad \text{a.s.} \tag{42}$$

$$\text{iv) } r_n \le 4(p+q+r) \left[\sum_{i=0}^{n} (\|y_i\|^2 + \|u_i\|^2) + \beta n \right]$$

where β, K_{AB}, K_{AC}, K_{BA}, and K_{BC} are given in Assumption 3.

Proof: Conclusion i) follows from (8) and (7) immediately, while conclusion iv) follows from Lemma 1 iii), (13), and (16) if we note that

$$r_n = 1 + \sum_{i=0}^{n} \|\phi_i^0 + \phi_i^\xi\|^2$$
$$\le 1 + 2\sum_{i=0}^{n} \|\phi_i^0\|^2 + 2\sum_{i=0}^{n} \|\phi_i^\xi\|^2$$
$$\le 1 + 2(p+q+r) \left[\sum_{i=0}^{n} (\|y_i\|^2 + \|u_i\|^2) + \beta n \right] + O(\epsilon^2) r_n$$
$$\le 4(p+q+r) \left[\sum_{i=0}^{n} (\|y_i\|^2 + \|u_i\|^2) + \beta n \right]$$

provided that ϵ is appropriately small.

We now proceed to prove conclusions ii) and iii). We note that ii) and iii) are reformulations of stability and minimum phase assumptions on the plant, respectively.

By (1) we have

$$\sum_{i=0}^{n} \|y_{i+1}\|^2 \le 3 \sum_{i=0}^{n} \|A^{-1}(z^{-1})B(z^{-1})u_i\|^2 + 3\sum_{i=0}^{n} \|A^{-1}(z^{-1})C(z^{-1})w_{i+1}\|^2 + 3\sum_{i=0}^{n} \|A^{-1}(z^{-1})\eta_i\|^2. \tag{43}$$

Since $A(z^{-1})$ is stable, we have the following expansion for $A^{-1}(z^{-1})B(z^{-1})$:

$$A^{-1}(z^{-1})B(z^{-1}) = \sum_{j=0}^{\infty} H_j z^{-j} \tag{44}$$

with $\|H_j\| = O(\lambda^j)$, $j \ge 0$, for some $\lambda \in (0, 1)$.

We now show that

$$\sum_{i=0}^{n} \|A^{-1}(z^{-1})B(z^{-1})u_i\|^2 \le K_{AB}^2 \sum_{i=0}^{n} \|u_i\|^2. \tag{45}$$

This inequality is established for the scalar case in [21, p. 960, eq. (18)]; the present matrix case can also be proved by using a similar method. The key steps are first to define

$$u_j(m) = \begin{cases} u_j, & j \le m \\ 0, & j > m \end{cases}$$

and show that for any $m > 0$

$$\sum_{(t,k=0)}^{\infty} \sum_{s=m+1}^{(m+t)\wedge(m+k)} u_{s-t}^\tau H_t^\tau H_k u_{s-k} = \sum_{s=m+1}^{\infty} \left\| \sum_{k=0}^{\infty} H_k u_{s-k}(m) \right\|^2 \ge 0$$

then to prove that

$$K_{AB}^2 \sum_{i=0}^{n} \|u_i\|^2 \ge \frac{1}{2\pi} \int_0^{2\pi} \left\| \sum_{j=0}^{n} A^{-1}(e^{i\phi})B(e^{i\phi})u_j e^{ij\phi} \right\|^2 d\phi$$
$$= \sum_{(t,k=0)}^{\infty} \sum_{s=0}^{(k+n)\wedge(t+n)} u_{s-t}^\tau H_t^\tau H_k u_{s-k}$$
$$\ge \sum_{(t,k=0)}^{\infty} \sum_{s=0}^{n} u_{s-t}^\tau H_t^\tau H_k u_{s-k}$$
$$= \sum_{i=0}^{n} \|A^{-1}(z^{-1})B(z^{-1})u_i\|^2. \tag{46}$$

Similar to (45) we have

$$\sum_{i=0}^{n} \|A^{-1}(z^{-1})C(z^{-1})w_{i+1}\|^2 \le K_{AC}^2 \sum_{i=0}^{n} \|w_{i+1}\|^2. \tag{47}$$

Further by (40) it is easy to see that

$$\sum_{i=0}^{n} \|A^{-1}(z^{-1})\eta_i\|^2 = O(\epsilon^2) \left[\sum_{i=0}^{n} (\|y_i\|^2 + \|u_i\|^2) + n \right]. \tag{48}$$

Putting (46), (47), (48), and (7) into (43), we can get (41) for appropriately small ϵ. Finally, (42) can be established by the same way as (41). This proves the lemma.

Lemma 4: Under the conditions of Theorem 2, the ELS algorithm has some degree of robustness in the sense that for small ϵ

$$\limsup_{n\to\infty} \|\theta_n - \theta\| = O(\epsilon), \qquad \text{a.s.} \tag{49}$$

Proof: We first note that if $\tau_k < \infty$ for some k, then by (30) and (31) we have

$$\begin{aligned}
&\sup_{\tau_k \le n < \sigma_k} \frac{1}{n} \sum_{i=1}^{n} \|u_i\|^2 \\
&= \sup_{\tau_k \le n < \sigma_k} \frac{1}{n} \left(\sum_{i=\tau_1}^{\sigma_1 - 1} + \sum_{i=\sigma_1}^{\tau_2 - 1} + \cdots + \sum_{i=\sigma_{k-1}}^{\tau_k - 1} + \sum_{i=\tau_k}^{n} \right) \|u_i\|^2 \\
&\le \frac{1}{\tau_k} \Bigg[2(\sigma_1 - \tau_1)\sigma^2 + 2\frac{\tau_2}{2} + 2(\tau_2 - \sigma_1)\sigma^2 \\
&\quad + \cdots + 2(\tau_k - \sigma_{k-1})\sigma^2 \Bigg] + \sup_{\tau_k \le n < \sigma_k} \frac{1}{n} \sum_{i=\tau_k}^{n} \|u_i\|^2 \\
&\le 2(\sigma^2 + 1) + \sup_{\tau_k \le n < \sigma_k} \frac{1}{n} \sum_{i=\tau_k}^{n} \|u_i\|^2 \\
&\le 2(\sigma^2 + 1) + 2 \sup_{\tau_k \le n < \sigma_k} \left[\frac{1}{n} (n - \tau_k + 1)\sigma^2 + \frac{1}{n} \sum_{i=\tau_k}^{n} \|u_i^1\|^2 \right] \\
&\le 4\sigma^2 + 2 + 2 \sup_{\tau_k \le n < \sigma_k} \left[\frac{1}{n} (M_o n + \|u_{\tau_k}^1\|^2) \right] \\
&\le 2(2\sigma^2 + 1 + M_o + M_1)
\end{aligned} \tag{50}$$

and further, if $\sigma_k < \infty$ for some k, then by (35), (36), and (50)

$$\begin{aligned}
&\sup_{\tau_k \le n < \sigma_{k+1}} \frac{1}{n} \sum_{i=1}^{n} \|u_i\|^2 \\
&\le \sup_{\tau_k \le n < \sigma_k} \frac{1}{n} \sum_{i=1}^{n} \|u_i\|^2 + \sup_{\sigma_k \le n < \tau_{k+1}} \frac{1}{n} \sum_{i=\sigma_k}^{n} \|u_i\|^2 \\
&\le 2(2\sigma^2 + 1 + M_o + M_1) + \sup_{\sigma_k \le n < \tau_{k+1}} \frac{1}{n} [(n - \sigma_k + 1)\sigma^2] \\
&\le 5\sigma^2 + 2(1 + M_o + M_1).
\end{aligned} \tag{51}$$

By Theorem 1 and Lemma 3 ii) we know that for (49) it is sufficient to show that there exists a constant $M > 0$ independent of ϵ, such that

$$\sum_{i=1}^{n} \|u_i\|^2 \le Mn, \qquad n > 0. \tag{52}$$

We verify (52) by considering the following three cases.

i) If $\tau_k < \infty$, $\sigma_k = \infty$ for some k, then by (50) and (51) we have

$$\begin{aligned}
&\sup_{n\ge 1} \frac{1}{n} \sum_{i=1}^{n} \|u_i\|^2 \\
&\le \sup_{1 \le n < \tau_k} \frac{1}{n} \sum_{i=1}^{n} \|u_i\|^2 + \sup_{\tau_k \le n < \infty} \frac{1}{n} \sum_{i=1}^{n} \|u_i\|^2 \\
&\le 9\sigma^2 + 4(1 + M_o + M_1)
\end{aligned}$$

and (52) is true for $M = 9\sigma^2 + 4(1 + M_o + M_1)$.

ii) If $\sigma_k < \infty$, $\tau_{k+1} = \infty$ for some k, then by (51) we have

$$\begin{aligned}
&\sup_{n\ge 1} \frac{1}{n} \sum_{i=1}^{n} \|u_i\|^2 \\
&\le \sup_{1 \le n < \tau_k} \frac{1}{n} \sum_{i=1}^{n} \|u_i\|^2 + \sup_{\tau_k \le n < \infty} \frac{1}{n} \sum_{i=1}^{n} \|u_i\|^2 \\
&\le 10\sigma^2 + 4(1 + M_o + M_1)
\end{aligned} \tag{53}$$

and (52) holds for $M = 10\sigma^2 + 4(1 + M_o + M_1)$.

iii) If $\tau_k < \infty$, $\sigma_k < \infty$, for all k, then by (51) we see that (52) is also true for $M = 5\sigma^2 + 2(1 + M_o + M_1)$.

This completes the proof of the lemma.

Lemma 5: Under the conditions of Theorem 2, if ϵ is small and n is large enough, then

$$\|u_n^1\|^2 \le 32(\alpha+1)^2\gamma^2(p+q+r) \left\{ (4K_{AB}^2 + 1) \cdot \frac{1}{n} \sum_{i=1}^{n} \|u_i\|^2 + (4K_{AC}^2 + 1)\beta + 2 \right\} n \tag{54}$$

where u_n^1 is defined by (32), and α, β, γ, K_{AB}, and K_{AC} are given in Assumption 3.

Proof: By ii) and iv) of Lemma 3 it is easy to see that

$$r_n \le 4(p+q+r) \left\{ (4K_{AB}^2 + 1) \cdot \sum_{i=0}^{n} \|u_i\|^2 + (4K_{AC}^2 + 1)\beta n \right\} + O(\epsilon^2) n$$

and then for small ϵ

$$\begin{aligned}
\|\phi_n\|^2 &\le r_n \\
&\le 4(p+q+r) \left\{ (4K_{AB}^2 + 1) \frac{1}{n} \sum_{i=0}^{n} \|u_i\|^2 + (4K_{AC}^2 + 1)\beta + 1 \right\} n.
\end{aligned} \tag{55}$$

By (32) we have

$$B_1 u_n^1 = (B_1 - B_{1n}) u_n^1 - (\theta_n^\tau \phi_n - B_{1n} u_n) + y_{n+1}^*. \tag{56}$$

Since all zeros of $\det B(z^{-1})$ are inside the closed unit disk, it follows that B_1 is nondegenerate, and by Lemma 4 we know that for small ϵ and appropriately large n

$$\|B_1^{-1}\| \|\theta_n - \theta\| \le \frac{1}{2}, \quad \|\theta_n - \theta\| < 1.$$

Therefore, by (56) we have

$$\begin{aligned}
\|u_n^1\| &\le \|B_1^{-1}\| [\|B_1 - B_{1n}\| \|u_n^1\| + (\|\theta\| + 1)\|\phi_n\| + \|y_{n+1}^*\|] \\
&\le \frac{1}{2} \|u_n^1\| + [(\|\theta\| + 1)\|\phi_n\| + \|y_{n+1}^*\|] \|B_1^{-1}\|.
\end{aligned} \tag{57}$$

Consequently, by (55) and (57) we see

$$\begin{aligned}
\|u_n^1\|^2 &\le 8(\alpha+1)^2\gamma^2 \|\phi_n\|^2 + 8\|y_{n+1}^*\|^2 \|B_1^{-1}\|^2 \\
&\le 32(\alpha+1)^2\gamma^2(p+q+r) \left\{ (4K_{AB}^2 + 1) \frac{1}{n} \sum_{i=0}^{n} \|u_i\|^2 + (4K_{AC}^2 + 1)\beta + 2 \right\} n
\end{aligned}$$

for sufficiently large n, since $\{y_n^*\}$ is a bounded sequence. This completes the proof of Lemma 5.

Lemma 6: Under the conditions of Theorem 2, there is some $k > 0$ such that

$$\tau_k < \infty, \ \sigma_k = \infty$$

for small ϵ.

Proof: We need only to prove the impossibility of the following two cases:

i) $\sigma_k < \infty$, $\tau_{k+1} = \infty$, for some k,

ii) $\tau_k < \infty$, $\sigma_k < \infty$, for any k.

If case i) occurs, then by (35) and (36)

$$u_n = v_n, \qquad \forall n \geq \sigma_k$$

and, hence, for n large enough, we will have

$$\frac{1}{n} \sum_{i=0}^{n} \|u_i\|^2 \leq \sigma^2 + 1$$

and then by Lemma 5 we know that for small ϵ and for all sufficiently large n

$$\|u_n^1\|^2 \leq 32(\alpha+1)^2\gamma^2(p+q+r)\{(4K_{AB}^2+1)(\sigma^2+1) + (4K_{AC}^2+1)\beta + 2\}n. \quad (58)$$

From (31) and (34), we see that (58) contradicts $\tau_{k+1} = \infty$. Hence, case i) is impossible.

If case ii) holds, then by Lemma 3 and (35) and (36) we have for any $k > 1$.

$$\begin{aligned}\frac{1}{\sigma_k}\sum_{i=0}^{\sigma_k} \|u_i\|^2 &\leq \frac{\sigma^2}{\sigma_k} + \frac{1}{\sigma_k}\sum_{i=0}^{\sigma_k-1}\|u_i\|^2 \\ &\leq \sigma^2 + \frac{1}{\sigma_k}\left[4K_{BA}^2 \sum_{i=0}^{\sigma_k-1}\|y_{i+1}\|^2 + 4K_{BC}^2\beta(\sigma_k-1) + O(\epsilon^2)(\sigma_k-1)\right] \\ &\leq \sigma^2 + 4K_{BC}^2\beta + O(\epsilon^2) + \frac{1}{\sigma_k}\left[4K_{BA}^2\sum_{i=0}^{\sigma_k-1}\|y_{i+1}\|^2\right]. \end{aligned} \quad (59)$$

By Lemma 4, it is easy to see that for small ϵ and large enough n

$$\det B_{1n} \neq 0$$

since $\det B_1 \neq 0$.

Therefore, by (32), (35), and (36) it is seen that for appropriately large k and all $i \in [\tau_\kappa, \sigma_k)$

$$\theta_i^\tau \phi_i = y_{i+1}^* + B_{1i}v_i$$

and by (1) and (13) we have for any $i \in [\tau_\kappa, \sigma_k)$

$$\begin{aligned} y_{i+1} - y_{i+1}^* &= -\theta_i^\tau\phi_i + B_{1i}v_i + \theta^\tau\phi_i^0 + w_{i+1} + \eta_i \\ &= B_{1i}v_i + w_{i+1} + \eta_i - \theta_i^\tau\phi_i^\xi + \tilde{\theta}_i^\tau\phi_i^0, \end{aligned} \quad (60)$$

where $\tilde{\theta}_i = \theta - \theta_i$.

From (60), Lemma 1 iii), and Lemmas 3 and 4 we see that

$$\begin{aligned} \sum_{i=\tau_k}^{\sigma_k-1} \|y_{i+1}\|^2 &\leq 6\sum_{i=\tau_k}^{\sigma_k-1}\{\|y_{i+1}^*\|^2 + \|B_{1i}v_i\|^2 + \|w_{i+1}\|^2 + \|\eta_i\|^2 + \|\theta_i^\tau\phi_i^\xi\|^2 + \|\tilde{\theta}_i^\tau\phi_i^0\|^2\} \\ &\leq 6\sum_{i=\tau_k}^{\sigma_k-1}[l_o^2 + (\|B_1\|+1)^2\sigma^2 + \|w_{i+1}\|^2 + \|\eta_i\|^2 + (\|\theta\|+1)^2\|\phi_i^\xi\|^2 + O(\epsilon^2)\|\phi_i^0\|^2] \\ &\leq 6\sigma_k[l_o^2 + (\|B_1\|+1)^2\sigma^2 + \beta] + O(\epsilon^2)\left[\sum_{i=0}^{\sigma_k-1}(\|y_i\|^2 + \|u_i\|^2) + \sigma_k\right]. \end{aligned} \quad (61)$$

By an argument used in the proof of (50) it is easy to see

$$\frac{1}{\tau_k}\sum_{i=0}^{\tau_k-1}\|u_i\|^2 \leq 2(\sigma^2+1).$$

Consequently, by Lemma 3 we have

$$\begin{aligned} \sum_{i=0}^{\tau_k-1}\|y_{i+1}\|^2 &\leq 4K_{AB}^2\sum_{i=0}^{\tau_k-1}\|u_i\|^2 + 4K_{AC}^2\beta\tau_k + O(\epsilon^2)\tau_k \\ &\leq 8K_{AB}^2(\sigma^2+1)\tau_k + 4K_{AC}^2\beta\tau_k + O(\epsilon^2)\tau_k. \end{aligned} \quad (62)$$

Combining (61) and (62) leads to

$$\sum_{i=0}^{\sigma_k-1}\|y_{i+1}\|^2 \leq \sigma_k\{6[l_o^2 + (\alpha+1)^2\sigma^2 + \beta] + 8K_{AB}^2(\sigma^2+1) + 4K_{AC}^2\beta + O(\epsilon^2)\} + O(\epsilon^2)\sum_{i=0}^{\sigma_k-1}(\|y_i\|^2 + \|u_i\|^2).$$

From this it follows that for small ϵ

$$\sum_{i=0}^{\sigma_k-1}\|y_{i+1}\|^2 \leq 2\sigma_k\left\{6[l_o^2 + (\alpha+1)^2\sigma^2 + \beta] + 8K_{AB}^2(\sigma^2+1) + 4K_{AC}^2\beta + \frac{1}{2}\right\} + O(\epsilon^2)\sum_{i=0}^{\sigma_k-1}\|u_i\|^2. \quad (63)$$

Finally, putting (63) into (59), we find that for appropriately small ϵ,

$$\frac{1}{\sigma_k}\sum_{i=0}^{\sigma_k}\|u_i\|^2 \leq 2\sigma^2 + 8K_{BC}^2\beta + 1 + 16K_{BA}^2 \left\{6[l_o^2 + (\alpha+1)^2\sigma^2 + \beta] + 8K_{AB}^2(\sigma^2+1) + 4K_{AC}^2\beta + \frac{1}{2}\right\} \quad (64)$$

and then by Lemma 5 for sufficiently large k

$$\|u^1_{\sigma_k}\|^2 \le 32(\alpha+1)^2\gamma^2(p+q+r)\left\{(4K^2_{AB}+1)[2\sigma^2+8K^2_{BC}\beta+1 + 16K^2_{BA}\left(6l_o^2+6(\alpha+1)^2\sigma^2+6\beta+8K^2_{AB}(\sigma^2+1) + 4K^2_{AC}\beta+\frac{1}{2}\right)\right] + (4K^2_{AC}+1)\beta+2\Bigg\}\sigma_k. \quad (65)$$

From (64) and (65) it is easy to see that

$$\begin{aligned}\sum_{i=\tau_k+1}^{\sigma_k}\|u^1_i\|^2 &= \sum_{i=\tau_k+1}^{\sigma_k-1}\|u^1_i\|^2+\|u^1_{\sigma_k}\|^2 \\ &\le 2\sum_{i=\tau_k+1}^{\sigma_k-1}\|u_i\|^2+2\sum_{i=\tau_k+1}^{\sigma_k-1}\|v_i\|^2+\|u^1_{\sigma_k}\|^2 \\ &\le [32(\alpha+1)^2\gamma^2(p+q+r)+2]\Bigg\{(4K^2_{AB}+1)\Bigg[2\sigma^2 + 8K^2_{BC}\beta+1+16K^2_{BA}\left(6l_o^2+6(\alpha+1)^2\sigma^2+6\beta + 8K^2_{AB}(\sigma^2+1)+4K^2_{AC}\beta+\frac{1}{2}\right)\Bigg] \\ &\quad + (4K^2_{AC}+1)\beta+2(\sigma^2+1)\Bigg\}\sigma_k \\ &\le M_o\sigma_k.\end{aligned}$$

But by the definition (30) for σ_k

$$\sum_{i=\tau_k+1}^{\sigma_k}\|u^1_i\|^2 > M_o\sigma_k.$$

The obtained contradiction proves the impossibility of case ii).

Proof of Theorem 2: By Lemma 4, Lemma 6, and (41), (52) there is $\epsilon^* > 0$ such that for any $\epsilon \in [0, \epsilon^*)$, conclusions i) and iii) hold true and there is some k such that

$$\tau_k < \infty, \ \sigma_k = \infty. \quad (66)$$

We now prove conclusion ii).

By (37), (39), Lemmas 1 and 3, and the independence of $\{v_i\}$ and $\{w_i\}$ from (60) and (66), it is not difficult to see that for any $\epsilon \in [0, \epsilon^*)$,

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n}\|y_i-y_i^*\|^2 \le \text{tr } R+O(\epsilon^2)+\|B_1\|^2\sigma^2.$$

This completes the proof of Theorem 2.

V. Conclusions

For open-loop stable stochastic systems with unmodeled dynamics and possibly unbounded random noise, we have designed an adaptive control algorithm so that the closed-loop system is stable and both the estimation error and the tracking error approach their minima when the unmodeled dynamics diminish. The estimates for the parameters appearing in the nominal model are given by the extended least-squares algorithm, whereas the control law is switched at stopping times to slow down the possible undesirable divergence rate of the signals, and is also disturbed by a sequence of random vectors bounded by an arbitrary small but fixed constant to provide sufficient excitation for robustness of identification.

Further research efforts are suggested to weaken the conditions imposed on the modeled part of system (1).

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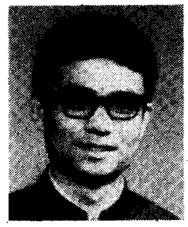

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