INTERNATIONAL JOURNAL OF ADAPTIVE CONTROL AND SIGNAL PROCESSING, VOL. 3, 1–14 (1989)

ADAPTIVE CONTROL FOR TIME-VARYING SYSTEMS: A COMBINATION OF MARTINGALE AND MARKOV CHAIN TECHNIQUES

LEI GUO AND SEAN P. MEYN

Department of Systems Engineering, RSPhysS, Australian National University, GPO Box 4, Canberra, ACT 2601, Australia

SUMMARY

Adaptive control problems of a first-order randomly time-varying stochastic system are considered. A class of adaptive controllers based on the Kalman filter is introduced and is analysed using a combination of martingale and Markov chain techniques. It is shown that both the expected value and sample path averages of the square of the output of the closed-loop system remain bounded and that the long-run cost is a continuous functional of the parameters of the controller and the distribution of the disturbance process. *These results hold even when the Gaussian assumption used in previous papers is removed and the a priori estimate of the noise variance is incorrect.*

KEY WORDS Random parameters Adaptive Stability Robustness Martingales Markov chains

1. INTRODUCTION

The principal objective in adaptive control theory is to find controllers that perform satisfactorily for systems which possess time-varying and only partially known dynamics.

There are few precise results for time-varying *stochastic* systems. In fact, the frequently used stochastic algorithms, e.g. stochastic gradient and least squares, can only be shown to be successful for some restricted classes of parameter variations. Specifically, it is required that the time-varying parameter is a constant plus a bounded martingale difference[1] or that the parameter varies in a small ball.[2] This is, as is well known, due to the fact that these algorithms, have the so-called *long-memory* property (i.e. the adaptation gain tends to zero). For systems which exhibit more complicated parameter variation, e.g. drifting parameters, it is believed that short-memory algorithms (i.e. algorithms with non-vanishing gain) will be more effective. However, the traditional analytical techniques break down since the system noise will have a more significant effect on the parameter estimates if such algorithms are used. Furthermore, stochastic Lyapunov techniques cannot be applied since none of the variables in the system or controller can be expected to converge in this case.

To the best of our knowledge, the first rigorous analysis of a short-memory minimum-variance adaptive controller applied to a time-varying stochastic system appeared in the paper by Meyn and Caines[3] where the ergodic theory of Markov chains is applied. In this paper it

This paper was recommended for publication by editor R. Bitmead

0890–6327/89/010001–14$07.00

Received 2 March 1988
Revised 18 July 1988

is shown that assuming a state-space model for the parameter process and Gaussianity of the noise process the optimal control may be computed using the Kalman filter. Under the appropriate conditions the following limits are shown to exist for all initial conditions:

$$\lim_{N\to\infty} E[y_N^2] = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} y_k^2 = \int y^2 \, d\pi$$

where π is an invariant probability on the state-space generating the output y. In Solo[4] it is shown that some of these results may also be established by applying standard limit theorems for martingales and utilizing the conditional Gaussian property of the output process. In both of the works of Meyn and Caines[3] and Solo[4] a Gaussianity assumption on the noise process is found to be crucial to the analysis.

The main objective of this paper is to use a combination of martingale and Markov chain techniques to show that the adaptive control system considered in References 3 and 4 is robust with respect to noise. We show:

(i) that both the expected value and sample path averages of the square of the output of the closed-loop system remain bounded even when the distribution of the disturbance process is non-Gaussian and the *a priori* estimate of the noise variances used in the implementation of the algorithm is incorrect.

(ii) that the long-run cost is a continuous functional of the controller parameters and the distribution of the disturbance process. Hence, if the disturbance is approximately Gaussian and the *a priori* estimate of its variance is approximately correct, then the control performance will be nearly optimal.

2. PROBLEM STATEMENT

The system analysed in References 3 and 4 is described by the following controlled time-varying AR(1) model:

$$y_{k+1} = \theta_k y_k + u_k + w_{k+1} \qquad k \geqslant 0 \tag{1}$$

with unknown parameters generated from a stable Markov model

$$\theta_{k+1} = \alpha\theta_k + e_{k+1} \qquad |\alpha| < 1 \tag{2}$$

We assume that the noise sequences $\{e_k\}$ and $\{w_k\}$ are mutually independent and also independent themselves, with zero mean, and satisfy

$$\sup_k E[|w_k|^{4+2\delta} + |e_k|^{4+2\delta}] < \infty \tag{3}$$

$$\sup_k E[|e_k|^{2+\delta}] < 1 \tag{4}$$

for some positive constant δ. The initial conditions are assumed to satisfy

$$E|y_0|^{4+2\delta} < \infty \qquad E|\theta_0|^{4+2\delta} < \infty \tag{5}$$

Denote

$$\sigma_w^2 = \sup_k E|w_k|^2 \qquad \sigma_e^2 = \sup_k E|e_k|^2 \tag{6}$$

An immediate application of (3) and (4) yields

$$\sigma_w^2 < \infty \qquad \sigma_e^2 < 1 \tag{7}$$

Let us now consider the following estimation algorithm (Kalman filter) for the unknown parameter θ_k:

$$\hat{\theta}_{k+1} = \alpha\hat{\theta}_k + \frac{\alpha P_k y_k}{\sigma_0^2 + P_k y_k^2}(y_{k+1} - \hat{\theta}_k y_k - u_k) \tag{8}$$

$$P_{k+1} = \alpha^2 P_k + \sigma_1^2 - \frac{\alpha^2 P_k^2 y_k^2}{\sigma_0^2 + P_k y_k^2} \tag{9}$$

where $\hat{\theta}_0$ and $P_0 > 0$, $\sigma_0^2 > 0$, $\sigma_1^2 > 0$ are deterministic constants and can be arbitrarily chosen (here σ_0^2 and σ_1^2 may be regarded as *a priori* estimates of σ_w^2 and σ_e^2 respectively).

Our objective is to minimize the variance of the output of the system, so we apply the 'certainty equivalent' minimum-variance controller

$$u_k = -\hat{\theta}_k y_k \tag{10}$$

It is known (e.g. References 4 and 5) that if the noise $\{w_k, e_k\}$ is a Gaussian white noise sequence, and if $\sigma_0^2 = \sigma_w^2$ and $\sigma_1^2 = \sigma_e^2$, then with appropriately chosen initial conditions the estimate $\hat{\theta}_k$ generated by (8) and (9) is the best estimate for θ_k, and P_k is the estimation error covariance:

$$\hat{\theta}_k = E[\theta_k \mid \mathscr{F}_{k-1}] \qquad P_k = E[(\tilde{\theta}_k)^2 \mid \mathscr{F}_{k-1}] \tag{11}$$

where $\mathscr{F}_k$ is the σ-algebra generated by $\{y_0, y_1, \ldots, y_k\}$ and $\tilde{\theta}_k$ is the estimation error

$$\tilde{\theta}_k = \theta_k - \hat{\theta}_k \tag{12}$$

In this case the control law (10) minimizes the criterion $E[y_{k+1}^2 \mid \mathscr{F}_k]$, and the stability of the system (1) with controller (10) applied has been proved[3,4] thanks to the important property (11).

In the non-Gaussian case the propery (11) fails and the estimate $\hat{\theta}_k$ generated by (8) and (9) may be far from optimal. Nevertheless, in this paper we will show that the control law (8)–(10) is still stabilizing and has other interesting properties.

3. ROBUST STABILITY

Theorem 1

Consider the time-varying system (1)–(5) with the control law defined by (8)–(10) applied. Then the closed-loop system is stable and

$$\sup_k E|y_k|^{2+\delta} < \infty \tag{13}$$

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} y_k^2 \leqslant M < \infty \quad \text{a.s.} \tag{14}$$

Where δ is as in (3)–(5) and M is a *deterministic* constant which can be chosen as

$$M = \frac{\sigma_w^2}{1-\sigma_e^2} + \frac{\alpha^2}{1-\sigma_e^2}\left\{\frac{\sigma_0 + 2\sigma_1}{2\sigma_1(1-|\alpha|)^2}\left[\frac{|\alpha|\sigma_1\mu_{4w}}{2\sigma_0(1-\alpha^2)^{1/2}} + \mu_{4e}\right]^2 + \left[\frac{\sigma_1}{2\sigma_0(1-\alpha^2)^{1/2}} + 1\right](\sigma_{4w})^2\right\}^2 \tag{15}$$

where by definition

$$\mu_{4w} = \sup_k\{E|w_k|^4\}^{1/4} \qquad \mu_{4e} = \sup_k\{E|e_k|^4\}^{1/4} \tag{16}$$

Remarks

(1) Theorem 1 can be easily generalized to more general control problems, e.g. adaptive tracking and adaptive pole assignment, by using techniques similar to those developed in this paper.
(2) None of the signals in the closed-loop system can be expected to be stationary or ergodic because of the assumptions made on the noise process $\{w_n, v_n\}$. This is especially so when the adaptive tracking problem is concerned, where the reference signals are only assumed to be bounded and deterministic.
(3) If the estimation algorithm (8), (9) (Kalman filter) is replaced by the short-memory gradient algorithm, results similar to Theorem 1 are still obtainable.[6]

The proof of Theorem 1 is separated into several lemmas.

Lemma 1

Under the conditions of Theorem 1 and the denotation (12)

$$\sup_k E[|\tilde{\theta}_k|^{4+2\delta}] < \infty \tag{17}$$

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} |\tilde{\theta}_k|^4 \leqslant (1-|\alpha|)^{-4}\left[\frac{|\alpha|\sigma_1\mu_{4w}}{2\sigma_0(1-\alpha^2)^{1/2}} + \mu_{4e}\right]^4 \quad \text{a.s.} \tag{18}$$

where δ, μ_{4w} and μ_{4e} are given in (3) and (16).

Proof. Let us first consider the upper and lower bounds for P_k. By (9) it follows that for any $k \geqslant 0$

$$\sigma_1^2 \leqslant P_{k+1} \leqslant \alpha^2 P_k + \sigma_1^2 \leqslant \frac{\sigma_1^2}{1-\alpha^2} + \alpha^{2(k+1)}P_0 \tag{19}$$

Now by (1) and (10) the output may be expressed

$$y_{k+1} = \tilde{\theta}_k y_k + w_{k+1} \tag{20}$$

From this and (1), (8) and (10) we get

$$\begin{aligned}\tilde{\theta}_{k+1} &= \alpha\tilde{\theta}_k - \frac{\alpha P_k y_k(\tilde{\theta}_k y_k + w_{k+1})}{\sigma_0^2 + P_k y_k^2} + e_{k+1}\\ &= \frac{\alpha\tilde{\theta}_k\sigma_0^2}{\sigma_0^2 + P_k y_k^2} - \frac{\alpha P_k y_k w_{k+1}}{\sigma_0^2 + P_k y_k^2} + e_{k+1}\end{aligned} \tag{21}$$

Consequently, by applying the elementary inequality

$$\frac{|b|}{a^2+b^2} \leqslant \frac{1}{2|a|} \quad (a \neq 0) \tag{22}$$

to the second term on the RHS of the above equality, it is seen that

$$|\tilde{\theta}_{k+1}| \leqslant |\alpha|\,|\tilde{\theta}_k| + \frac{|\alpha|(P_k)^{1/2}}{2\sigma_0}|w_{k+1}| + |e_{k+1}| \tag{23}$$

With this together with (3), the fact that $|\alpha| < 1$ and

$$\limsup_{N\to\infty} P_N \leqslant \frac{\sigma_1^2}{1-\alpha^2} \qquad \text{(by (19))} \tag{24}$$

it is easy to convince oneself that (17) holds.

We now proceed to prove (18). By (3) and the martingale convergence theorem[7] it is known that the series

$$\sum_{k=1}^{\infty} (|w_k|^4 - E|w_k|^4)/k$$

is convergent, and so by the Kronecker lemma

$$\frac{1}{N}\sum_{k=1}^{N} (|w_k|^4 - E|w_k|^4) \to 0 \qquad \text{as } N\to\infty$$

This together with (16) gives

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} |w_k|^4 \leqslant (\mu_{4w})^4 \quad \text{a.s.} \tag{25}$$

Similarly

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} |e_k|^4 \leqslant (\mu_{4e})^4 \quad \text{a.s.} \tag{26}$$

From (23)–(26) it is not difficult to see that

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} |\tilde{\theta}_k|^4 < \infty \quad \text{a.s.} \tag{27}$$

To complete the proof of (18) we have to establish the upper bound. Applying the Minkowski inequality to (23) we see that

$$\left\{\frac{1}{N}\sum_{k=1}^{N} |\tilde{\theta}_{k+1}|^4\right\}^{1/4} \leqslant |\alpha| \left\{\frac{1}{N}\sum_{k=1}^{N} |\tilde{\theta}_k|^4\right\}^{1/4} + \frac{|\alpha|}{2\sigma_0}\left\{\frac{1}{N}\sum_{k=1}^{N} (P_k)^2 |w_{k+1}|^4\right\}^{1/4} + \left\{\frac{1}{N}\sum_{k=1}^{N} |e_{k+1}|^4\right\}^{1/4}$$

Then noting (24)–(27) and taking 'lim sup' on both sides of the above inequality, we finally get

$$(1-|\alpha|)\limsup_{N\to\infty} \left\{\frac{1}{N}\sum_{k=1}^{N} |\tilde{\theta}_k|^4\right\}^{1/4} \leqslant \frac{|\alpha|\sigma_1\mu_{4w}}{2\sigma_0(1-\alpha^2)^{1/2}} + \mu_{4e}$$

which is tantamount to (18). This completes the proof. □

We remark that similar results for the matrix case have recently been established in Guo *et al.*[8]

Denote

$$f_k = \frac{\sigma_0^2\tilde{\theta}_k - P_k y_k w_{k+1}}{\sigma_0^2 + P_k y_k^2} \tag{28}$$

The importance of f_k will be seen in the forthcoming analysis. Let us establish some of its useful properties first.

Lemma 2

Under the conditions of Theorem 1, the variable f_k defined by (28) has the following properties:

(i) $\sup_k E[|f_k y_{k+1}|^{2+\delta}] < \infty$

(ii) $$\limsup_{N\to\infty} \left\{\frac{1}{N}\sum_{k=1}^{N} |f_k y_{k+1}|^2\right\}^{1/2} \leqslant \frac{\sigma_0 + 2\sigma_1}{2\sigma_1(1-|\alpha|)^2}\left[\frac{|\alpha|\sigma_1\mu_{4w}}{2\sigma_0(1-\alpha^2)^{1/2}} + \mu_{4e}\right]^2 + \left[\frac{\sigma_1}{2\sigma_0(1-\alpha^2)^{1/2}} + 1\right](\mu_{4w})^2 \qquad \text{a.s.}$$

Proof. By (19), (20), (22) and (28) it follows that

$$\begin{aligned}
|f_k y_{k+1}| &\leqslant |\tilde{\theta}_k| \left|\frac{y_k\tilde{\theta}_k + w_{k+1}}{\sigma_0^2 + P_k y_k^2}\right| \sigma_0^2 + \left|\frac{P_k y_k w_{k+1}(y_k\tilde{\theta}_k + w_{k+1})}{\sigma_0^2 + P_k y_k^2}\right| \\
&\leqslant |\tilde{\theta}_k|^2 \frac{|y_k|\sigma_0^2}{\sigma_0^2 + P_k y_k^2} + |\tilde{\theta}_k||w_{k+1}|\frac{\sigma_0^2}{\sigma_0^2 + P_k y_k^2} \\
&\quad + |\tilde{\theta}_k||w_{k+1}|\frac{P_k y_k^2}{\sigma_0^2 + P_k y_k^2} + |w_{k+1}|^2 \frac{P_k|y_k|}{\sigma_0^2 + P_k y_k^2} \\
&\leqslant |\tilde{\theta}_k|^2 \frac{\sigma_0}{2(P_k)^{1/2}} + 2|\tilde{\theta}_k||w_{k+1}| + |w_{k+1}|^2 \frac{(P_k)^{1/2}}{2\sigma_0} \\
&\leqslant \left(\frac{\sigma_0}{2\sigma_1} + 1\right)|\tilde{\theta}_k|^2 + \left\{\frac{\sigma_1}{2\sigma_0(1-\alpha^2)^{1/2}} + 1 + \frac{\alpha^{(k+1)}(P_0)^{1/2}}{2\sigma_0}\right\}|w_{k+1}|^2
\end{aligned} \tag{29}$$

thus property (i) follows easily from (3), (29) and Lemma 1. To conclude (ii) we apply the Minkowski inequality to (29) to get

$$\begin{aligned}
\left\{\frac{1}{N}\sum_{k=1}^{N}|f_k y_{k+1}|^2\right\}^{1/2} &\leqslant \left(\frac{\sigma_0}{2\sigma_1} + 1\right)\left(\frac{1}{N}\sum_{k=1}^{N}|\tilde{\theta}_k|^4\right)^{1/2} \\
&\quad + \left[\frac{\sigma_1}{2\sigma_0(1-\alpha^2)^{1/2}} + 1\right]\left\{\frac{1}{N}\sum_{k=1}^{N}|w_{k+1}|^4\right\}^{1/2} \\
&\quad + \frac{(P_0)^{1/2}}{2\sigma_0}\left\{\frac{1}{N}\sum_{k=1}^{N}\alpha^{2(k+1)}|w_{k+1}|^4\right\}^{1/2}
\end{aligned}$$

It is easy to show that the last term tends to zero as $N \to \infty$. Consequently, by Lemma 1 and (25),

$$\limsup_{N\to\infty}\left\{\frac{1}{N}\sum_{k=1}^{N}|f_k y_{k+1}|^2\right\}^{1/2} \leqslant \frac{\sigma_0 + 2\sigma_1}{2\sigma_1(1-|\alpha|)^2}\left[\frac{|\alpha|\sigma_1\mu_{4w}}{2\sigma_0(1-\alpha^2)^{1/2}} + \mu_{4e}\right]^2 + \left[\frac{\sigma_1}{2\sigma_0(1-\alpha^2)^{1/2}} + 1\right](\mu_{4w})^2$$

Thus the proof of Lemma 2 is complete. □

We are now in a position to prove the first assertion of Theorem 1.

Lemma 3

Under the conditions of Theorem 1,

$$\sup_k E[|y_k|^{2+\delta}] < \infty$$

where δ is given as in (3)–(5).

Proof. By (21) and (28) we know that

$$\tilde{\theta}_{k+1} = \alpha f_k + e_{k+1}$$

so by (20) it follows that

$$y_{k+1} = e_k y_k + \alpha y_k f_{k-1} + w_{k+1} \tag{30}$$

Now let us denote the L_p-norm with $p = 2 + \delta$ for a random variable x as

$$\| x \|_p = \{E|x|_p\}^{1/p}$$

Then by (30) and the independence of e_k and y_k we have

$$\| y_{k+1} \|_p \leqslant \| e_k \|_p \| y_k \|_p + |\alpha| \| y_k f_{k-1} \|_p + \| w_{k+1} \|_p \tag{31}$$

Note that $\sup_k \| e_k \|_p < 1$ by (4), hence by lemma 2(i) and (31) it is easy to get the desired result

$$\sup_k \| y_k \|_p < \infty \quad \square$$

Lemma 4

Under the conditions of Theorem 1,

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} y_k^2 \leqslant M < \infty \qquad \text{a.s.} \tag{32}$$

where M is given by (15).

Proof. The proof technique is similar to that of Reference 4. By (30) we have

$$y_{k+1}^2 = (E|e_k|^2) y_k^2 + (\alpha y_k f_{k-1})^2 + w_{k+1}^2 + g_k \tag{33}$$

where

$$g_k = (e_k^2 - E|e_k|^2) y_k^2 + 2\{\alpha e_k f_{k-1} y_k^2 + e_k y_k w_{k+1} + \alpha y_k f_{k-1} w_{k+1}\} \tag{34}$$

We note that $\{g_k, \mathscr{F}_k\}$ forms a martingale difference sequence, with $\mathscr{F}_k$ being the σ-algebra generated by $\{e_i, w_{i+1}, i \leqslant k\}$.

By (34) and the Schwarz inequality

$$\begin{aligned} E|e_k f_{k-1} y_k^2|^{1+(\delta/2)} &\leqslant E|e_k|^{1+(\delta/2)} \{E|y_k|^{1+(\delta/2)} |y_k f_{k-1}|^{1+(\delta/2)}\} \\ &\leqslant \{E|y_k|^{2+\delta}\}^{1/2} \{E|y_k f_{k-1}|^{2+\delta}\}^{1/2} \end{aligned}$$

From this together with (34), (3) and Lemmas 2 and 3 it follows that

$$\sup_k E[|g_k|^{1+(\delta/2)}] < \infty$$

Therefore the series $\Sigma_{k=1}^{\infty} (g_k/k)$ is convergent almost surely by the martingale convergence theorem,[7] and applying the Kronecker lemma yields

$$\frac{1}{N}\sum_{k=1}^{N} g_k \to 0 \quad \text{a.s. as } N\to\infty \tag{35}$$

Finally, summing both sides of (34) and noting (6), we obtain

$$(1-\sigma_e^2)\frac{1}{N}\sum_{k=1}^{N} y_k^2 \leqslant \frac{1}{N}(\sigma_e^2 y_0^2 - y_{N+1}^2) + \frac{\alpha^2}{N}\sum_{k=0}^{N}(y_k f_{k-1})^2 + \frac{1}{N}\sum_{k=0}^{N} w_{k+1}^2 + \frac{1}{N}\sum_{k=0}^{N} g_k \tag{36}$$

The first term on the RHS converges to zero almost surely by Lemma 3 and the Borel–Cantelli lemma. Thus by (3), (6), (35) and (36) it is easy to see that

$$\limsup_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} y_k^2 \leqslant \frac{1}{1-\sigma_e^2}\left\{\sigma_w^2 + (\alpha^2)\limsup_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N}(y_k f_{k-1})^2\right\}$$

The desired upper bound M can be obtained by combining this will Lemma 2 (ii). This completes the proof of Lemma 4 and hence the proof of Theorem 1. □

4. ERGODICITY AND STRUCTURAL ROBUSTNESS

We will now show that by strengthening the assumptions made on the disturbance process, the closed-loop system equations (1) and (8)–(10) give rise to a Markov chain Φ satisfying every condition introduced in the Appendix. In particular, the state process Φ satisfies condition GA, is stable in probability and, under the appropriate conditions on the disturbance process, is weakly stochastically controllable. These facts will be used to give a complete description of the asymptotic properties of the output process of the closed-loop system.

We henceforth make the following additional assumption (A).

(A) The disturbance process (w_k, e_k) is independent and identically distributed (i.i.d.) and the corresponding distributions μ_w and μ_e possess continuous densities which are positive at the origin.

Under this condition the closed-loop system equations give rise to a Markov state process

$$\Phi_k = \begin{bmatrix} P_k \\ \tilde\theta_k \\ y_k \end{bmatrix} = \begin{bmatrix} \sigma_1^2 + \alpha^2\sigma_0^2 P_{k-1}(P_{k-1}y_{k-1}^2 + \sigma_0^2)^{-1} \\ \alpha\tilde\theta_{k-1} - \alpha P_{k-1}y_{k-1}(\tilde\theta_{k-1}y_{k-1} + w_k)(\sigma_0^2 + P_{k-1}y_{k-1}^2)^{-1} + e_k \\ \tilde\theta_{k-1}y_{k-1} + w_k \end{bmatrix} \tag{37}$$

evolving on $\mathbf{X} = R_+ \times R \times R$ which satisfies conditions A1–A4 of the Appendix.

We are fortunate enough to have the following extremely useful result.

Lemma 5

Suppose that conditions (3), (4) and (A) are satisfied for the state process (37). Then:

(i) Φ is weakly stochastically controllable.
(ii) Φ satisfies condition GA of the Appendix.
(iii) For fixed α, δ, $\rho \in (0,1)$, $B>0$ and a compact set C contained in $(0,\infty)$, the family of systems

$$\left\{\Phi\colon \int t^{2+\delta}\sigma_e(\mathrm{d}t) < \rho,\ \int t^{4+2\delta}[\mu_e(\mathrm{d}t) + \mu_w(\mathrm{d}t)] < B,\ \sigma_0^2, \sigma_1^2 \in C\right\}$$

is uniformaly stable in probability.

Proof. By Theorem A1 in the Appendix we know that for (i) it suffices to show that for some $T > 1$ and some sequence $\{(e_i, w_i), 1 \leqslant i \leqslant T\}$ the controllability matrix for this system is full rank. It is easy to verify[9] that in fact the second-order controllability matrix has the form

$$\mathbf{C}_2 = \begin{bmatrix} 0 & \dfrac{-2\alpha^2\sigma_0^2 P_1^2 y_1}{(P_1 y_1^2 + \sigma_0^2)^2} & 0 & 0 \\ \# & \# & 1 & \# \\ \# & \# & 0 & 1 \end{bmatrix}$$

which is full rank whenever $y_1 = \tilde{\theta}_0 y_0 + w_1$ is non-zero. This shows that for each $(P_0, \tilde{\theta}_0, y_0) \in R_+ \times R^2$ the matrix $\mathbf{C}_2$ is full rank for a.e. (Lebesgue) $\{(e_1, w_1), (e_2, w_2)\} \in R^2 \times R^2$, and so by Theorem A1 the closed-loop system is weakly stochastically controllable.

To see that Φ satisfies condition GA for $w^* = 0$, observe that for any $k \in \mathbf{Z}_+$ and $x \in \mathbf{X}$ the asymptotic behaviour of the state readout map $S_x^k(\,.\,)$ evaluated at 0 may be analysed by 'turning off' the noise in equations (1), (8)–(10) yielding

$$\lim_{k \to \infty} S_x^k(0, \ldots, 0) = [\sigma_1^2/(1 - \alpha^2), 0, 0]^{\mathrm{T}}$$

Part (iii) of the lemma follows from the proofs of Theorem 1 and Lemma 1, and from equation (19), which imply that for some $M_0 = M_0(\alpha, \rho, \delta, B, x)$

$$\limsup_{k \to \infty} E_x[\| \Phi_k \|^{2+\delta}] \leqslant M_0 < \infty \tag{38}$$

for all $x \in \mathbf{X}$ and all realizations Φ satisfying the conditions of the lemma. Since $\| \,.\, \|^{2+\delta}$ is a moment on $\mathbf{X}$, this implies that this collection of systems is uniformly stable in probability. □

Applying (38), Lemma 5, Theorem A2 and its corollary yields:

Theorem 2

Suppose that conditions (3), (4) and (A) hold for the state process (37). Then for every initial condition $x \in \mathbf{X}$ and any $\varepsilon > 0$

$$\lim_{k \to \infty} P_x\{|y_k| > \varepsilon\} = \pi\{|y| > \varepsilon\}$$

$$\lim_{N \to \infty} E[y_N^2] = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} y_k^2 = \int y^2 \, \mathrm{d}\pi$$

where y denotes the function $y : \mathbf{X} \to R$ defined by $y(x) = (0, 0, 1)x$, $x \in \mathbf{X}$, and π is the unique invariant probability on $\mathscr{B}(\mathbf{X})$.

We now consider a parametrized family of state processes $\{\Phi^s : 0 \leqslant s \leqslant 1\}$. We assume that for each $s \in [0, 1]$ the state process Φ^s is generated by the recursion (37) where the parameters $\sigma_0^2(s)$ and $\sigma_1^2(s)$ and the distributions μ_w^s and μ_e^s depend on s, and all other parameters are fixed. We further assume that for fixed constants $\rho \in (0, 1)$ and $\delta, B > 0$

$$\int t^{2+\delta} \mu_e^s(\mathrm{d}t) < \rho \qquad \int t^{4+2\delta} [\mu_e^s(\mathrm{d}t) + \mu_w^s(\mathrm{d}t)] < B \tag{39}$$

for all $s \in [0, 1]$. It follows by Lemma 5 that the family of state processes $\{\Phi^s : 0 \leqslant s \leqslant 1\}$ is uniformly stable in probability.

Theorem 3

Suppose that conditions (3), (4) and (A) hold for the parametrized family of state processes $\{\Phi^s : 0 \leqslant s \leqslant 1\}$. Then if the parameters and distributions converge:

$$[\sigma_0^2(s), \sigma_1^2(s)] \xrightarrow[s \to 0]{} [\sigma_0^2(0), \sigma_1^2(0)]$$

$$[\mu_w^s, \mu_e^s] \xrightarrow[s \to 0]{\text{weakly}} [\mu_w^0, \mu_e^0]$$

it follows that the invariant probabilities and long-run costs converge:

$$\pi_s \xrightarrow[s \to 0]{\text{weakly}} \pi_0$$

$$E_{\pi_s}[y^2] \xrightarrow[s \to 0]{} E_{\pi_0}[y^2]$$

$$P_{\pi_s}\{|y| > \varepsilon\} \xrightarrow[s \to 0]{} P_{\pi_0}\{|y| > \varepsilon\} \quad \varepsilon > 0$$

Proof. The first and second limits follow directly from Theorem A3 and its corollary. The last limit follows from Theorem A3 and the fact that the invariant probability π_0 is absolutely continuous with respect to Lebesgue measure. □

Hence, for example, if the disturbance processes are approximately Gaussian and the variance estimates σ_0^2 and σ_1^2 approximate σ_w^2 and σ_e^2 respectively, then the control performance will be close to the optimal one.

5. CONCLUSIONS

In this paper, by using a combination of martingale and Markov chain techniques, we have shown that the first-order stochastic adaptive control system of Meyn and Caines[3] is robust with respect to noise. The martingale techniques are used to establish the closed-loop stability, while the Markov chain techniques are used in the analysis of the performance.

Martingale methods have previously been used extensively in stochastic control theory; typically the martingale convergence theorem in applied in order to prove the convergence of a stochastic Lyapunov function. As in Reference 4, this paper shows that martingales can be a useful tool even in problems where it is unrealistic to search for stochastic Lyapunov functions.

The theory of Markov chains has received less attention in adaptive control theory, the principal reason being that in problems where a variable in a Markovianization of the input–state–output process converges almost surely (for instance, the variable r_k in the stochastic gradient algorithm), the ergodic theory of Markov chains as it now stands can say very little. However, in situations where none of the variables converges, a noise controllability condition may often be established. In this case the stochastic stability theory presented in this paper may be applied to prove the existence of limits of loss functions on the state process for all initial conditions and to establish parameter robustness theorems for the state process. Since it has often been noted that it is undesirable to have vanishing gains in adaptive algorithms, this suggests that the theory of Markov chains may become a valuable tool in stochastic control theory.

We are presently attempting to generalize our results to more general cases. In the ARX(p, q) case the estimation algorithm derived from the Kalman filter is extremely similar to the short-memory gradient algorithm and appears to have some very desirable properties[8] that would make it a useful estimation algorithm in practice. However, since the stability of a time-varying ARMAX system controlled by a short-memory gradient-based adaptive algorithm has never been established, this appears to be a challenging problem.

APPENDIX

Here we review a general theory for Markovian systems of the form

$$\Phi_{k+1} = F(\Phi_k, w_{k+1}) \quad k \in \mathbf{Z}_+ \tag{40}$$

where for all k, $\Phi_k \in \mathbf{X}$ = an open subset of R^n, $w_k \in R^p$ and $F: \mathbf{X} \times R^p \to \mathbf{X}$ is continuously differentiable (C^1). It is assumed that the initial condition Φ_0 and the disturbance process $w \triangleq \{w_k\}$ satisfy:

A1. (Φ_0, w) are random variables on the probability space $(\Omega, \mathscr{F}, P_{\Phi_0})$.
A2. Φ_0 is independent of w.
A3. w is an independent and identically distributed (i.i.d) process.
A4. The distribution μ_w of w_k, $k \in \mathbf{Z}_+$, possess a continuous density.

When assumptions A1–A4 are satisfied, the recursion (40) gives rise to a Markov chain Φ on $\mathbf{X}$, and this fact enables us to exploit the many ergodic theorems available for this class of stochastic processes.

Results obtainable from the ergodic theory of Markov chains

We will now summarize some results from the stochastic stability theory of References 9 and 10. In these works it is shown that if (40) satisfies a weak form of stability, and a noise controllability condition holds, then the steady state behaviour of Φ is determined by invariant probabilities on the state process.

(a) Controllability

Here we introduce a useful formulation of stochastic controllability.

The state readout map $S_x^k: \mathbf{R}^{kp} \to \mathbf{X}$ of the system (40) is defined inductively for $k \in \mathbf{Z}_+$, $x \in \mathbf{X}$ and $z = (z_1, \ldots, z_k)^{\mathrm{T}} \in \mathbf{R}_{kp}$ by

$$S_x^k = F(S_x^{k-1}(z_1, \ldots, z_{k-1}), z_k) \quad k \geqslant 1 \qquad S_x^0 = x$$

The state readout map is so named because for all $k \in \mathbf{Z}_+$, $\Phi_k = S_x^k(w_1, \ldots, w_k)$ when $\Phi_0 = x$.

Given two measures ν and μ on $\mathscr{B}(\mathbf{X})$, we say that ν is absolutely continuous with respect to μ if $\nu(A) = 0$ whenever $\mu(A) = 0$. We let $\mathbf{I}_A\mu$ denote the measure defined for $A, B \in \mathscr{B}(\mathbf{X})$ by $(\mathbf{I}_A\mu)\{B\} = \mu(AB)$.

Definition. The system (40) is called *weakly stochastically controllable* if for each initial condition $x \in \mathbf{X}$ there exists $T = T(x) \in \mathbf{Z}_+$ and an open set O_x contained in $\mathbf{X}$ such that $\mathbf{I}_{O_x}\mu^{\mathrm{Leb}}$ is absolutely continuous with respect to $P^T(x,.)$, where $P^T(x,.)$ is the measure on $\mathscr{B}(\mathbf{X})$ induced by $\Phi_T = \mathbf{S}_x^T(w_1, \ldots, w_T)$.

For $y \in \mathbf{X}$ and a sequence $\{z_k : z_k \in R^p, k \in \mathbf{Z}_+\}$ let $\{\mathbf{A}_k, \mathbf{B}_k : k \in \mathbf{Z}_+\}$ denote the matrices

$$\mathbf{A}_k = \left[\frac{\partial F}{\partial x}\right]_{(S_y^k, z_{k+1})} \qquad \mathbf{B}_k = \left[\frac{\partial F}{\partial z}\right]_{(S_y^k, z_{k+1})}$$

and let $\mathbf{C}_y^k = \mathbf{C}_y^k\ (z_1, \ldots, z_k)$ denote the generalized controllability matrix

$$\mathbf{C}_y^k = [\mathbf{A}_{k-1} \ldots \mathbf{A}_1\mathbf{B}_0 \mid \mathbf{A}_{k-1} \ldots \mathbf{A}_2 B_1 \mid \ldots \mid \mathbf{A}_{k-1}\mathbf{B}_{k-1} \mid \mathbf{B}_{k-1}]$$

We let O_w denote the open set $\{x \in R^p : p_w(x) > 0\}$ where $p_w(x)$ is the density of μ_w. The following result gives a necessary and sufficient condition for weak stochastic controllability in terms of the controllability matrix $\mathbf{C}_y^T$ which is analogous to the controllability condition used in linear system theory.

Theorem A1. Suppose that Φ is of the form (40) and that conditions A1–A4 hold. Then the system (40) is weakly stochastically controllable if and only if for all initial conditions $y \in \mathbf{X}$ there exists $T \geqslant 1$ such that

$$\operatorname{rank} \mathbf{C}_y^T(\lambda) = n \quad \text{for some } \lambda \in O_w^T$$

(b) Stability

Definition. The system (40) is called *stable in probability* if for each deterministic initial condition $x \in \mathbf{X}$ and each $\varepsilon > 0$ there exists a compact subset C contained in $\mathbf{X}$ such that

$$\liminf_{k \to \infty} P_x\{\Phi_k \in C\} \geqslant 1 - \varepsilon$$

We remark that if the state space is closed and for some $p > 0$

$$\limsup_{k \to \infty} E_x[|\Phi_k|_p] < \infty \quad \text{for all initial conditions } x \in \mathbf{X}$$

then Φ is stable in probability.

In most cases stochastic systems of the form (40) which are stable in probability exhibit the following related property. Given a system of the form (40) and a point $\omega^* \in O_w$, we will call the deterministic system

$$d_{k+1} = F(d_k, \omega^*), \; k \in \mathbf{Z}_+$$

with initial condition $d_0 \in \mathbf{X}$ the *freely evoling system.*

Definition. The system (40) satisfies *condition GA* if some $d^* \in \mathbf{X}$ is globally attracting for the freely evolving system. That is:

Condition GA. For some fixed $\omega^* \in O_w$ and $d^* \in \mathbf{X}$, and each initial condition $x \in \mathbf{X}$,

$$\lim_{k \to \infty} d_k = \lim_{k \to \infty} S_x^k(\omega^*, \ldots, \omega^*) = d^*$$

Let C denote the set of bounded and continuous functions $f: \mathbf{X} \to R$ and let $\mathcal{M}$ denote the set of probabilities on $\mathcal{B}(\mathbf{X})$, the Borel field on $\mathbf{X}$. A sequence $\{\mu_k : k \in \mathbf{Z}_+\}$ in $\mathcal{M}$ converges *weakly* to $\mu_\infty \in \mathcal{M}$ if

$$\lim_{k \to \infty} \int f \, \mathrm{d}\mu_k = \int f \, \mathrm{d}\mu_\infty$$

for all $f \in C$, and this will be denoted

$$\mu_k \xrightarrow[k \to \infty]{\text{weakly}} \mu_\infty$$

An alternative topology on $\mathcal{M}$ is defined by the total variation norm

$$|\nu - \mu|_{\text{tv}} \triangleq \sup \left| \int f \, \mathrm{d}\nu - \int f \, \mathrm{d}\mu \right| \quad \nu, \mu \in \mathcal{M}$$

where the supremum is taken over all Borel functions f for which $|f(x)| \leqslant 1$ for all $x \in \mathbf{X}$. We say $\{\mu_k : k \in \mathbf{Z}_+\}$ converges *in total variation norm* to μ_∞ if it converges in this norm or, equivalently,

$$\lim_{k \to \infty} \sup_{A \in \mathcal{B}(\mathbf{X})} |\mu_k\{A\} - \mu_\infty\{A\}| = 0$$

Most of the important results concerning the asymptotic behaviour of Markov processes require the existence of an invariant probability (see e.g. References 11–13). If π is an invariant probability and Φ_0 has distribution π, then $\Phi_k \sim \pi$ for all $k > 0$, and in fact Φ is a strictly stationary process in this case. This fact may used to establish the existence of the limit of sample path averages of functions of Φ for a large class of initial conditions.[11]

To establish ergodic theorems which hold for all initial conditions, stronger assumptions are needed. A Markov chain Φ is said to be aperiodic and positive Harris recurrent[10,14] if a unique invariant probability π exists such that for every initial condition $x \in \mathbf{X}$, and every positive Borel measurable function $f:\mathbf{X}\to R$,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} f(\Phi_k)=\int f\,\mathrm{d}\pi \quad \text{a.s.}\,[P_x]$$

$$\lim_{k\to\infty}|\mu_k-\pi|_{\mathrm{tv}}=0$$

where μ_k denotes the distribution of the random variable Φ_k at time k.

This differs considerably from the standard definition, but is in fact equivalent. In Reference 14 an aperiodic positive Harris recurrent Markov chain is simply called ergodic.

The following result relates the notion of Harris recurrence with the stability and controllability condition introduced above.

Theorem A2. Suppose that conditions A1–A4 hold and that Φ is weakly stochastically controllable, stable in probability and condition GA holds. Then Φ is aperiodic and positive Harris recurrent.

The following corollary follows easily:

Corollary A2. Let $y:\mathbf{X}\to R$ be Borel measurable and suppose that the Markov chain Φ satisfies the conditions of Theorem A2. Then for all initial conditions $x\in\mathbf{X}$ and all $\varepsilon\geqslant 0$

$$\lim_{k\to\infty} P_x\{|y_k|>\varepsilon\}=\pi\{|y|>\varepsilon\}$$

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} y_k^2=\int y^2\,\mathrm{d}\pi \quad \text{a.s.}\,[P_x]$$

If further for some $\delta>0$ and $x\in\mathbf{X}$, $\sup_k E_x[|y_k|^{2+\delta}]\leqslant M(x)<\infty$ then

$$\lim_{N\to\infty} E_x[y_N^2]=\int y^2\,\mathrm{d}\pi\leqslant M^{2/(2+\varepsilon)}<\infty$$

where y_k is defined by $y_k=y(\Phi_k)$.

(c) Structural robustness

Here we consider a parameterized family of systems $\{\Phi^s:0\leqslant s\leqslant 1\}$. For each $s\in[0,1]$ we assume that Φ^s is of the form (40) satisfying conditions A1–A4.

Definition. A parametrized family of systems $\{\Phi^s:0\leqslant s\leqslant 1\}$ of the form (40) is called uniformly stable in probability if for each deterministic initial condition $x\in\mathbf{X}$ and each $\varepsilon>0$ there exists a compact subset C contained in $\mathbf{X}$ such that

$$\liminf_{k\to\infty} P_x\{\Phi_k^s\in C\}\geqslant 1-\varepsilon \quad \text{for all } 0\leqslant s\leqslant 1$$

We remark that the uniformity refers to the parameter s and not the initial condition x. This differs from the definition of uniform stability given in Reference 10.

Theorem A3. Suppose that for each $0\leqslant s\leqslant 1$ the Markov chain Φ^s satisfies the conditions of Theorem A2 and the following additional assumptions hold:

(i) $\{\Phi^s:0\leqslant s\leqslant 1\}$ is uniformly stable in probability.
(ii) $F^s\to F^0$ uniformly on compact sets in $\mathbf{X}\times R^p$ as $s\to 0$.
(iii)
$$\mu_w^s \xrightarrow[s\to 0]{\text{weakly}} \mu_w^0$$

Then

$$\pi_s \xrightarrow[s\to 0]{\text{weakly}} \pi_0$$

Corollary A3. Let $y : \mathbf{X} \to R$ be continuous and suppose that the collection of Markov chains Φ^s satisfies the conditions of Theorem A3. Suppose further that for some $\varepsilon > 0$ and initial condition $x \in \mathbf{X}$, $\sup_{k \in \mathbf{Z}_+, s \in [0,1]} E_x[|y(\Phi_k)|^{2+\delta}] < \infty$. Then

$$\lim_{s \to 0} \int y^2 \, \mathrm{d}\pi_s = \int y^2 \, \mathrm{d}\pi_0 < \infty$$

REFERENCES

1. Chen, H. F. and P. E. Caines, 'On the adaptive control of a class of systems with random parameters and disturbances', *Automatica*, **21**, 737–741 (1985).
2. Chen, H. F. and L. Guo, 'A robust stochastic adaptive controller', *IEEE Trans. Automatic Control*, to appear (1988).
3. Meyn, S. P. and P. E. Caines, 'A new approach to stochastic adaptive control', *IEEE Trans. Automatic Control.*, **AC-32**, 220–226 (1987).
4. Solo, V., 'Stochastic adaptive control and martingale limit theory', *IEEE Trans. Automatic Control*, to appear (1988).
5. Mayne, D. Q., 'Optimal non-stationary estimation of the parameters of a linear system with Gaussian inputs', *J. Electron. Control*, **14**, 101 (1963).
6. Meyn, S. P., 'Ergodic theorems for discrete time stochastic systems using a generalized stochastic Lyapunov function', submitted for publication (1988).
7. Chow, Y. S., 'Local convergence of martingales and the law of large numbers', *Ann. Math. Stat.*, **36**, 552–558 (1965).
8. Guo, L., L. Xia and J. B. Moore, 'Tracking randomly varying parameters: analysis of a standard algorithm', submitted for publication (1988).
9. Meyn, S. P., 'Asymptotic behavior of stochastic systems possessing Markovian realizations', *Ph.D.Thesis*, Departmant of Electrical Engineering, McGill University, Montreal, 1987.
10. Meyn, S. P. and P. E. Caines, 'Asymptotic behavior of stochastic systems possessing Markovian realizations', submitted for publication (1988).
11. Doob, J. L. *Stochastic Process*, Wiley, New York, 1953.
12. Foguel, S. R., *The Ergodic Theory of Markov Processes*, Van Nostrand Mathematical Studies #21, Van Nostrand Reinhold Co., New York, 1969.
13. Orey, S., *Limit Theorems for Markov Chain Transition Probabilities*, Van Nostrand Reinhold Mathematical Studies #34, Van Nostrand, London, 1971.
14. Nummelin, E., *General Irreducible Markov Chains and Non-Negative Operators*, Cambridge University Press, Cambridge, 1984.