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NONSTATIONARY TIME SERIES IDENTIFICATION†

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Abstract—In this paper strongly consistent estimates are given for both unknown parameters and orders of the nonstationary time series, where the nonstationarity arises because: (1) at any time n a feedback control is added to the usual ARMA process which is successfully applied to modelling economic systems and (2) det $A(z)$ may have zeros on the unit circle in addition to those located outside the unit disk, where $A(z)$ is the matrix polynomial corresponding to the AR part of the ARMA process.

1. INTRODUCTION

Most of the papers in the time series analysis which is popularly used in modelling economic systems [1] are devoted to analysing stationary processes and the Yule–Walker equation and other statistical methods are applied to estimating unknown parameters in the model [2, 3]. However, for some modelling problems in economic systems or in engineering the control (maybe feedback control) action should be taken into account, and in this case the usual ARMA process becomes controlled ARMA process sometimes called ARMAX process, which, in general, is no longer stationary. The nonstationarity may also occur in the ordinary ARMA process with unstable autoregressive part. These two kinds of nonstationarity are discussed and the consistency of estimates for orders and for unknown parameters of the model is analysed in this paper.

Consider the following controlled multidimensional ARMA process:

$$A(z)\mathbf{y}_n = B(z)\mathbf{u}_n + C(z)\mathbf{w}_n, \quad n \geqslant 0,$$
$$\mathbf{y}_n = \mathbf{w}_n = \mathbf{0}, \quad \mathbf{u}_n = \mathbf{0}, \quad n < 0, \tag{1.1}$$

where $\mathbf{y}_n, \mathbf{u}_n$ and $\mathbf{w}_n$ are the m-output, l-input and m-driven noise respectively, and where $A(z)$, $B(z)$ and $C(z)$ are matrix polynomials in backwards shift operator z:

$$A(z) = \mathbf{I} + \mathbf{A}_1 z + \cdots + \mathbf{A}_{p_0} z^{p_0}, \quad p_0 \geqslant 0, \tag{1.2}$$

$$B(z) = \mathbf{B}_1 z + \cdots + \mathbf{B}_{q_0} z^{q_0}, \quad q_0 \geqslant 0, \tag{1.3}$$

$$C(z) = \mathbf{I} + \mathbf{C}_1 z + \cdots + \mathbf{C}_{r_0} z^{r_0}, \quad r_0 \geqslant 0, \tag{1.4}$$

with unknown matrix coefficients $\mathbf{A}_i, \mathbf{B}_j, \mathbf{C}_k$ $(1 \leqslant i \leqslant p_0, 1 \leqslant j \leqslant q_0, 1 \leqslant k \leqslant r_0)$ and unknown finite true orders (p_0, q_0, r_0). Here, by the true orders (p_0, q_0, r_0) we mean that the triple (p_0, q_0, r_0) satisfies

$$(p_0, q_0, r_0) = \underset{0 \leqslant p, q, r < \infty}{\arg\max} \{p + q + r : \mathbf{A}_p \neq \mathbf{0}, \mathbf{B}_q \neq \mathbf{0}, \mathbf{C}_r \neq \mathbf{0}\}. \tag{1.5}$$

The driven noise $\{\mathbf{w}_n\}$ is assumed to be a martingale difference sequence with respect to an increasing sequence of σ-fields $\{\mathscr{F}_n\}$ (i.e. $\mathbf{w}_n$ is $\mathscr{F}_n$-measurable and $E(\mathbf{w}_n | \mathscr{F}_{n-1}) = 0$ for every n) and satisfies

$$\sup_n E[\|\mathbf{w}_{n+1}\|^\beta | \mathscr{F}_n] < \infty, \quad \text{a.s. for some } \beta > 2, \tag{1.6}$$

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and

$$\liminf_{n\to\infty} \lambda_{\min}\left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{w}_i\mathbf{w}_i^{\tau}\right) > 0, \quad \text{a.s.,} \tag{1.7}$$

where and hereafter $\lambda_{\min}(\mathbf{X})$ [$\lambda_{\min}(\mathbf{X})$] denotes the minimum [maximum] eigenvalue of a matrix $\mathbf{X}$, and $\|\mathbf{X}\| = \{\lambda_{\max}(\mathbf{X}\mathbf{X}^{\tau})\}^{1/2}$ its norm.

For the system input $\{\mathbf{u}_n\}$, it is assumed that

$$\mathbf{u}_n \text{ is } \mathscr{F}_n\text{-measurable for any } n \geqslant 0, \tag{1.8}$$

so that $\mathbf{u}_n$ may be any feedback control that depends on the past inputs and outputs $\{\mathbf{y}_i, \mathbf{u}_{i-1}, i \leqslant n\}$.

The problem of estimating the unknown parameters (coefficients) of the controlled ARMA model (1.1) has been extensively studied for the past two decades and strong consistency of various stochastic identification algorithms has been proved under the standard condition of persistence of excitation for the system with known orders (p_0, q_0, r_0). This kind of results is well summarized in books of Ljung and Söderström [4], Goodwin and Sin [5] and Chen [6].

Recently, in consistency analysis of parameter identification research attention is paid to the more realistic (and more difficult) cases where the system signals may not be persistently excited and strong consistency has been established for stochastic gradient algorithms (Chen and Guo [7, 8]) as well as for extended least squares algorithms (Lai and Wei [9, 10], Chen [6] and Chen and Guo [11]). These results are successfully applied to stochastic adaptive control problems of simultaneously identifying system parameters and controlling the system for tracking in [7, 10] and for minimizing the quadratic cost [8, 11].

As far as the order estimation is concerned, most of the important results are developed for uncontrolled stationary ARMA processes. The estimates (p_n, r_n) for the unknown orders (p_0, r_0) are usually given by minimizing some criterion, for example, AIC($\mathbf{p}$, $\mathbf{r}$) (Akaike [12]), BIC($\mathbf{p}$, $\mathbf{r}$) (Akaike [13], Rissanen [14] and Schwarz [15]), and ΦIC($\mathbf{p}$, $\mathbf{r}$) (Hannan and Quinn [16]). However, all these results cannot be applied to feedback control systems, described by (1.1), which essentially differs from the ARMA model by additional control terms, which are crucial for all real control systems, and may depend upon the past system inputs and outputs as implied by (1.8). Efforts towards estimating unknown orders of feedback control systems have recently been made in Chen and Guo [17, 18] where, having introduced a new criterion to be minimized, the authors have obtained consistent estimates for orders of (possibly non-persistently excited) stochastic feedback control systems.

Section 2 concerns identification of coefficients of feedback control systems. Section 3 deals with the order estimation for controlled ARMA processes and Section 4 presents parameter estimation results without using SPR condition. Finally, Section 5 treats ARMA process with $A(z)$ unstable.

2. PARAMETER ESTIMATION FOR CONTROLLED TIME SERIES

We will assume from now on that the upper bounds for the unknown orders p_0, q_0 and r_0 are available, i.e.

Assumption 1

The true orders (p_0, q_0, r_0) belong to a known finite set M:

$$M = \{(p, q, r)\colon 0 \leqslant p \leqslant p^*, 0 \leqslant q \leqslant q^*, 0 \leqslant r \leqslant r^*\}.$$

For any $(p, q, r) \in M$, set

$$\theta(p, q, r) = [-\mathbf{A}_1 \dots -\mathbf{A}_p \mathbf{B}_1 \dots \mathbf{B}_q \mathbf{C}_1 \dots \mathbf{C}_r]^{\tau}, \tag{2.1}$$

where by definition

$$\mathbf{A}_i = \mathbf{0}, \quad \mathbf{B}_j = \mathbf{0}, \quad \mathbf{C}_k = \mathbf{0}, \quad \text{for } i > p_0, j > q_0, k > r_0.$$

The "extended least squares" estimate $\theta_n(p,q,r)$ for $\theta(p,q,r)$ is recursively defined by

$$\theta_{n+1}(p,q,r)=\theta_n(p,q,r)+\mathbf{a}_n\mathbf{P}_n\varphi_n(p,q,r)[\mathbf{y}_{n+1}^\tau-\varphi_n^\tau(p,q,r)\theta_n(p,q,r)] \tag{2.2}$$

$$\mathbf{P}_{n+1}=\mathbf{P}_n-a_n\mathbf{P}_n\varphi_n(p,q,r)\varphi_n^\tau(p,q,r)\mathbf{P}_n, \tag{2.3}$$

$$\mathbf{a}_n=(1+\varphi_n^\tau(p,q,r)\mathbf{P}_n\varphi_n(p,q,r))^{-1}, \tag{2.4}$$

$$\varphi_n^\tau(p,q,r)=[\mathbf{y}_n^\tau\ldots\mathbf{y}_{n-p+1}^\tau\,\mathbf{u}_n^\tau\ldots\mathbf{u}_{n-q+1}^\tau\,\hat{\mathbf{w}}_n^\tau\ldots\hat{\mathbf{w}}_{n-r+1}^\tau], \tag{2.5}$$

$$\hat{\mathbf{w}}_n=\mathbf{y}_n-\theta_n^\tau(p,q,r)\varphi_{n-1}(p,q,r), \tag{2.6}$$

where the initial values $\theta_0(p,q,r)$ and $\mathbf{P}_0>\mathbf{0}$ are arbitrarily chosen. To be fixed, we take $\mathbf{P}_0=d\mathbf{I}$ with $d=mp+lq+mr$.

For strong consistency of this algorithm we need the following standard condition on the noise model (1, 4).

Assumption 2

The transfer matrix $C^{-1}(z)-\frac{1}{2}\mathbf{I}$ is strictly positive real (SPR), i.e.

$$C^{-1}(e^{i\theta})+C^{-\tau}(e^{-i\theta})-\mathbf{I}>\mathbf{0},\quad\forall\theta\in[0,2\pi].$$

Relaxations of this condition are considered later on for a class of control systems.

Let us now introduce the following set M^* consisting of three "edge" points:

$$M^*=\{(p_0,q^*,r^*),(p^*,q_0,r^*),(p^*,q^*,r_0)\}. \tag{2.7}$$

The importance of M^* will be seen shortly from Theorems 2.1, 2.2 and 3.1. Convergence or divergence rates of the above mentioned algorithm are given in the following theorem:

Theorem 2.1

For the controlled time series described by (1.1)–(1.8) assume that Assumptions 1 and 2 hold, and that the orders (p,q,r) used in the algorithm (2.1)–(2.6) belong to M. Then as $n\to\infty$,

$$\|\theta_n(p,q,r)-\theta(p,q,r)\|^2=O\left(\frac{\log\lambda_{\max}^{(p,q,r)}(n)}{\lambda_{\min}^{(p,q,r)}(n)}\right),\quad\text{a.s.}. \tag{2.8}$$

Furthermore, if $(p,q,r)\in M^*$ and

$$\log\mu_{\max}^{(p,q,r)}(n)=o(\mu_{\min}^{(p,q,r)}(n)),\ \text{a.s.} \tag{2.9}$$

then

$$\|\theta_n(p,q,r)-\theta(p,q,r)\|^2=O\left(\frac{\log\mu_{\max}^{(p,q,r)}(n)}{\mu_{\min}^{(p,q,r)}(n)}\right),\ \text{a.s.}, \tag{2.10}$$

where $\lambda_{\max}^{(p,q,r)}(n)$ $[\mu_{\max}^{(p,q,r)}(n)]$ and $\lambda_{\min}^{(p,q,r)}(n)$ $[\mu_{\min}^{(p,q,r)}(n)]$ are the maximum and minimum eigenvalues of

$$\sum_{i=0}^{n-1}\varphi_i(p,q,r)\varphi_i^\tau(p,q,r)+\mathbf{P}_0^{-1}\quad\text{and}\quad\sum_{i=0}^{n-1}\varphi_i^0(p,q,r)\varphi_i^{0\tau}(p,q,r)+\mathbf{P}_0^{-1}, \tag{2.11}$$

respectively. Here $\varphi_n(p,q,r)$ is given by (2.5) and $\varphi_n^0(p,q,r)$ is defined as

$$\varphi_n^0(p,q,r)=[\mathbf{y}_n^\tau\ldots\mathbf{y}_{n-p+1}^\tau\,\mathbf{u}_n^\tau\ldots\mathbf{u}_{n-q+1}^\tau\,\mathbf{w}_n^\tau\ldots\mathbf{w}_{n-r+1}^\tau]^\tau. \tag{2.12}$$

Proof. For (2.8) it suffices to show

$$\operatorname{tr}\tilde{\theta}_{n+1}^\tau(p,q,r)\mathbf{P}_{n+1}^{-1}\tilde{\theta}_{n+1}(p,q,r)=O(\log\lambda_{\max}^{(p,q,r)}(n)), \tag{2.13}$$

where $\tilde{\theta}_n(p,q,r)=\theta(p,q,r)-\theta_n(p,q,r)$.

Set

$$\boldsymbol{\xi}_{n+1}=\mathbf{y}_{n+1}-\mathbf{w}_{n+1}-\theta_{n+1}^\tau(p,q,r)\varphi_n(p,q,r).$$

It is easy to see

$$\mathbf{y}_{n+1}^{\tau} - \varphi_n^{\tau}(p, q, r)\theta_{n+1}(p, q, r) = \mathbf{a}_n(\mathbf{y}_{n+1}^{\tau} - \varphi_n^{\tau}(p, q, r)\theta_n(p, q, r))$$

and

$$C(z)\boldsymbol{\xi}_{n+1} = \tilde{\theta}_{n+1}^{\tau}(p, q, r)\varphi_n(p, q, r).$$

By Assumption 2 there are constants $k_0 > 0$, $k_1 \geqslant 0$ such that

$$\mathbf{s}_n = \sum_{i=0}^{n} \varphi_i^{\tau}(p, q, r)\tilde{\theta}_{i+1}(p, q, r)(\boldsymbol{\xi}_{i+1} - \tfrac{1}{2}(1 + k_0)\tilde{\theta}_{i+1}^{\tau}(p, q, r)\varphi_i(p, q, r)) + k_1 \geqslant \mathbf{0}, \quad \forall n \geqslant 0. \quad (2.14)$$

Then we have

$$\begin{aligned}
\operatorname{tr} \tilde{\theta}_{k+1}^{\tau}(p, q, r)\mathbf{P}_{k+1}^{-1}\tilde{\theta}_{k+1}(p, q, r) &= \| \varphi_k^{\tau}(p, q, r)\tilde{\theta}_{k+1}(p, q, r) \|^2 - 2(\boldsymbol{\xi}_{k+1}^{\tau} + \mathbf{w}_{k+1}^{\tau})(\tilde{\theta}_{k+1}(p, q, r) \\
&\quad + \mathbf{P}_k\varphi_k(p, q, r)(\boldsymbol{\xi}_{k+1}^{\tau} + \mathbf{w}_{k+1}^{\tau}))^{\tau}\varphi_k(p, q, r) \\
&\quad + \varphi_k^{\tau}(p, q, r)\mathbf{P}_k\varphi_k \| \boldsymbol{\xi}_{k+1} + \mathbf{w}_{k+1} \|^2 \\
&\quad + \operatorname{tr} \tilde{\theta}_k^{\tau}(p, q, r)\mathbf{P}_k^{-1}\tilde{\theta}_k(p, q, r) \\
&\leqslant \operatorname{tr} \tilde{\theta}_k^{\tau}(p, q, r)\mathbf{P}_k^{-1}\tilde{\theta}_k(p, q, r) - 2(\varphi_k^{\tau}(p, q, r)\tilde{\theta}_{k+1}(p, q, r) \\
&\quad \times (\boldsymbol{\xi}_{k+1} - \tfrac{1}{2}(1 + k_0)\tilde{\theta}_{k+1}^{\tau}(p, q, r)\varphi_n(p, q, r)) \\
&\quad - k_0 \| \tilde{\theta}_{k+1}^{\tau}(p, q, r)\varphi_k(p, q, r) \|^2 \\
&\quad - 2\mathbf{w}_{k+1}^{\tau}\tilde{\theta}_{k+1}^{\tau}(p, q, r)\varphi_k(p, q, r).
\end{aligned}$$

Summing up both sides of the last inequality and using (2.14) we obtain

$$\begin{aligned}
\operatorname{tr} \theta_{k+1}^{\tau}(p, q, r)\mathbf{P}_{k+1}^{-1}\tilde{\theta}_{k+1}(p, q, r) &\leqslant O(1) - k_0 \sum_{i=0}^{n} \| \tilde{\theta}_{i+1}^{\tau}(p, q, r)\varphi_i(p, q, r) \|^2 \\
&\quad - 2 \sum_{i=0}^{n} \mathbf{w}_{i+1}^{\tau}\tilde{\theta}_{i+1}^{\tau}(p, q, r)\varphi_i(p, q, r), \quad \text{a.s.} \quad (2.15)
\end{aligned}$$

Set

$$\boldsymbol{\eta}_n = \mathbf{y}_{n+1} - \theta_n^{\tau}(p, q, r)\varphi_n(p, q, r) - \mathbf{w}_{n+1}.$$

Noticing that $\boldsymbol{\eta}_n$ is $\mathscr{F}_n$-measurable and applying Lemma 2 in [11] we find that

$$\begin{aligned}
\left| \sum_{i=0}^{n} \mathbf{w}_{i+1}^{\tau} \tilde{\theta}_{i+1}^{\tau}(p, q, r)\varphi_i(p, q, r) \right| &= \left| \sum_{i=1}^{n} \mathbf{w}_{i+1}^{\tau}(\tilde{\theta}_i^{\tau}(p, q, r) - \mathbf{a}_i(\mathbf{w}_{i+1} + \boldsymbol{\eta}_i)\varphi_i^{\tau}(p, q, r)\mathbf{P}_i)\varphi_i(p, q, r) \right| \\
&\leqslant O\left(\left(\left[\sum_{i=0}^{n} \| \tilde{\theta}_i^{\tau}(p, q, r)\varphi_i(p, q, r) \|^2 \right)\right]^{\gamma}\right) \\
&\quad + O\left(\sum_{i=0}^{n} \mathbf{a}_i\varphi_i^{\tau}(p, q, r)\mathbf{P}_i\varphi_i(p, q, r) \|\mathbf{w}_{i+1}\|^2 \right), \quad \gamma \in (\tfrac{1}{2}, 1). \quad (2.16)
\end{aligned}$$

Paying attention to

$$\varphi_i^{\tau}(p, q, r)\mathbf{P}_i\varphi_i(p, q, r) = \frac{\det(\mathbf{P}_{i+1}^{-1}) - \det(\mathbf{P}_i^{-1})}{\det(\mathbf{P}_i^{-1})},$$

we can show

$$\sum_{i=0}^{\infty} \frac{\mathbf{a}_i\varphi_i^{\tau}(p, q, r)\mathbf{P}_i\varphi_i(p, q, r)}{(\log(\det \mathbf{P}_{i+1}^{-1}))^c} < \infty, \quad \text{for any } c > 1,$$

and

$$\sum_{i=0}^{\infty} \frac{\mathbf{a}_i\varphi_i^{\tau}(p, q, r)\mathbf{P}_i\varphi_i(p, q, r)}{\log(\det \mathbf{P}_{i+1}^{-1})} (\| \mathbf{w}_{i+1} \|^2 - E(\| \mathbf{w}_{i+1} \|^2/\mathscr{F}_i)) < \infty.$$

Then by Kronecker lemma we have

$$\sum_{i=0}^{n} \mathbf{a}_i \varphi_i^\tau(p, q, r)\mathbf{P}_i \varphi_i(p, q, r) \| \mathbf{w}_{i+1} \|^2 = O(\log \lambda_{\max}^{(p,q,r)}(n + 1)), \tag{2.17}$$

and (2.13) follows from (2.15)–(2.17).

By showing that

$$\sum_{i=0}^{n} \| \xi_{i+1} \|^2 = O(\log \lambda_{\max}^{(p,q,r)}(n)),$$

the second part of the theorem is easy to be verified. □

Remark 1

In the second part of the theorem the requirement $(p, q, r) \in M^*$ cannot be replaced by $(p, q, r) = (p^*, q^*, r^*)$. As an example, let us take $(p, q, r) = (p_0 + 1, q_0 + 1, r_0 + 1)$ and assume that (1.1) is one-dimensional. Then by (1.1)–(1.4) and (2.12) we see

$$\mathbf{x}^\tau \varphi_n^0(p, q, r) \equiv \mathbf{0} \quad \text{for all } n,$$

where

$$\mathbf{x}^\tau = [-\mathbf{1}\ -\mathbf{A}_1 \ldots -\mathbf{A}_{p_0} \mathbf{0} \mathbf{B}_1 \ldots \mathbf{B}_{q_0} \mathbf{1} \mathbf{C}_1 \ldots \mathbf{C}_{r_0}].$$

This implies that $\mu_{\min}^{(p,q,r)}(n) \not\to \infty$ as $n \to \infty$, and consequently the assumption (2.9) fails.

We now proceed to demonstrate that if $(p, q, r) \in M^*$ the excitation requirement (2.9) can indeed be satisfied for a large class of feedback control systems.

Let $\{\mathbf{v}_n\}$ be a sequence of l-dimensional mutually independent random vectors independent of $\{\mathbf{w}_n\}$ with properties

$$E\mathbf{v}_n = \mathbf{0}, \quad E\mathbf{v}_n\mathbf{v}_n^\tau = \frac{1}{n^\epsilon}\mathbf{I}, \quad \|\mathbf{v}_n\|^2 \leq \frac{\sigma^2}{n^\epsilon}, \quad \epsilon \in \left[0, \frac{1}{2(t+1)}\right), \tag{2.18}$$

where $t = (m + 1)p^* + q^* + r^* - 1$ and $\sigma^2 > 0$ is a constant.

Without loss of generality, we assume

$$\mathscr{F}_n = \sigma\{\mathbf{w}_i, \mathbf{v}_i, i \leq n\} \quad \text{and} \quad \mathscr{F}'_n = \sigma\{\mathbf{w}_i, \mathbf{v}_{i-1}, i \leq n\}.$$

Let $\mathbf{u}_n^0$ be an l-dimensional $\mathscr{F}'_n$-measurable desired control. Obviously, any feedback (adaptive) control is of this kind. The attenuating excitation technique developed in Chen and Guo [8, 11] suggests to take the actual input for the system as

$$\mathbf{u}_n = \mathbf{u}_n^0 + \mathbf{v}_n \tag{2.19}$$

instead of $\mathbf{u}_n = \mathbf{u}_n^0$. This method is very successful in simultaneously minimality of control performance and consistency of parameter estimate [8, 11].

The following identifiability condition is needed in the sequel:

Assumption 3

$A(z)$, $B(z)$ and $C(z)$ have no common left factor and $\mathbf{A}_{p_0}$, $\mathbf{B}_{q_0}$ and $\mathbf{C}_{r_0}$ are of full row rank.

We note that the rank requirement for $\mathbf{A}_{p_0}$, $\mathbf{B}_{q_0}$ and $\mathbf{C}_{r_0}$ in Assumption 3 is automatically satisfied for scalar systems [see (1.5)].

Theorem 2.2

Let the "attenuating excitation control" (2.19) be applied to the system (1.1)–(1.7). Suppose that Assumptions 1 and 3 are satisfied, and that there is a non-negative number δ,

$$\delta \in \left[0, \frac{1 - 2\epsilon(t+1)}{2t+3}\right]$$

such that

$$\frac{1}{n}\sum_{i=0}^{n}(\|\mathbf{y}_i\|^2+\|u_i^0\|^2)=O(n^\delta), \quad \text{a.s.} \tag{2.20}$$

Then for all $(p,q,r)\in M^*$,

$$\liminf_{n\to\infty}\mu_{\min}^{(p,q,r)}(n)/n^\alpha>0, \quad \text{a.s.}, \tag{2.21}$$

where $\alpha=1-(t+1)(\epsilon+\delta)$, and M^* and $\mu_{\min}^{(p,q,r)}(n)$ are the same as those in Theorem 2.1.

Proof. By (1.6) and the Martingale convergence theorem it is easy to see that the series

$$\sum_{i=1}^{\infty}(\|\mathbf{w}_{i+1}\|^2-E[\|\mathbf{w}_{i+1}\|^2|\mathscr{F}_i])/i$$

is convergent, and hence the Kronecker lemma leads to

$$\frac{1}{n}\sum_{i=1}^{n}(\|\mathbf{w}_{i+1}\|^2-E[\|\mathbf{w}_{i+1}\|^2|\mathscr{F}_i])\to 0, \quad \text{as } n\to\infty.$$

which in conjunction with (1.6) gives

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\|\mathbf{w}_i\|^2<\infty, \quad \text{a.s.}$$

Thus the desired result (2.17) can be proved along the lines of the proof of Theorem 2 in [18]. Details are omitted here. □

Remark 2

By (2.20) and (2.21), it is easy to see that under conditions of Theroem 2.2, (2.9) is satisfied for all $(p,q,r)\in M^*$, and hence the strong consistency of parameter estimates is guaranteed. Theorem 2.1 implies that for estimating the unknown parameters of (1.1), it is not necessary to assume that all of the three true orders $\{p_0,q_0,r_0\}$ are known. Indeed, given any one of the three orders $\{p_0,q_0,r_0\}$ and the upper bounds for the other two, the unknown parameters as well as the other two true orders can be identified theoretically if the system is suitably excited.

3. ORDER ESTIMATION FOR CONTROLLED TIME SERIES

We now consider the case where none of the three orders $\{p_0,q_0,r_0\}$ is available.

We need the following excitation condition on the signals of the system (1.1), which is slightly stronger than condition (2.9) used for parameter estimation.

Assumption 4

A sequence of positive numbers $\{\mathbf{c}_n\}$ can be found such that

$$\log\mu_{\max}^{(p,q,r)}(n)/\mathbf{c}_n\xrightarrow[n\to\infty]{}0,$$

and

$$\mathbf{c}_n/\mu_{\min}^{(p,q,r)}(n)\xrightarrow[n\to\infty]{}0, \quad \text{a.s.}$$

hold for any $(p,q,r)\in M^*$, where M^*, $\mu_{\max}^{(p,q,r)}(n)$ and $\mu_{\min}^{(p,q,r)}(n)$ are the same as those defined in Theorem 2.1.

Set

$$\hat{\mathbf{w}}_n^*=\mathbf{y}_n-\theta_n^\tau(p^*,q^*,r^*)\varphi_n(p^*,q^*,r^*), \tag{3.1}$$

$$f_n(p,q,r)=[\mathbf{y}_n^\tau\ldots\mathbf{y}_{n-p+1}^\tau\,\mathbf{u}_n^\tau\ldots\mathbf{u}_{n-q+1}^\tau\,\hat{\mathbf{w}}_n^{*\tau}\ldots\hat{\mathbf{w}}_{n-r+1}^{*\tau}]^\tau, \tag{3.2}$$

and

$$\alpha_n(p,q,r)=\left(\sum_{i=0}^{n-1} f_i(p,q,r)f_i^{\tau}(p,q,r)+\mathbf{P}_0^{-1}\right)^{-1}\sum_{i=0}^{n-1} f_i(p,q,r)\mathbf{y}_{i+1}^{\tau}, \tag{3.3}$$

for any $(p,q,r)\in M$, where $\mathbf{P}_0^{-1}$, $\theta_n(p^*,q^*,r^*)$ and $\varphi_n(p^*,q^*,r^*)$ are defined by (2.2)–(2.6) with (p,q,r) replaced by (p^*,q^*,r^*).

The new criterion CIC (where the first "C" emphasizes that the criterion is designed for control systems) introduced in [18] is

$$\mathrm{CIC}(p,q,r)_n=\sigma_n(p,q,r)+(p+q+r)\mathbf{c}_n, \tag{3.4}$$

where the subscript n denotes the data size, and where $\mathbf{c}_n$ is given in Assumption 4, and $\sigma_n(p,q,r)$ is a residual given by

$$\sigma_n(p,q,r)=\sum_{i=0}^{n-1}\|\mathbf{y}_{i+1}-\alpha_n^{\tau}(p,q,r)f_i(p,q,r)\|^2. \tag{3.5}$$

Finally, the estimate (p_n,q_n,r_n) for (p_0,q_0,r_0) is obtained by minimizing $\mathrm{CIC}(p,q,r)_n$, i.e.

$$(p_n,q_n,r_n)=\arg\min_{(p,q,r)\in M}\mathrm{CIC}(p,q,r)_n. \tag{3.6}$$

We note that the second term on the right-hand side of (3.4) heavily depends on signals of the system, and such a dependence is necessary because the input $\mathbf{u}_n$ is an arbitrary feedback control. This is the essential difference between CIC and the well known AIC, BIC and ΦIC.

Theorem 3.1

If Assumptions 1, 2 and 4 hold for the system described by (1.1)–(1.8) and the estimation procedure (3.1)–(3.6), then the order estimate (p_n,q_n,r_n) for (p_0,q_0,r_0) given by (3.6) is strongly consistent:

$$(p_n,q_n,r_n)\xrightarrow[n\to\infty]{}(p_0,q_0,r_0)\quad \text{a.s.} \tag{3.7}$$

Proof. The key steps of the proof is to establish the following expansion:

$$\mathrm{CIC}(p,q,r)_n-\mathrm{CIC}(p_0,q_0,r_0)_n$$

$$\geqslant\begin{cases}\mathbf{c}_n(p+q+r-p_0-q_0-r_0+o(1)), & \text{a.s., if } (s,t,\lambda)=(p,q,r), \quad (3.8)\\ \lambda_{\min}^{(s,t,\lambda)}(n)\dfrac{\alpha_0}{4}+o(1)), & \text{a.s., otherwise,} \quad (3.9)\end{cases}$$

for any $(p,q,r)\in M$, where

$$(s,t,\lambda)=(p\vee p_0,q\vee q_0,r\vee r_0),$$
$$\alpha_0=\min\{\|\mathbf{A}_{p_0}\|^2,\|\mathbf{B}_{q_0}\|^2,\|\mathbf{C}_{r_0}\|>0$$

and $a\vee b$ means $\max(a,b)$.

In the case where $(s,t,\lambda)=(p,q,r)$, (1.1) can be rewritten as

$$\mathbf{y}_{n+1}=\theta^{\tau}(p,q,r)\varphi_n^0(p,q,r)+\mathbf{w}_{n+1}$$

and we can show that

$$\sigma_n(p,q,r)=O(\log\bar{r}_n^0)+\sum_{i=0}^{n-1}\mathbf{w}_{i+1} \tag{3.10}$$

where

$$\bar{r}_n^0=\sum_{i=0}^{n-1}\|\varphi_i^0(p^*,q^*,r^*)\|^2+1.$$

From this it is easy to conclude (3.8).

In the case where $(s, t, \lambda) \neq (p, q, r)$ we note that

$$\lambda_{\min}^{(s,t,\lambda)}(n) \geq \{\lambda_{\min}^{(p_0,q^*,r^*)}(n), \quad \lambda_{\min}^{(p^*,q_0,r^*)}(n), \quad \lambda_{\min}^{(p^*,q^*,r_0)}(n)\}$$

then we can show

$$\sigma_n(p, p, r) \geq \frac{\alpha_0}{4}\lambda_{\min}^{(s,t,\lambda)}(n)(1 + o(1)) + \sum_{i=0}^{n-1} \mathbf{w}_{i+1},$$

which together with (3.10) implies (3.9).

The proof is completed by showing that any limit point of (p_n, q_n, r_n) coincides with (p_0, q_0, r_0) and this can be done by using (3.9) and (3.10). For details we refer to [18]. □

Example 1

Let us consider again the control systems described in Theorem 2.2. In such situations, it is easy to see that Assumption 4 is satisfied and $\mathbf{c}_n$ may be taken as n^a, $(\log n)^b$, $n^a(\log n)^b$ for appropriately chosen a and b.

4. RELAXATION OF SPR CONDITION

The SPR condition (Assumption 2) plays a crucial role in the preceding analysis for consistency of estimates (Theorems 2.1 and 3.1). Here we will relax this condition for a class of control systems.

To this end, let us further assume that the driven noise $\{\mathbf{w}_n\}$ in system (1.1) is a Gaussian white noise sequence with

$$E\mathbf{w}_n = 0, \quad E\mathbf{w}_n\mathbf{w}_n^\tau = \mathbf{R}_w > 0. \tag{4.1}$$

Define a "pre-whitened" process $\{\mathbf{z}_n\}$ as follows:

$$\mathbf{z}_n = \mathbf{y}_n + \mathbf{e}_n, \tag{4.2}$$

where $\{\mathbf{e}_n\}$ is a Gaussian white noise sequence which is independent of $\{\mathbf{u}_n, \mathbf{w}_n\}$ and with properties:

$$E\mathbf{e}_n = 0, \quad E\mathbf{e}_n\mathbf{e}_n^\tau = \sigma_e^2\mathbf{I}_m, \quad \sigma_e^2 > 0. \tag{4.3}$$

This "pre-whitening" idea was first proposed by Moore [20] and expanded on by Guo *et al.* [19]. The following theorem is a specialization of Theorem 2.1 in Guo *et al.* [19].

Theorem 4.1

Consider the controlled moving average process described by (1.1)–(1.5) with $p_0 = 0$, where the input $\{\mathbf{u}_n\}$ is assumed to be independent of $\{\mathbf{w}_n\}$ with $\{\mathbf{w}_n\}$ satisfying (4.1). If in the estimation algorithm (2.1)–(2.6), (p, q, r) is taken as $(0, q^*, r^*)$ and $\mathbf{y}_n$ is replaced by $\mathbf{z}_n$ for all n, and if in the pre-whitening of (4.2) and (4.3) σ_e^2 is chosen to satisfy

$$\sigma_e^2 > r_0 \|\mathbf{R}_w\| \cdot \|[\mathbf{C}_1 \ldots \mathbf{C}_{r_0}]\|^2 - \lambda_{\min}(\mathbf{R}_w)$$

then the following convergence rate holds

$$\|\hat{\mathbf{B}}_n - \mathbf{B}^*\| = O\left(\left(\frac{\log \lambda_{\max}^{(0,q^*,r^*)}(n)}{\lambda_{\min}^{(0,q^*,r^*)}(n)}\right)^{1/2}\right) \quad \text{a.s.}$$

where

$$\mathbf{B}^* = [\mathbf{B}_1 \ldots \mathbf{B}_{q_0} \underbrace{0\,0 \ldots 0}_{q^* - q_0}]$$

and $\hat{\mathbf{B}}_n$ is the estimate for $\mathbf{B}^*$ and is given by $\theta_n(0, q^*, r^*)$ (that is the first $m \times [l \times q^*]$ block of $\theta_n^\tau(0, q^*, r^*)$) and where $\lambda_{\max}^{(0,q^*,r^*)}(n)$, $\lambda_{\min}^{(0,q^*,r^*)}(n)$ are defined in a similar way as those in Theorem 2.1 only with $\mathbf{y}_i$ replaced by $\mathbf{z}_i$ (for all i) in the definition of $\varphi_i(0, q^*, r^*)$.

Remark 3

By using pre-whitening technique the order of the system described in this section can also be established without SPR condition. Also, the unknown parameters $\mathbf{C}_i (0 \leqslant i \leqslant r_0)$ can be identified by a parallel processing involving on-line spectral factorization [21].

It is worth noting that the "dither" $\{\mathbf{e}_n\}$ used in the "pre-whitening" method is added only to the estimation algorithm, while in the "attenuating excitation" method (see Theorem 2.2) the "dither" $\{\mathbf{v}_n\}$ is added into the system via the input (2.19). Relaxation of SPR condition for more general feedback control systems belongs to further study.

5. IDENTIFICATION OF A CLASS OF NONSTATIONARY ARMA PROCESSES

Consider the following uncontrolled multidimensional time series:

$$A(z)\mathbf{y}_n = C(z)\mathbf{w}_n, \quad n \geqslant 0, \tag{5.1}$$

where $A(z)$ and $C(z)$ are defined by (1.2) and (1.4) respectively, and $\{\mathbf{w}_n\}$ is as in (1.6)–(1.7).

This model is obviously a specialization of (1.1), and hence results presented in the proceeding sections can be directly applied.

The known parameter set M assumed in Assumption 1 now becomes

$$M = \{(p, r): 0 \leqslant p \leqslant p^*, 0 \leqslant r \leqslant r^*\} \tag{5.2}$$

and (2.1) and (2.7) now read

$$\theta(p, r) = [-\mathbf{A}_1 \ldots -\mathbf{A}_p \, \mathbf{C}_1 \ldots \mathbf{C}_r], \quad (p, r) \in M, \tag{5.3}$$

$$M^* = \{(p_0, r^*), (p^*, r_0)\}. \tag{5.4}$$

Similarly, (3.4)–(3.6) should be rewritten as

$$\mathrm{CIC}(p, r)_n = \sigma_n(p, r) + (p + r)\mathbf{c}_n, \tag{5.5}$$

$$\sigma_n(p, r) = \sum_{i=0}^{n-1} \|\mathbf{y}_{i+1} - \alpha_n^{\tau}(p, r) f_i(p, r)\|^2, \tag{5.6}$$

$$(p_n, r_n) = \arg\min_{(p, r) \in M} \mathrm{CIC}(p, r)_n, \quad n \geqslant 1 \tag{5.7}$$

where $f_i(p, r)$ and $\alpha_n(p, r)$ are defined by (3.2) and (3.3) but with all control terms $\mathbf{u}_i$ deleted, and where in (5.5) $\mathbf{c}_n$ is any sequence of positive numbers (e.g. $\mathbf{c}_n = \sqrt{n}$) satisfying

$$\mathbf{c}_n/n \to 0 \quad \text{and} \quad \log n/\mathbf{c}_n \to 0, \quad \text{as } n \to \infty. \tag{5.8}$$

For the system structure we make the following assumptions.

(H_1) Zeros of det $A(z)$ lie outside the unit disk or on the unit circle.
(H_2) $A(z)$ and $C(z)$ are left coprime and $\mathbf{A}_{p_0}$ and $\mathbf{C}_{r_0}$ are of full rank.
(H_3) $C^{-1}(z) - \frac{1}{2}\mathbf{I}$ is strictly positive real.

It is easy to see that system (5.1) under condition (H_1–H_3) is not necessarily stationary, because zeros of det $A(z)$ are allowed to lie on the unit circle.

Theorem 3.1

Consider the ARMA model (5.1)–(5.4) and assume that conditions (H_1–H_3) are satisfied.

(i) Let $\theta_n(p, r)$ be defined in the same way as $\theta_n(p, q, r)$ via (2.2)–(2.6) but with all control terms $\mathbf{u}_i$ deleted, if (p, r) is chosen so that $(p, r) \in M^*$, then as $n \to \infty$,

$$\|\theta_n(p, r) - \theta(p, r)\| = O((\log n/n)^{1/2}), \quad \text{a.s.} \tag{5.9}$$

(ii) The order estimate (p_n, r_n) for (p_0, r_0) given by (5.5)–(5.8) is strongly consistent, i.e.

$$(p_n, r_n) \xrightarrow[n \to \infty]{} (p_0, r_0), \quad \text{a.s.} \tag{5.10}$$

We first prove a lemma.

Lemma. For the ARMA model (5.1) and (5.2) if condition (H_2) is satisfied, then there is a constant $c_0 > 0$ such that for all $(p, r) \in M^*$,

$$\lambda_{\min}\left(\sum_{i=0}^{n} \varphi_i^0(p, r)\varphi_i^{0\tau}(p, r)\right) \geqslant c_0\lambda_{\min}\left(\sum_{i=0}^{n} \mathbf{W}_i\mathbf{W}_i^\tau\right), \tag{5.11}$$

where

$$\varphi_i^0(p, r) = [\mathbf{y}_i^\tau \dots \mathbf{y}_{i-p+1}^\tau \, \mathbf{w}_i^\tau \dots \mathbf{w}_{i-r+1}^\tau]^\tau \tag{5.12}$$

and

$$\mathbf{W}_i = [\mathbf{w}_i^\tau \dots \mathbf{w}_{i-mp^*-r^*+1}^\tau]^\tau. \tag{5.13}$$

Proof. This lemma can be proved by a similar argument as that used in the proof of Lemma 1 in Chen and Guo [22], so we only point out the key steps in the proof here.

Write

$$\det A(z) = \mathbf{a}_0 + \mathbf{a}_1 z + \dots + \mathbf{a}_{mp_0} z^{mp_0}$$

and define

$$\boldsymbol{\Psi}_n = (\det A(z))\varphi_n^0(p, r), \quad (p, r) \in M^* \tag{5.14}$$

By the Schwarz inequality it can be shown

$$\lambda_{\min}\left(\sum_{i=0}^{n} \boldsymbol{\Psi}_i\boldsymbol{\Psi}_i^\tau\right) \leqslant (mp_0 + 1) \sum_{j=0}^{mp_0} \mathbf{a}_j^2 \, \lambda_{\min}\left(\sum_{i=0}^{n} \varphi_i^0(p, r)\varphi_i^{0\tau}(p, r)\right).$$

Hence for (5.11) we need only to show that there exists $c_1 > 0$ such that for any $(\mathrm{p}, r) \in M^*$,

$$\lambda_{\min}\left(\sum_{i=0}^{n} \boldsymbol{\Psi}_i\boldsymbol{\Psi}_i^\tau\right) \geqslant c_1\lambda_{\min}\left(\sum_{i=0}^{n} \mathbf{W}_i\mathbf{W}_i^\tau\right). \tag{5.15}$$

For any vector $\mathbf{x} \in \mathbb{R}^{m(p+r)}$, let us write it in its vector-component form

$$\mathbf{x} = [\mathbf{x}^{(0)\tau} \dots \mathbf{x}^{(p-1)\tau}\mathbf{x}^{(p)\tau} \dots \mathbf{x}^{(p+r-1)\tau}]^\tau,$$

with $\mathbf{x}^{(i)} \in \mathbb{R}^m$, $0 \leqslant i \leqslant p + r - 1$, and set

$$H_x(z) = \mathbf{x}^{(0)\tau}(\operatorname{Adj} A(z))C(z) + \dots + \mathbf{x}^{(p-1)\tau}(\operatorname{Adj} A(z))C(z)z^{p-1}$$
$$+ \mathbf{x}^{(p)\tau} \det A(z) + \dots + \mathbf{x}^{(p+r-1)\tau} z^{r-1} \det A(z) \triangleq \sum_{i=0}^{s} g_i^\tau(\mathbf{x}) z^i, \quad s = mp^* + r^* - 1. \tag{5.16}$$

By (5.1), (5.12), (5.14) and (5.16), a simple manipulation leads to

$$\mathbf{x}^\tau \sum_{i=0}^{n} \boldsymbol{\Psi}_i\boldsymbol{\Psi}_i^\tau \mathbf{x} = \sum_{i=0}^{n} (H_x(z)\mathbf{W}_i)^2 \geqslant \min_{\|\mathbf{x}\|=1} \|g(\mathbf{x})\| \, \lambda_{\min}\left(\sum_{i=0}^{n} \mathbf{W}_i\mathbf{W}_i^\tau\right), \quad \mathbf{x} \in \mathbb{R}^{m(p+r)} \tag{5.17}$$

for any $\mathbf{x} \in \mathbb{R}^{m(p+r)}$, $\|\mathbf{x}\| = 1$, and any $(p, r) \in M^*$, where

$$g(\mathbf{x}) = [g_0^\tau(\mathbf{x}), \dots, g_s^\tau(\mathbf{x})]^\tau$$

with $g_i(\mathbf{x})$ $(0 \leqslant i \leqslant s)$ given by (5.16).

Thus for (5.15) it suffices to show that

$$\min_{\|\mathbf{x}\|=1} \|g(\mathbf{x})\| \neq 0, \quad \forall (p, r) \in M^*, \quad \mathbf{x} \in \mathbb{R}^{m(p+r)},$$

which can be guaranteed by condition (H_2) (see [22] and [18]). □

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1

By condition (H_1) from (5.1) and (1.6) it is not difficult to see that there exists a constant $b \geqslant 1$ such that

$$\sum_{i=0}^{n} \|\mathbf{y}_i\|^2 = O(n^b), \quad \text{a.s.}$$

from here and (1.6) and (5.12), we know that

$$\lambda_{\max}\left(\sum_{i=0}^{n} \varphi_i^0(p,r)\varphi_i^{0\tau}(p,r)\right) = O(n^b), \quad \text{a.s.}, \tag{5.18}$$

for any $(p, r) \in M^*$.

On the other hand, by (1.6), (1.7) and (5.11) it is evident that

$$\liminf_{n\to\infty} \lambda_{\min}\left(\frac{1}{n}\sum_{i=1}^{n} \varphi_i^0(p,r)\varphi_i^{0\tau}(p,r)\right) \neq 0, \quad \text{a.s.}, \tag{5.19}$$

for any $(p, r) \in M^*$.

Combining (5.18), (5.19), we see

$$\log \lambda_{\max}\left(\sum_{i=0}^{n} \varphi_i^0(p,r)\varphi_i^{0\tau}(p,r)\right) \Big/ \lambda_{\min}\left(\sum_{i=0}^{n} \varphi_i^0(p,r)\varphi_i^{0\tau}(p,r)\right) = O\left(\frac{\text{lon}\, n}{n}\right) \tag{5.20}$$

for any $(p, r) \in M^*$.

Finally, the desired results (5.9) and (5.10) are derived by applying results in Theorems 2.1 and 2.3. This completes the proof. □

6. CONCLUSION

We have used the system-theoretic methods for nonstationary time series analysis and have given consistency analysis for estimates of both orders and unknown coefficients of the model which in this paper is restricted to the linear and time-invariant one. Even in this framework there are many problems still left open. For example: how to weaken the SPR condition for general controlled ARMA processes? How to recursively estimate orders of feedback systems and how to relax the restriction that the upper bounds for orders are *a priori* available? Further, the robustness issue is of great importance, i.e. how to model the system if the real data differ from (1.1) by some unmodelled dynamics [23, 24]? If control $\mathbf{u}_n$ is designed to minimize some cost, then the problem belongs to the field of stochastic adaptive control.

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