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The Limit Passace in the Stochastic Adaptive LQ Control Problem*

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Abstract

For MIMO stochastic adaptive control systems with quadratic loss function with weighting εI for control u this paper considers the exchangeability of "$\lim_{\varepsilon\to 0}$" with "$\inf_u \limsup_{n\to\infty}$" and establishes that the minimal value of the quadratic loss function tends to the minimal tracking error as $\varepsilon\to 0$.

1. Description of the system

We consider the stochastic system described by

$$A(z)y_{n+1}=B(z)u_n+C(z)w_{n+1} \tag{1}$$

where y_n, u_n and w_n denote the m—output, l—input and m—driven noise respectively, $A(z)$, $B(z)$ and $C(z)$ are matrix polynomials in shift-back operator z

$$A(z)=I+A_1z+\cdots+A_pz^p \tag{2}$$

$$B(z)=B_1+B_2z+\cdots B_qz^{q-1} \tag{3}$$

$$C(z)=I+C_1z+\cdots+C_rz^r \tag{4}$$

with unknown coefficient

$$\theta^{\tau}=[-A_1\cdots-A_p, B_1,\cdots B_q, C_1\cdots C_r]$$

Let the driven noise be a mds $\{w_n, F_n\}$ with properties

$$\sup_i E\{\|w_{i+1}\|^2/F_i\}<\infty \tag{5}$$

and

$$\frac{1}{n}\sum_{i=1}^{n} w_iw_i^{\tau} \longrightarrow R>0. \quad \text{a.s.} \tag{6}$$

The stochastic adaptive LQ control problem consists in simultaneously identifying the unknown θ and minimizing the loss function

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$\limsup_{n} J_n(u)$, where

$$J_n(u)=\frac{1}{n}\sum_{i=1}^{n}[(y_i-y_i^*)^\tau Q_1(y_i-y_i^*)+u_i^\tau Q_2u_i], \tag{7}$$

where $Q_1\geqslant 0, Q_2>0$, $\{y_n^*\}$ is a bounded deterministic reference signal and where the control u_i at time i is required to depend on $\{u_j, j<i, y_k, k\leqslant i\}$ only.

2. Statement of the Problem

Let $P^\tau D=H^\tau Q_1H$ be any decomposition of $H^\tau Q_1H$ with

$$H=[\underbrace{I\quad 0\cdots 0\quad}_{ms}]\}m \tag{8}$$

and set

$$A=\begin{pmatrix}-A_1 & I & 0\cdots 0\\ \vdots & \ddots & \\ \vdots & \cdots & I\\ -A_s & 0 & 0\cdots 0\end{pmatrix},\quad B=\begin{pmatrix}B_1\\ \vdots\\ B_s\end{pmatrix},\quad C=\begin{pmatrix}I\\ C_1\\ \vdots\\ C_{s-1}\end{pmatrix},\quad s=\max(p,q,r+1) \tag{9}$$

and $\quad A_i=0,\ B_j=0,\ C_k=0\quad$ for $i>p, j>q, k>r$.

Condition A. The triple (A,B,D) is controllable and observable.

It is well known [1] that under condition A the Riccati equation

$$S=A^\tau SA-A^\tau SB(Q_2+B^\tau SB)^{-1}B^\tau SA+H^\tau Q_1H \tag{10}$$

has an unique positive definite solution $S>0$ in the class of non—negative definite matrices and the matrix

$$F=A-B(Q_2+B^\tau SB)^{-1}B^\tau SA \tag{11}$$

is stable.

It is shown in[2,3] that

$$\inf_u \limsup_{n\to\infty} J_n(u)=\mathrm{tr}\quad SCRC^\tau+\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n-1}[y_i^{*\tau}Q_1y_i^*$$

$$-\ b_{i+1}^\tau B(Q_2+B^\tau SB)^{-1}B^\tau b_{i+1}], \tag{12}$$

where

$$b_i=-\sum_{j=0}^{\infty}F^{j\tau}H^\tau Q_1y_{i+j}^*=Fb_{i+1}-H^\tau Q_1y_i^* \tag{13}$$

For the special case $Q_1=I$ and $Q_2=\varepsilon I$ we rewrite $J_n(u)$ as

$$J_n^\varepsilon(u)=\frac{1}{n}\sum_{i=1}^{n}(\|y_i-y_i^*\|^2+\varepsilon\|u_i\|^2). \tag{14}$$

The above mentioned results say that

$$\inf_{u\in U}\limsup_{n\to\infty} J_n^\varepsilon(u)=\mathrm{tr}\ S^\varepsilon CRC^\tau+\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n-1}(\|y_i^*\|^2$$

$$-b_{i+1}^{\varepsilon\tau}B(\varepsilon I+B^\tau S^\varepsilon B)^{-1}Bb_{i+1}^\varepsilon) \tag{15}$$

where S^ε is defined by (10) with Q_1,Q_2 replaced by I and εI respectively and b_i^ε is defined by (13) with F replaced by

$$F_\varepsilon=A-B(\varepsilon I+B^\tau S^\varepsilon B)^{-1}B^\tau S^\varepsilon A \tag{16}$$

On the other hand, the following performance

$$\limsup_{n\to\infty} J_n^0(u)=\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\|y_i-y_i^*\|^2 \tag{17}$$

corresponds to the stochastic adaptive tracking problem for which from [4],[5],[6] we know that

$$\inf_u\limsup_{n\to\infty} J_n^0(u)=\mathrm{tr}\ R. \tag{18}$$

It is natural to ask if the limit passage holds true, i.e. if

$$\lim_{\varepsilon\to 0}\inf_u\limsup_{n\to\infty} J_n^\varepsilon(u)=\inf_u\limsup_{n\to\infty} J_n^0(u)\ ? \tag{19}$$

We note that the right—hand side of (15) greatly differs from that of(18)and the optimal stochastic adaptive controls for tracking and for quadratic loss function are structurally different from each other.

3. Main Result

Theorem Assume that $1=m$, Condition A is satisfied and $B(z)$ is stable with B_1 being of full rank. Then

$$\lim_{\varepsilon\to 0}\Big[\mathrm{tr}\ S^\varepsilon CRC^\tau+\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n-1}(\|y_i^*\|^2$$

$$-b_{i+1}^{\varepsilon\tau}B(\varepsilon I+B^\tau S^\varepsilon B)^{-1}B^\tau b_{i+1}^\varepsilon)\Big]=\mathrm{tr}\ R, \tag{20}$$

i. e. (19) is valid.

Proof It is well—known[1] that for the deterministic system

$$x_{k+1}=Ax_k+Bu_k$$

with performance index

$$J_\varepsilon(u)=\sum_{i=0}^{\infty}(x_i^\tau H^\tau Hx_i+u_i^\tau\varepsilon u_i)$$

the minimum value of $J_\varepsilon(u)$ is $\min_u J_\varepsilon(u) = x_0^\tau S^\varepsilon x_0$. From this we see that as $\varepsilon \to 0$, S^ε non-increasingly converges to a finite limit S^0

$$S^\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} S^0 \geqslant 0 \tag{21}$$

By the matrix inverse formula

$$(S^{-1} + BR^{-1}B^\tau)^{-1} = S - SB(R + B^\tau SB)^{-1}B^\tau S$$

we can rewrite (10) with $Q_1 = I, Q_2 = \varepsilon I$ as

$$S^\varepsilon = A^\tau((S^\varepsilon)^{-1} + B\varepsilon^{-1}B^\tau)^{-1}A + H^\tau H. \tag{22}$$

From this we find that

$$B^\tau S^\varepsilon B \geqslant B^\tau H^\tau HB = B_1^\tau B_1 > 0 \qquad \forall \varepsilon > 0$$

hence $B^\tau S^0 B \geqslant B_1^\tau B_1 > 0$ and

$$\|S^\varepsilon B(\varepsilon I + B^\tau S^\varepsilon B)^{-1}B^\tau S^\varepsilon - S^0 B(B^\tau S^0 B)^{-1}B^\tau S^0\| \underset{\varepsilon \to 0}{\longrightarrow} 0 \tag{23}$$

Letting $\varepsilon \longrightarrow 0$ in (22) and noticing (21), (23) we obtain

$$S^0 = A^\tau S^0 A - A^\tau S^0 B(B^\tau S^0 B)^{-1}B^\tau S^0 A + H^\tau H \tag{24}$$

It is easy to verify that $H^\tau H$ satisfies equation (24) and

$$(S^0 - H^\tau H) = (A + BL^0)^\tau(S^0 - H^\tau H)(A + BL^*) \tag{25}$$

where

$$L^0 = -(B^\tau H^\tau HB)^{-1}B^\tau S^0 A,$$

$$L^* = -(B^\tau H^\tau HB)^{-1}B^\tau H^\tau HA = -[B_1^{-1}0\cdots 0A].$$

Then we have

$$S^0 - H^\tau H = (A + BL^0)^{n\tau}(S^0 - H^\tau H(A + BL^*)^n, \quad \forall n > 1 \tag{26}$$

we note that

$$A + BL^* = A - \begin{pmatrix} B_1 \\ \vdots \\ B_s \end{pmatrix}[B_1^{-1}0\cdots 0],$$

$$A = \begin{pmatrix} 0 & 0 & 0\cdots & 0 \\ B_2B_1^{-1}A_1 - A_2, & -B_2B_1^{-1}, & I\cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ & & & I \\ B_sB_1^{-1}A_1 - A_s, & -B_sB_1^{-1}, & 0\cdots & 0 \end{pmatrix}. \tag{27}$$

and

$$\det(\lambda I-(A+BL^*))=\lambda^m\det\left(\lambda I-\begin{pmatrix}-B_2B_1^{-1} & I & 0\cdots 0\\ \vdots & & \ddots \\ & & & I\\ -B_s B_1^{-1} & 0 & \cdots & 0\end{pmatrix}\right)$$

$$=\lambda^m\det(\lambda^{s-1}I+\lambda^{s-2}B_2B_1^{-1}+\cdots B_sB_1^{-1})$$

$$=\det\ B_1^{-1}\lambda^m\det(\lambda^{s-1}B_1+\lambda^{s-2}B_2+\cdots+B_s).$$

By Condition A from this we know that $A+BL^*$ is stable and hence

$$\|(A+BL^*)^n\|=0(\gamma^n),\ \text{for some}\ \gamma\in(0,1)$$

What we can say for $A+BL^0$ is that all its eigenvalues are less than or equal to 1, since $A-B(\varepsilon I+B^\tau S^\varepsilon B)^{-1}B^\tau S^\varepsilon A$ is stable for any $\varepsilon>0$. Then $\|(A+BL^0)^n\|=0(n^{m s})$.

Therefore, from (26) we assert

$$\|S^0-H^\tau H\|=0(n^{m s})0(\gamma^n)\xrightarrow[n\to\infty]{}0$$

and

$$S^0=H^\tau H,\ L^*=L^0 \tag{28}$$

Thus, $\operatorname{tr} S^\varepsilon CRC^\tau\xrightarrow[\varepsilon\to 0]{}\operatorname{tr}R$. So to complete the proof we only need to show that the second term in left—hand side of (20) tends to 0 as $\varepsilon\to 0$, i. e.

$$\lim_{\varepsilon\to 0}\ \limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n}[\|y_i^*\|^2-b_{i+1}^{\varepsilon\tau}B(\varepsilon I+B^\tau S^\varepsilon B)^{-1}B^\tau b_{i+1}^\varepsilon]=0 \tag{29}$$

Since stable matrices F_ε converge to a stable matrix $A+BL^0$ as $\varepsilon\to 0$, the series $\sum_{j=1}^{\infty}\|F_\varepsilon^{j\tau}H^\tau\|$ uniformly converges in ε belonging to some interval $\varepsilon\in[0,\alpha]$ with α small. Then noticing $H(A+BL^0)=0$ from (27), we obtain from (13) that

$$\sup_{i\geq 0}\|b_{i+1}^\varepsilon-H^\tau y_{i+1}^*\|=\sup_{i\geq 0}\|\sum_{j=1}^{\infty}F_\varepsilon^{j\tau}H^\tau y_{i+j+1}^*\|\xrightarrow[\varepsilon\to 0]{}0$$

and hence

$$\sup_{i\geq 0}\|y_i^{*\tau}y_i^*-b_{i+1}^{\varepsilon\tau}B(\varepsilon I+B^\tau S^\varepsilon B)^{-1}B^\tau b_{i+1}^\varepsilon\|=\sup_{i\geq 0}\|y_i^{*\tau}$$

$$\cdot(I-HB(\varepsilon I+B^\tau S^\varepsilon B)^{-1}B^\tau H^\tau)y_i^*+y_i^{*\tau}HB(\varepsilon I+B^\tau S^\varepsilon B)^{-1}$$

$$\cdot B^\tau H^\tau y_i^*-b_{i+1}^{\varepsilon\tau}B(\varepsilon I+B^\tau S^\varepsilon B)^{-1}B^\tau b_{i+1}^\varepsilon\|\to 0$$

which implies (29).

4. Conclusions and Further Comments

In this short paper we have shown the continuity of minimal values of the quadratic loss function for stochastic adaptive control systems as the weighting matrix εI for control goes to zero. However, it is still open, if the optimal adaptive control itself converges in some sense. Further, if this is the case, does the optimal adaptive LQ control tend to the optimal adaptive tracking control as $\varepsilon \to 0$? Since the minimal value of the quadratic loss function is the same for both adaptive and non-adaptive control systems, the above mentioned questions also arise for systems with known θ.

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随机适应LQ问题中的极限过渡

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摘 要

对二次损失函数下的多输入多输出随机适应控制系统，本文讨论对控制作用u的加权阵为εI时“$\lim\limits_{\varepsilon\to 0}$”和“$\inf\limits_{u}\ \limsup\limits_{n\to\infty}$”的极限交换问题，并证明二次损失函数的最小值当$\varepsilon\to 0$时趋于最小跟踪误差。