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Convergence and Logarithm Laws of Self-tuning Regulators*

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Convergence problems of several basic least-squares self-tuning regulators are solved completely. Some recent related results are also surveyed and extended.

Key Words—Stochastic systems; least-squares; self-tuning regulator; convergence analysis.

Abstract—This paper starts with a survey of some recent results on least-squares (LS) and LS-based self-tuning regulators (STR). Several long-standing problems concerning the basic properties of LS-based STR, such as stability, optimality, consistency and the best convergence rate, are solved within a unified framework. Some previously related results are also subsumed and extended. Various new techniques for analysing LS-based adaptive tracking systems are presented, which may also be useful for analysing other adaptive control problems.

1 INTRODUCTION

Over the past three decades, a great deal of research effort has been devoted to the area of adaptive control, and much progress has been made in both theory and applications (see e.g. Goodwin and Sin, 1984; Kumar and Varaiya, 1986; Åström and Wittenmark, 1989; Chen and Guo, 1991; and references therein). However, because of the strong nonlinearity of the closed-loop equations of adaptive systems and the complexity of the stochastic processes involved, many fundamental theoretical problems still remain open (see e.g. Åström, 1983), among which the convergence of the least-squares (LS) based self-tuning regulator (STR) is probably the most notable.

The STR was originally proposed by Kalman (1958), and developed for stochastic minimum variance control problems in the landmark paper of Åström and Wittenmark (1973). Since the STR is very flexible with respect to the underlying design method and is easy to implement with microprocessors, it has received considerable attention. Apparently, Åström and Wittenmark (1973) were the first to attempt an analysis of adaptive minimum-variance control constructed using LS-type estimates. They showed that if the LS parameter estimates converge to some limit then the adaptive controller must be optimal. However, as they noted, since the closed-loop system is characterized as a nonlinear stochastic system, it is very difficult to rigorously prove that the estimates are indeed convergent, although simulations of numerous examples indicate that they do. Later, Ljung (1977) presented an interesting analysis of the STR via the ODE approach; however, some conditions on the input–output signals of the closed-loop system are required, which are found to be hard to either remove or verify.

The first significant progress in this direction was made by Goodwin *et al.* (1981). They showed that if the LS algorithm is replaced by a stochastic approximation (SA) algorithm then the resulting adaptive control system is asymptotically optimal. However, as pointed out by Sin and Goodwin (1982), it seems that in practically all applications of stochastic adaptive control a least-squares iteration is used, since it generally has much superior rates of convergence compared with stochastic approximation. In view of this, Sin and Goodwin (1982) and Chen (1984) studied a modified least-squares (MLS) algorithm for parameter estimation, and showed that the resulting adaptive control system is asymptotically optimal; Kumar and Moore (1982) introduced a sequence of weighting coefficients in the LS algorithm, which is chosen according to a certain stability measure to guarantee

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convergence. However, it is not known whether the control laws in Sin and Goodwin (1982) and Kumar and Moore (1982) enjoy the same nice convergence rate as that of the LS-based adaptive algorithm. Moreover, the modifications introduced there are not shown to be really necessary.

In order to improve the convergence rate, Lai and Wei (1986) studied another type of modification of the Åström–Wittenmark scheme, and, by introducing occasional blocks of white-noise probing inputs, they proved a sharp convergence rate for the control law when the underlying system is open-loop stable. Guo and Chen (1988) also modified the LS-based control law by introducing a sequence of stopping times as well as a sequence of decaying excitation signals; they established the convergence rates for both the controller and the parameter estimator. The main restriction in Lai and Wei (1986) and Guo and Chen (1988) is that the convergence of the adaptive schemes is only proved for open-loop stable systems. Further, by using the technique of 'Bayesian embedding', Kumar (1990) succeeded in showing that, outside an exceptional set of true parameter vectors of Lebesgue measure zero, the LS-based adaptive controller converges. However, as implicitly mentioned by Kumar (1990), the main problems with the 'Bayesian embedding' approach are:

(i) the Gaussianity and independency assumptions on the noise are essential;
(ii) the convergence is not guaranteed when the true parameter vector is contained in the exceptional set.

Recently, on the basis of some ideas in the analysis of time-varying stochastic systems by Guo (1990), a novel approach has been found by Guo and Chen (1991), which leads to a complete stability proof for a class of LS-based STR. Later, some results of Guo and Chen (1991) were generalized to multidelay systems by Ren and Kumar (1991) and to model reference adaptive control by Meyn and Brown (1993). However, several interesting issues are not addressed by Guo and Chen (1991):

(i) in the case where b_1 is not available, it is not known if it is possible to prove the convergence of the LS-based STR without making any modifications to the LS estimate for b_1;
(ii) the methods for establishing the logarithm law or the best convergence rate of the LS-based STR are not discussed;
(iii) the consistency of LS estimate in adaptive control systems is not clear when no external excitations are introduced.

These issues will be discussed here.

This paper is based on the work of Guo (1993) and Guo and Chen (1991). Its purpose is threefold:

(i) to present a unified solution for several long-standing problems concerning the basic asymptotic properties of LS-based STR;
(ii) to give a survey (and extension) of some recent theoretical progresses on convergence of both LS estimator and LS-based STR;
(iii) to refine some key analytical techniques used for LS-based adaptive systems, so that they may be easily comprehended and (it is hoped) applied to other adaptive control problems.

For simplicity of mathematical calculations, and for easy understanding of the underlying key proof ideas, we only consider SISO systems with unit delay and white noise disturbances in this paper. More general MIMO systems with multidelay and coloured noises may be treated in a similar way, as partly illustrated in Guo and Chen (1991), Guo (1994) and Ren and Kumar (1991).

2. PROBLEM FORMULATION

Consider the following SISO system:

$$A(z)y_n = B(z)u_{n-1} + w_n, \quad n \geq 0, \tag{1}$$

where $\{y_n\}$, $\{u_n\}$ and $\{w_n\}$ are, respectively, the system output, input and noise processes (without loss of generality, we assume $y_n = w_n = u_n = 0\ \forall n < 0$), and $A(z)$ and $B(z)$ are polynomials in the backward-shift operator z:

$$A(z) = 1 + a_1 z + \cdots + a_p z^p, \quad p \geq 0,$$
$$B(z) = b_1 + b_2 z + \cdots + b_q z^{q-1}, \quad q \geq 1,$$

with a_i and b_j unknown coefficients and p and q upper bounds on the true orders.

We introduce the unknown parameter vector

$$\theta = [-a_1 \ \cdots \ -a_p \ \ b_1 \ \cdots \ b_q]^{\mathrm{T}} \tag{2}$$

and the corresponding regressor

$$\varphi_n = [y_n \ \cdots \ y_{n-p+1} \ \ u_n \ \cdots \ u_{n-q+1}]^{\mathrm{T}}. \tag{3}$$

The system (1) may be succinctly written as

$$y_{n+1} = \theta^{\mathrm{T}}\varphi_n + w_{n+1}, \quad n \geq 0. \tag{4}$$

Our control objective is, at any time instant n, to construct a feedback control u_n based on the past measurements $\{y_0 \ \cdots \ y_n \ u_0 \ \cdots \ u_{n-1}\}$ so that the following averaged tracking error is asymptotically minimized:

$$J_n \triangleq \frac{1}{n}\sum_{i=1}^{n}(y_i - y_i^*)^2, \tag{5}$$

where $\{y_i^*\}$ is a known reference signal.

In order to analyse the above control problem, we introduce the following standard conditions:

(A.1) The noise sequence $\{w_n, \mathscr{F}_n\}$ is a Martingale difference sequence (where $\{\mathscr{F}_n\}$ is a sequence of nondecreasing σ-algebras) with conditional variance σ^2, i.e.

$$E[w_{n+1}^2 \mid \mathscr{F}_n] = \sigma^2 > 0 \quad \text{a.s.} \tag{6}$$

We also assume that there exists $\beta > 2$ such that

$$\sup E[|w_{n+1}|^\beta \mid \mathscr{F}_n] < \infty \quad \text{a.s.} \tag{7}$$

(A.2) $B(z) \neq 0$ $\forall z$ with $|z| \leq 1$ (the minimum-phase condition).

(A.3) $\{y_n^*\}$ is a bounded random/deterministic reference sequence that is independent of $\{w_n\}$.

It is well-known (see e.g. Goodwin and Sin, 1984; Kumar and Varaiya, 1986) that for the control problem (5), the Condition A.2 is necessary for the internal stability of the closed-loop system.

We first consider the case where the parameter θ is known. Since $\{w_n\}$ is unpredictable, it is obvious that the minimum value of $\lim_{n\to\infty} J_n$ is σ^2 and the corresponding optimal control law satisfies $y_{n+1}^* = E[y_{n+1} \mid \mathscr{F}_n]$, or, according to (4) and A.1

$$\theta^{\mathrm{T}} \varphi_n = y_{n+1}^* \tag{8}$$

From this, the optimal control u_n can be expressed explicitly as

$$u_n = \frac{1}{b_1}(a_1 y_n + \cdots + a_p y_{n-p+1} + b_2 u_{n-1} - \cdots - b_q u_{n-q+1} + y_{n+1}^*) \tag{9}$$

Substituting this into (1), we obtain the following ideal closed-loop equation in the case where θ is known:

$$y_n - y_n^* - w_n \equiv 0 \quad \forall n \geq 0 \tag{10}$$

Next, we consider the case where θ is unknown. In this case the recursive least-squares (LS) method is usually used, which gives estimates for the unknown parameter θ as follows:

$$\theta_{n+1} = \theta_n + a_n P_n \varphi_n (y_{n+1} - \varphi_n^{\mathrm{T}} \theta_n), \tag{11}$$

$$P_{n+1} = P_n - a_n P_n \varphi_n \varphi_n^{\mathrm{T}} P_n, \tag{12}$$

$$a_n = (1 + \varphi_n^{\mathrm{T}} P_n \varphi_n)^{-1}, \tag{13}$$

where the initial values θ_0 and $P_0 > 0$ can be arbitrarily chosen.

According to the 'certainty equivalence principle', replacing the θ in (8) by its LS estimate θ_n, we obtain the following standard STR:

$$\theta_n^{\mathrm{T}} \varphi_n = y_{n+1}^* \tag{14}$$

or

$$u_n = \frac{1}{b_{1n}}(a_{1n} y_n + \cdots + a_{pn} y_{n-p+1} - b_{2n} u_{n-1} - \cdots - b_{qn} u_{n-q+1} + y_{n+1}^*), \tag{15}$$

where a_{in} and b_{jn} are the components of θ_n, i.e.

$$\theta_n \triangleq [-a_{1n} \ \cdots \ -a_{pn} \ \ b_{1n} \ \cdots \ b_{qn}]^{\mathrm{T}}.$$

Since the closed-loop equation in the ideal case is (10), it is natural to expect that in the adaptive case under (14) the closed-loop equation satisfies

$$y_n - y_n^* - w_n \approx 0 \quad \forall n, \quad \text{in a certain sense.}$$

Usually, we expect that the accumulated 'closed-loop tracking error'

$$R_n \triangleq \sum_{i=1}^{n} (y_i - y_i^* - w_i)^2 \tag{16}$$

satisfies

$$R_n = o(n) \quad \text{a.s.} \tag{17}$$

Intuitively speaking, (17) means that the error '$y_n - y_n^* - w_n$' converges to zero in an averaging sense.

By the condition A.1, it is easy to prove that

$$R_n = o(n) \Leftrightarrow J_n \xrightarrow[n\to\infty]{} \sigma^2 \quad \text{a.s.},$$

where J_n and σ^2 are, respectively, defined by (5) and (6).

Therefore a sufficient and necessary condition for the optimality of STR is that (17) holds. Now, a natural question is whether or not (17) is really satisfied for LS-type STR? Moreover, how fast is the growth rate of R_n? The latter question is concerned essentially with the accuracy or convergence rate of the STR. We shall treat these problems in the subsequent sections.

3 ANALYSIS OF LS

The analysis of the LS algorithm is the first step in the study of any LS-based adaptive control.

The origins of LS can be traced back at least to Gauss (1863), and its recursive form can be found in the early works of Plackett (1950). A basic problem in the analysis of LS is strong consistency, i.e. determining the kind of conditions under which

$$\lim_{n\to\infty} \theta_n = \theta \quad \text{a.s.} \tag{18}$$

There is a vast literature on strong consistency of LS. In the field of dynamical system identification the early works are those by Ljung (1976), Moore (1978) and Solo (1979), among many others. They proved that if the condition number of P_n^{-1} is bounded, or if the input–output signals satisfy a persistence-of-excitation (PE) condition, then (18) holds. The PE condition was later significantly relaxed by Chen (1982). In a subsequent breakthrough paper, Lai and Wei (1982) succeeded in obtaining the weakest condition (in some sense) for the strong consistency of the LS algorithm. Various generalizations of their results are also available (see e.g. Chen and Guo, 1986; Lai and Wei, 1986). Unfortunately, it has been found that the verifications for even the weakest consistency condition of Lai and Wei (1982) are surprisingly difficult when the input–output signals are generated by the closed-loop equations determined by the LS-type STR. This is so even if the reference signal $\{y_i^*\}$ is sufficiently rich. Nevertheless, analysing properties of the LS algorithm is a preliminary step towards the study of LS-based STR.

In this section we shall consider the linear regression model

$$y_{n+1} = \theta^{\mathrm{T}}\varphi_n + w_{n+1}, \quad n \geq 0, \tag{19}$$

which is of the form (4); however, here y_n and φ_n are not necessarily constrained to satisfy (1) or (3). What we actually require here is that φ_n be d-dimensional and $\mathscr{F}_n$-measurable.

Lemma 3.1. Let the condition A.1 be satisfied for the linear model (19). Then the LS algorithm (11)–(13) satisfies the following identity as $n \to \infty$:

$$\tilde{\theta}_{n+1}^{\mathrm{T}} P_{n+1}^{-1} \tilde{\theta}_{n+1} + [1 + o(1)] \sum_{k=0}^{n} a_k (\varphi_k^{\mathrm{T}} \tilde{\theta}_k)^2 = \sigma^2 \sum_{k=0}^{n} a_k \varphi_k^{\mathrm{T}} P_k \varphi_k + o(\log (\det P_{n+1}^{-1})) + O(1) \quad \text{a.s.},$$

where $\tilde{\theta}_k = \theta - \theta_k$.

This lemma requires no excitation conditions on the regression process $\{\varphi_n\}$, and is based on the work of Lai (1986), Wei (1987), Chen and Guo (1986) and Guo and Chen (1991). The proof is supplied in the Appendix. From this basic lemma, a number of useful results can now be derived.

Corollary 3.1. Under the conditions of Lemma 3.1, we have

(i) $$\sum_{k=0}^{n} \frac{(\varphi_k^{\mathrm{T}} \tilde{\theta}_k)^2}{1 + \varphi_k^{\mathrm{T}} P_k \varphi_k} = O(\log r_n) \quad \text{a.s.},$$

(ii) $$\tilde{\theta}_{n+1}^{\mathrm{T}} P_{n+1}^{-1} \tilde{\theta}_{n+1} = O(\log r_n) \quad \text{a.s.},$$

where

$$r_n = 1 + \sum_{i=0}^{n} \|\varphi_i\|^2.$$

Proof. By (12) and (13), it is evident that

$$P_{n+1}^{-1} = P_0^{-1} + \sum_{k=0}^{n} \varphi_k \varphi_k^{\mathrm{T}}. \tag{20}$$

So, denoting $d = \dim(\varphi_k)$, we have

$$\log[\det(P_{n+1}^{-1})] \leq d \log \lambda_{\max}(P_{n+1}^{-1}) \leq d \log r_n + O(1). \tag{21}$$

Consequently Corollary 3.1 follows immediately from Lemma 3.1 by noting (21) and the relation (A.3) from the Appendix.

Corollary 3.2. Under the conditions of Lemma 3.1, the following results hold:

(i) $$\|\theta_{n+1} - \theta\|^2 = O\left(\frac{\log r_n}{\lambda_{\min}(P_{n+1}^{-1})}\right) \quad \text{a.s.};$$

(ii) $\|\theta_{n+1}\| = o(\sqrt{\log r_n})$ a.s. on $\{r_n \to \infty\}$.

Proof. Note that

$$\|\tilde{\theta}_{n+1}\|^2 \leq \tilde{\theta}_{n+1}^{\mathrm{T}} \left[\frac{P_{n+1}^{-1}}{\lambda_{\min}(P_{n+1}^{-1})}\right] \tilde{\theta}_{n+1}.$$

Hence (i) follows from Corollary 3.1(ii). So we need only prove assertion (ii).

By (11) and (19), we have

$$\theta_{n+1} = \theta_n + a_n P_n \varphi_n (\varphi_n^{\mathrm{T}} \tilde{\theta}_n) + a_n P_n \varphi_n w_{n+1}. \tag{22}$$

Note that, by (12)

$$\sum_{k=1}^{\infty} a_k \varphi_k^{\mathrm{T}} P_k^2 \varphi_k = \sum_{k=1}^{\infty} \operatorname{tr}(P_k - P_{k+1}) < \infty. \tag{23}$$

So, by the Martingale convergence theorem (see e.g. Chen and Guo, 1991, Theorem 2.3), the series $\sum_{k=1}^{n} a_k P_k \varphi_k w_{k+1}$ converges a.s.. Also, by (23) and the Dini theorem (see Knopp, 1928, p. 293), there exists a positive random sequence $\{\beta_k\}$ such that $\beta_k \to 0$ and $\sum_{k=1}^{\infty} \beta_k^{-1} a_k \varphi_k^{\mathrm{T}} P_k^2 \varphi_k < \infty$

a.s. Hence, by (22), the Schwarz inequality and Corollary 3.1(i)

$$\|\theta_{n+1}\| \le \|\theta_1\| + \left\|\sum_{k=1}^{n} a_k P_k \varphi_k(\varphi_k^{\mathrm{T}}\tilde{\theta}_k)\right\| + O(1)$$

$$\le O(1) + \left(\sum_{k=1}^{n} a_k \varphi_k P_k^2 \varphi_k \beta_k^{-1}\right)^{1/2}$$

$$\times \left(\sum_{k=1}^{n} \beta_k a_k \|\tilde{\theta}_k^{\mathrm{T}}\varphi_k\|^2\right)^{1/2}$$

$$= O(1) + O\left(\left(\sum_{k=1}^{n} \beta_k a_k \|\varphi_k^{\mathrm{T}}\tilde{\theta}_k\|^2\right)^{1/2}\right)$$

$$= O(1) + o(\sqrt{\log r_n}) \qquad \square$$

Remark 3.1. The second assertion of Corollary 3.2 cannot be derived directly from the first, because the condition $\lambda_{\min}(P_{n+1}^{-1}) \to \infty$ is not assumed.

Corollary 3.3. If, in addition to the conditions of Lemma 3.1

$$\varphi_n^{\mathrm{T}} P_n \varphi_n \to 0, \quad \text{as } r_n \to \infty, \quad \text{a.s.} \qquad (24)$$

then

$$\limsup_{n\to\infty} \frac{1}{\log r_n} \sum_{k=0}^{n} (\varphi_k^{\mathrm{T}}\tilde{\theta}_k)^2 \le \sigma^2 d \quad \text{a.s.} \qquad (25)$$

Furthermore, if (24) is strengthened to

$$r_n = O(n), \quad \|\varphi_n\|^2 = O(n^{\delta}) \quad (0<\delta<1) \quad \text{a.s.} \qquad (26)$$

and

$$n = O(\lambda_{\min}(P_n^{-1})) \quad \text{a.s.} \qquad (27)$$

then

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=0}^{n} (\varphi_k^{\mathrm{T}}\tilde{\theta}_k)^2 = \sigma^2 d \quad \text{a.s.,} \qquad (28)$$

where $d = \dim(\varphi_k)$, σ^2 is the variance of $\{w_k\}$ and r_n is defined as in Corollary 3.1.

The proof is given in Appendix A.

Remark 3.2. The verification of the condition $\varphi_n^{\mathrm{T}} P_n \varphi_n \to 0$ is the key difficulty in establishing the logarithm law of LS-based STR. This condition is satisfied when $\{\varphi_n\}$ is bounded. Indeed, by (12), we have

$$\sum_{i=1}^{\infty} \frac{(\varphi_i^{\mathrm{T}} P_i \varphi_i)^2}{(1+\|\varphi_i\|^2)(1+\varphi_i^{\mathrm{T}} P_0 \varphi_i)}$$

$$\le \sum_{i=1}^{\infty} \left(\frac{\varphi_i^{\mathrm{T}} P_i \varphi_i}{1+\|\varphi_i\|^2}\right)\left(\frac{\varphi_i^{\mathrm{T}} P_i \varphi_i}{1+\varphi_i^{\mathrm{T}} P_i \varphi_i}\right)$$

$$= \sum_{i=1}^{\infty} \frac{\varphi_i^{\mathrm{T}}(P_i - P_{i+1})\varphi_i}{1+\|\varphi_i\|^2}$$

$$\le \sum_{i=1}^{\infty} \operatorname{tr}(P_i - P_{i+1}) < \infty,$$

which, in conjunction with the boundedness of $\{\varphi_i\}$, gives $\sum_{i=1}^{\infty} (\varphi_i^{\mathrm{T}} P_i \varphi_i)^2 < \infty$, and hence $\varphi_n^{\mathrm{T}} P_n \varphi_n \to 0$. When $\{\varphi_i\}$ is unbounded, the verification of $\varphi_n^{\mathrm{T}} P_n \varphi_n \to 0$ is much harder. Conditions (26) and (27) are one set of conditions guaranteeing such a property.

4 THE ÅSTRÖM–WITTENMARK STR: STABILITY AND OPTIMALITY

For the system (1), the leading coefficient b_1 in $B(z)$ is usually referred to as the high frequency gain. When b_1 is known, we need only estimate the following parameter vector

$$\theta = [-a_1 \quad \cdots \quad -a_p \quad b_2 \quad \cdots \quad b_q]^{\mathrm{T}}. \qquad (29)$$

In this case the corresponding regression vector φ_n should be defined as

$$\varphi_n = [y_n \quad \cdots \quad y_{n-p+1} \quad u_{n-1} \quad \cdots \quad u_{n-q+1}]^{\mathrm{T}}, \qquad (30)$$

and the system (1) may be written in the form

$$y_{n+1} - b_1 u_n = \theta^{\mathrm{T}} \varphi_n + w_{n+1} \qquad (31)$$

The LS estimate for the unknown parameter θ is

$$\theta_{n+1} = \theta_n + a_n P_n \varphi_n (y_{n+1} - b_1 u_n - \varphi_n^{\mathrm{T}} \theta_n), \qquad (32)$$

$$P_{n+1} = P_n - a_n P_n \varphi_n \varphi_n^{\mathrm{T}} P_n, \quad a_n = (1 + \varphi_n^{\mathrm{T}} P_n \varphi_n)^{-1}, \qquad (33)$$

where the initial conditions θ_0 and $P_0 > 0$ can be arbitrarily chosen.

According to (8) and the 'certainty equivalence principle', the LS STR is

$$u_n = \frac{1}{b_1}(a_{1n} y_n + \cdots + a_{pn} y_{n-p+1} - b_{2n} u_{n-1} - \cdots - b_{qn} u_{n-q+1} + y_{n+1}^*)$$

$$= \frac{1}{b_1}(y_{n+1}^* - \theta_n^{\mathrm{T}} \varphi_n), \qquad (34)$$

where a_{in} and b_{in} are the components of θ_n, i.e. $\theta_n = [-a_{1n} \quad \cdots \quad -a_{pn} \quad b_{2n} \quad \cdots \quad b_{qn}]^{\mathrm{T}}$.

Obviously, the closed-loop control system defined by (29)–(34) is a highly nonlinear equation of the output signals. Since the STR (34) was initially proposed and studied by Åström and Wittenmark (1973), it may be referred to as the Åström–Wittenmark STR.

Before proceeding further, we need to introduce some notation that will be used throughout the sequel:

$$\alpha_k \triangleq \frac{(\varphi_k^{\mathrm{T}}\tilde{\theta}_k)^2}{1+\varphi_k^{\mathrm{T}}P_k\varphi_k}, \quad \delta_k \triangleq \mathrm{tr}\,(P_k - P_{k+1}), \quad (35)$$

$$r_n \triangleq 1 + \sum_{i=0}^{n} \|\varphi_i\|^2, \quad \tilde{\theta}_k = \theta - \theta_k. \quad (36)$$

Moreover, we assume that $\{d_n\}$ is a nondecreasing positive deterministic sequence such that

$$w_n^2 = O(d_n) \quad \text{a.s.}, \quad d_{n+1} = O(d_n) \quad (37)$$

Note that under the condition A.1, d_n can be taken as

$$d_n = n^{\delta} \quad \forall \delta \in (2/\beta, 1), \quad (38)$$

where β is given by (7) (cf. Guo and Chen, 1991, p. 804). Furthermore, if additional assumptions are imposed on $\{w_n\}$ then d_n can be taken smaller. For instance, if $\{w_n\}$ is a bounded noise then $d_n \equiv 1$, and if $\{w_n\}$ is Gaussian white noise then $d_n = \log n$, etc.

We first present two preliminary lemmas. The key idea of the proof is to dominate the output signals of the nonlinear closed-loop system by the solution of a certain linear time-varying equation.

Lemma 4.1. Consider the closed-loop control system (29)–(34). If Conditions A.1–A.3 are fulfilled then there exists a positive random process $\{L_n\}$ such that

$$y_n^2 \le L_n \quad \forall n \quad \text{a.s.}, \quad (39)$$

and $\{L_n\}$ satisfies the following 'linear time-varying relationship':

$$L_{n+1} \le (\lambda + c\alpha_n\delta_n)L_n + \xi_n, \quad (40)$$

where the constants $\lambda \in (0, 1)$, $c > 0$, α_n and δ_n are defined by (35), and $\{\xi_n\}$ is a positive random process satisfying

$$\xi_n = O(d_n \log r_n), \quad (41)$$

with d_n and r_n defined, respectively, by (37) and (36).

Proof. Substituting (34) into (31) yields

$$y_{n+1} = \varphi_n^{\mathrm{T}}\tilde{\theta}_n + y_{n+1}^* + w_{n+1} \quad (42)$$

Then, by (35) and (37), we get

$$\begin{aligned} y_{n+1}^2 &\le 2(\varphi_n^{\mathrm{T}}\tilde{\theta}_n)^2 + O(d_n) \\ &= 2\alpha_n[1 + \varphi_n^{\mathrm{T}}P_n\varphi_n] + O(d_n) \\ &= 2\alpha_n[1 + \varphi_n^{\mathrm{T}}P_{n+1}\varphi_n \\ &\quad + \varphi_n^{\mathrm{T}}(P_n - P_{n+1})\varphi_n] + O(d_n) \\ &\le 2\alpha_n(2 + \delta_n\|\varphi_n\|^2) + O(d_n) \\ &= 2\alpha_n\delta_n\|\varphi_n\|^2 + O(d_n + \log r_n), \quad (43) \end{aligned}$$

where we have used the fact that $\varphi_n^{\mathrm{T}}P_{n+1}\varphi_n \le 1$ and $\alpha_n = O(\log r_n)$ (see Corollary 3.1(i)).

By the Condition A.2, it is easy to see from (1) that there exists $\lambda \in (0, 1)$ such that

$$u_{n-1}^2 = O\Big(\sum_{i=0}^{n} \lambda^{n-i}y_i^2\Big) + O(d_n) \quad (44)$$

Then

$$\begin{aligned} \|\varphi_n\|^2 &= \sum_{i=0}^{p-1} y_{n-i}^2 + \sum_{i=1}^{q-1} u_{n-i}^2 \\ &= O\Big(\sum_{i=0}^{n} \lambda^{n-i}y_i^2\Big) + O(d_n) \end{aligned}$$

Define $L_n = \sum_{i=0}^{n} \lambda^{n-i}y_i^2$. Then, (39) is satisfied. Consequently, by (43), we have

$$y_{n+1}^2 \le c\alpha_n\delta_nL_n + O(d_n \log r_n), \quad (45)$$

where $c > 0$ is a constant. From this and the definition of L_n, we have

$$\begin{aligned} &L_{n+1} \\ &= \lambda L_n + y_{n+1}^2 \le (\lambda + c\alpha_n\delta_n)L_n + O(d_n \log r_n), \end{aligned}$$

which is (40), and hence the lemma is proved. □

Next, we estimate the growth rate of the regression process $\{\varphi_k\}$ by analysing the 'linear time-varying' equation (40).

Lemma 4.2. Under the conditions of Lemma 4.1, we have

$$\|\varphi_n\|^2 = O(r_n^{\epsilon}d_n) \quad \text{a.s.} \quad \forall \epsilon > 0,$$

where r_n and d_n are defined by (36) and (37), respectively.

Proof. By (40)

$$\begin{aligned} L_{n+1} &\le \prod_{j=0}^{n} (\lambda + c\alpha_j\delta_j)L_0 + \sum_{i=0}^{n} \prod_{j=i+1}^{n} (\lambda + c\alpha_j\delta_j)\xi_i \\ &= \lambda^{n+1}\prod_{j=0}^{n}(1 + \lambda^{-1}c\alpha_j\delta_j)L_0 \\ &\quad + \sum_{i=0}^{n} \lambda^{n-i} \prod_{j=i+1}^{n} (1 + \lambda^{-1}\alpha_j\delta_j)\xi_i. \quad (46) \end{aligned}$$

We now proceed to analyse the products in (46). First, from

$$\begin{aligned} \sum_{j=0}^{\infty} \delta_j &= \sum_{j=0}^{\infty} (\mathrm{tr}\,P_j - \mathrm{tr}\,P_{j+1}) \\ &\le \mathrm{tr}\,P_0 < \infty, \quad (47) \end{aligned}$$

we know that $\delta_j \to 0$. Then, by Corollary 3.1(i), it is known that for any $\epsilon > 0$, there exists i_0 large enough that

$$\lambda^{-1}c\sum_{j=i}^{n} \delta_j\alpha_j \le \epsilon \log r_n \quad \forall n \ge i \ge i_0 \quad (48)$$

From this and the inequality $1+x \le e^x$, $x \ge 0$, we have, $\forall n \ge i \ge i_0$

$$\prod_{j=i}^{n} (1+\lambda^{-1} c\delta_j \alpha_j) \le \exp\left(\lambda^{-1} c \sum_{j=i}^{n} \alpha_j \delta_j\right) \le \exp(\epsilon \log r_n) = r_n^\epsilon \quad (49)$$

Substituting this into (46) and using (41), we have $L_{n+1} = O(r_n^\epsilon d_n \log r_n)$ $\forall \epsilon > 0$, whence, by the arbitrariness of ϵ, we have from (39), $y_{n+1}^2 = O(r_n^\epsilon d_n)$ $\forall \epsilon > 0$. It follows from this and (44) that $u_n^2 = O(r_n^\epsilon d_n)$, $\forall \epsilon > 0$. Hence the lemma is proved. □

Once the growth rate of $\{\varphi_n\}$ is determined, the optimality of STR can easily be derived using Corollary 3.1(i), as will be seen from the following theorem.

Theorem 4.1. Consider the adaptive control system (29)–(34). Let the Conditions A.1–A.3 be satisfied. Then the closed-loop system is stable and optimal, and

$$R_n = O(\log n) + O(\epsilon_n) \quad \text{a.s.} \quad (50)$$

with

$$\epsilon_n = (\log n) \max_{1 \le i \le n} \{\delta_i i^\epsilon d_i\} \quad \forall \epsilon > 0, \quad (51)$$

where R_n, δ_n and d_n are defined by (16), (35) and (37), respectively.

Proof. First, note that $\epsilon_n = O(n^\epsilon d_n)$ $\forall \epsilon > 0$. Hence, if (50) holds then, by (38), the optimality $R_n = o(n)$ holds obviously. Furthermore, by the optimality (17) and the Conditions A.1–A.3, it is immediately seen from (1) that

$$\sum_{i=0}^{n} (y_i^2 + u_i^2) = O(n) \quad \text{a.s.}$$

Hence, the stability property is also true.

Thus, we need only prove (50). By (35), Corollary 3.1(i) and Lemma 4.2, we derive from (42) that

$$\begin{aligned} R_{n+1} &= \sum_{i=0}^{n} (y_{i+1} - y_{i+1}^* - w_{i+1})^2 \\ &= \sum_{i=0}^{n} (\varphi_i^{\mathrm{T}} \tilde{\theta}_i)^2 = \sum_{i=0}^{n} \alpha_i (1 + \varphi_i^{\mathrm{T}} P_i \varphi_i) \\ &= O(\log r_n) + O\left(\sum_{i=0}^{n} \alpha_i \delta_i \|\varphi_i\|^2\right) \\ &= O(\log r_n) + O\left(\max_{1 \le i \le n} \{\delta_i r_i^\epsilon d_i\} \log r_n\right) \\ &\forall \epsilon > 0 \end{aligned} \quad (52)$$

Therefore, for (50), it suffices to prove that $r_n = O(n)$. By (52) and the assumptions on $\{y_i^*\}$ and $\{w_i\}$, it is evident that

$$\sum_{i=0}^{n+1} y_i^2 = O(n) + O(r_n^\epsilon d_n) \quad \forall \epsilon > 0$$

By this and Condition A.2, it follows from (1) that

$$\sum_{i=0}^{n} u_i^2 = O(n) + O(r_n^\epsilon d_n) \quad \forall \epsilon > 0$$

Therefore, by the last two relationships, we have for any $\epsilon > 0$

$$\begin{aligned} r_n &= 1 + \sum_{i=0}^{n} \|\varphi_i\|^2 = O(n) + O(r_n^\epsilon d_n) \\ &= O(n) + O(r_n^\epsilon n^\delta) \quad \forall \delta \in (2/\beta, 1) \end{aligned}$$

Take ϵ small enough that $\epsilon + \delta < 1$. We get

$$\begin{aligned} \frac{r_n}{n} &= O(1) + O\left(\left(\frac{r_n}{n}\right)^\epsilon \frac{1}{n^{1-\epsilon-\delta}}\right) \\ &= O(1) + o\left(\left(\frac{r_n}{n}\right)^\epsilon\right). \end{aligned}$$

It is seen from this that $r_n = O(n)$ holds. □

Remark 4.1. If $\delta_n = O(n^{-\delta})$ for some $\delta \in (2/\beta, 1)$ then the result of Theorem 4.1 implies $R_n = O(\log n)$. This theorem is an extension of Theorem 1 of Guo and Chen (1991) in that here the condition $p \ge 1$ on the system order is no longer required and the convergence rate is more precisely established.

5. THE ÅSTRÖM–WITTENMARK STR: THE LOGARITHM LAW

In Theorem 4.1 we have shown that the growth rate of R_n is bounded by $O(n^\epsilon d_n)$ $\forall \epsilon > 0$. However, a natural question is: what is the precise growth rate of R_n? This question in essence concerns the best convergence rate of STR, and will be answered by establishing a logarithm law in this section.

Lemma 5.1. Consider the system (1) and the R_n defined by (16). Let the Conditions A.1 and A.3 be satisfied and let $B(z)$ and $A(z)-1$ be coprime with $|a_p| + |b_q| \ne 0$. Moreover, assume that there exist a nondecreasing random sequence $\{\tau_n\}$ and a set D such that $R_{\tau_n+1} = o(\tau_n)$ on D. Then

$$\liminf_{n\to\infty} \lambda_{\min}\left(\frac{1}{\tau_n} \sum_{i=0}^{\tau_n} \varphi_i \varphi_i^{\mathrm{T}}\right) > 0 \quad \text{a.s. on } D,$$

where φ_i is defined by (30).

This lemma can be found in Theorem 2.1 of

Guo (1994). The next lemma is a direct corollary of the work of Jain *et al* (1975).

Lemma 5.2. Let $\{w_i, \mathscr{F}_i\}$ be a Martingale difference sequence satisfying the (7) and let $\{f_i, \mathscr{F}_i\}$ be an adapted process. If there is a constant $\epsilon \in [0, 1)$ such that $\sum_{i=1}^{n} f_i^2 = O(n)$ and $f_n^2 = O(n^\epsilon)$ a.s. then

$$\sum_{i=1}^{n} f_i w_{i+1} = O(\sqrt{n \log\log n}) \quad \text{a.s.}$$

Theorem 5.1. Under the conditions of Theorem 4.1, if $B(z)$ and $A(z) - 1$ are coprime with $|a_p| + |b_q| \neq 0$ then the following logarithm law holds for the closed-loop system:

$$\lim_{n\to\infty} \frac{R_n}{\log n} = (p + q - 1)\sigma^2 \quad \text{a.s.}$$

and

$$\|\theta_n - \theta\| = O\left(\sqrt{\frac{\log\log n}{n}}\right) \quad \text{a.s.,}$$

where R_n is defined by (16).

Proof. By Theorem 4.1, $R_n = o(n)$ a.s. and

$$\sum_{i=0}^{n} \|\varphi_i\|^2 = O(n) \quad \text{a.s.} \tag{53}$$

Hence, by Lemma 5.1,

$$\liminf_{n\to\infty} \lambda_{\min}\left(\frac{1}{n}\sum_{i=0}^{n} \varphi_i \varphi_i^{\mathrm{T}}\right) > 0 \quad \text{a.s.} \tag{54}$$

Also, by (53), Lemma 4.2 and (38), we have

$$\|\varphi_n\|^2 = O(n^\delta) \quad \text{a.s.} \quad \forall \delta \in (2/\beta, 1) \tag{55}$$

Hence, by Corollary 3.3, we have

$$\sum_{k=0}^{n} (\varphi_k^{\mathrm{T}} \tilde{\theta}_k)^2 \sim \sigma^2 (p + q - 1) \log n$$

From this and (42), the first assertion of the theorem follows.

To prove the second assertion, write θ_n as

$$\theta_n - \theta = P_n P_0^{-1}(\theta_0 - \theta) + P_n \sum_{i=0}^{n-1} \varphi_i w_{i+1} \tag{56}$$

By (53)–(55) and Lemma 5.2, we have

$$\sum_{i=0}^{n-1} \varphi_i w_{i+1} = O(\sqrt{n \log\log n}) \quad \text{a.s.}$$

Hence, by this and (54), the second assertion of the theorem follows from (56).

Remark 5.1. Theorem 5.1 means that the n-step accumulated tracking error of the closed-loop system is precisely $O(\log n)$, and the logarithmic law holds, i.e. $R_n \sim (p + q - 1)\sigma^2 \log n$. Obviously, this result is much deeper than the optimality (17).

6. LS-BASED STR

In this and the next sections we shall treat the general case where the high frequency gain b_1 is unknown. In this case the unknown parameter θ and the regressor φ_n should be defined by (2) and (3). We shall adopt the notations α_k, δ_k and r_k defined by (35) and (36).

Naturally, the estimate for θ should be given by the LS algorithm (11)–(13), and the STR should be defined by (14) or (15). However, the first problem that we shall meet now is that u_n may not be well defined, since the set $\{b_{1n} = 0\}$ may have a positive probability, unless additional assumptions are imposed on the distribution of the noise sequence $\{w_n\}$.

If we do not intend to make additional assumptions on the noise process, a natural and simple method for overcoming this difficulty is to modify b_{1n} or θ_n slightly. Let us denote the modified estimate by $\hat{\theta}_n$. Then the LS-based STR is naturally defined from

$$\hat{\theta}_n^{\mathrm{T}} \varphi_n = y_{n+1}^* \tag{57}$$

or

$$u_n = \frac{1}{\hat{b}_{1n}} (\hat{a}_{1n} y_n + \cdots + \hat{a}_{pn} y_{n-p+1} - \hat{b}_{2n} u_{n-1} - \cdots - \hat{b}_{qn} u_{n-q+1} + y_{n+1}^*), \tag{58}$$

where $\hat{a}_{in}$ and $\hat{b}_{jn}$ are the components of $\hat{\theta}_n$, which are the (modified) estimates for a_i and b_j, respectively.

We shall first list several requirements on $\{\hat{\theta}_n\}$ that are needed to establish a fairly general theorem; then we shall show that how these requirements on $\{\hat{\theta}_n\}$ can be fulfilled.

(H.1) $\quad \|\hat{\theta}_n\|^2 = O(\log r_{n-1})$ a.s.

(H.2) $\quad \displaystyle\sum_{i=1}^{n} \frac{(\varphi_i^{\mathrm{T}} \tilde{\theta}_i)^2}{1 + \varphi_i^{\mathrm{T}} P_i \varphi_i} = O(\log r_n)$ a.s.

(H.3) $\quad \liminf_{n\to\infty} \sqrt{\log (n + r_{n-1})}\, |\hat{b}_{1n}| \neq 0$ a.s.

Here $\{\varphi_n\}$ and $\{P_n\}$ are defined, respectively, by (3) and (12), $\tilde{\theta}_n \triangleq \theta - \hat{\theta}_n$, and $\hat{b}_{1n}$ is the estimate for b_1 given by $\hat{\theta}_n$.

We now proceed to analyse properties of the closed-loop system under the control law (57).

Similarly to Lemma 4.1, we first dominate the output signals by the solution of a 'linear time-varying' equation. The following lemma is proved in the Appendix.

Lemma 6.1. For the system (1), let the conditions A.1–A.3 hold and let the control law be defined by (57), with $\{\hat{\theta}_k\}$ satisfying H.1–H.3. Then there exists a positive random process $\{L_k\}$ such that

$$y_k^2 \le L_k \quad \forall k$$

and $\{L_k\}$ satisfies

$$L_{k+1} \le (\lambda + cf_k)L_k + \xi_k,$$

where the constants $\lambda \in (0, 1)$, $c > 0$ and

$$f_k = [\alpha_k \delta_k \log(k + r_k)]^2 + \alpha_k \delta_k,$$

$$\xi_k = O(d_k \log^4 (k + r_k)).$$

Similarly to Lemma 4.2, we can also prove the following lemma (see the Appendix).

Lemma 6.2. Under the conditions of Lemma 6.1, we have

$$\|\varphi_n\|^2 = O((n + r_n)^{\epsilon} d_n) \quad \text{a.s.} \quad \forall \epsilon > 0,$$

where r_n and d_n are defined by (36) and (37), respectively.

By Lemma 6.2, the following theorem can be proved in completely the same way as that for Theorem 4.1 (the details are not repeated here).

Theorem 6.1. For the system (1), let the conditions A.1–A.3 be satisfied, and let the control law be defined by (57), with $\{\hat{\theta}_k\}$ satisfying H.1–H.3. Then the closed-loop system is stable, optimal and has the rate of convergence

$$R_n = O(\log n + \varepsilon_n) \quad \text{a.s.}, \tag{59}$$

where R_n, r_n and ϵ_n are defined by (16), (36) and (51), respectively.

Now, by Corollaries 3.1 and 3.2, the standard LS satisfies H.1 and H.2, with $\hat{\theta}_k \equiv \theta_k$. Hence we have the following.

Corollary 6.1. Let the conditions A.1–A.3 be satisfied for the system (1). If the control law is defined by the (non-modified) LS STR (11)–(15) then the property (59) holds a.s. on the set

$$D \triangleq \Big\{ b_{1n} \neq 0 \ \forall n; \times \liminf_{n\to\infty} [\log(n + r_{n-1})]\, |b_{1n}| \neq 0 \Big\}. \tag{60}$$

Remark 6.1. By a refined (essentially the same) argument as that for Lemmas 6.1 and 6.2, it can be shown that Corollary 6.1 holds on a set larger than that defined by (60). For example, in Corollary 6.1 we may take $D = \bigcup_{m=1}^{\infty} D_m$, with

$$D_m = \Big\{ b_{1n} \neq 0 \ \forall n; \liminf_{n\to\infty} \times [\log(n + r_{n-1})]^m\, |b_{1n}| \neq 0 \Big\}.$$

Next, we consider suitable modifications on the standard LS, so that the property H.3 holds a.s.

Case 1. If the sign of b_1 is known, for example $b_1 > 0$, then a standard way to avoid the zero-divisor problem in (15) is to replace (11) by the following projected LS:

$$\hat{\theta}_{n+1} = \pi_n[\hat{\theta}_n + a_n P_n \varphi_n (y_{n+1} - \varphi_n^{\mathrm{T}} \hat{\theta}_n)], \tag{61}$$

where $\pi_n\{\cdot\}$ is a projection operator defined by

$$\pi_n\{x\} = \arg\min_{y \in D_n} \|P_{n+1}^{-1/2}(x - y)\|, \quad x \in \mathbb{R}^{p+q}, \tag{62}$$

with

$$D_n = \left\{ \theta : b_1 \ge \frac{1}{\sqrt{\log(n+1)}} \right\}, \quad n \ge 1. \tag{63}$$

Note that because of the simple form of the domain D_n, the computation of the projection in (61) is an easy task (cf. Goodwin and Sin, 1984, pp. 93–94).

Theorem 6.2. Let the conditions A.1–A.3 be satisfied for the system (1) with $b_1 > 0$, and let the control law be defined by (57), with $\{\hat{\theta}_n\}$ generated by (61). Then (59) holds with probability one.

Proof. First, by the definition of the projection in (62), $\{\hat{\theta}_k\}$ obviously satisfies the condition H.3. Next, we note that Corollary 3.1 obtained for the standard LS (11) can be derived *mutatis mutandis* for the projected LS (61). Hence $\{\hat{\theta}_k\}$ also satisfies H.1 and H.2, and consequently Theorem 6.1 is applicable. □

Case 2. If nothing is known about b_1, we may introduce a modification (only) to b_{1n}, which is similar to that of Guo and Chen (1991). However, the guaranteed convergence rate in that case will be poor (although it may not be so practically). Here, similarly to the work of Lozano and Zhao (1994) we consider the following form of modifications to θ_k:

$$\hat{\theta}_k = \theta_k + P_k^{1/2} e_{i_k}, \tag{64}$$

where θ_k is the LS estimate defined by (11), P_k is defined by (12), and $\{i_k\}$ is a sequence of integers taking values on $\{0, 1, \ldots, d\}$ with $d = p + q$, and is defined by

$$i_k = \arg\max_{0 \le i \le d} |b_{1k} + e_{p+1}^{\mathrm{T}} P_k^{1/2} e_i|, \tag{65}$$

where $e_0 = 0$, and e_i, $1 \le i \le d$, is the ith column of the $d \times d$ identity matrix.

Theorem 6.3. Let the conditions A.1–A.3 hold for the system (1), and let the control law be defined by (57), with $\hat{\theta}_k$ defined by (64) and (65). Then the closed-loop tracking system has the convergence rate (59).

Proof By Theorem 6.1, we need only show that $\{\hat{\theta}_k\}$ defined by (66) satisfies the requirements H.1–H.3. First, by Corollary 3.2(ii), H.1 is satisfied, since both $\{e_{i_k}\}$ and $\{P_k^{1/2}\}$ are bounded. Next, by Corollary 3.1(i) and the relation (A.3) from the Appendix, together with (21), we have

$$\begin{aligned}\sum_{k=1}^{n} \frac{[\varphi_k^{\mathrm{T}}(\theta - \hat{\theta}_k)]^2}{1 + \varphi_n^{\mathrm{T}} P_n \varphi_n} &= \sum_{k=1}^{n} a_k [\varphi_k^{\mathrm{T}}(\theta - \theta_k - P_k^{1/2} e_{i_k})]^2 \\ &= O\Big(\sum_{k=1}^{n} a_k [\varphi_k^{\mathrm{T}}(\theta - \theta_k)]^2\Big) + O\Big(\sum_{k=1}^{n} a_k \varphi_k^{\mathrm{T}} P_k \varphi_k\Big) \\ &= O(\log r_n).\end{aligned}$$

Hence, H.2 is also satisfied.

To prove H.3, let us set $\beta_k = P_k^{-1/2}(\theta - \theta_k)$. Then we have $\theta = \theta_k + P_k^{1/2}\beta_k$, and, by Corollary 3.1(ii), we know that $\|\beta_k\|^2 = O(\log r_{k-1})$. It follows from this that

$$\begin{aligned}|b_1|^2 &= |b_{1k} + e_{p+1}^{\mathrm{T}} P_k^{1/2} \beta_k|^2 \\ &= \left| [b_{1k} \quad e_{p+1}^{\mathrm{T}} P_k^{1/2}] \begin{bmatrix} 1 \\ \beta_k \end{bmatrix} \right|^2 \\ &\le \|[b_{1k} \quad e_{p+1}^{\mathrm{T}} P_k^{1/2}]\|^2 (1 + \|\beta_k\|^2) \\ &= O(\|[b_{1k} \quad e_{p+1}^{\mathrm{T}} P_k^{1/2}]\|^2 \log r_{k-1})\end{aligned}$$

Now, since the Condition A.2 implies that $b_1 \ne 0$, we know that there exists a random variable $c > 0$ such that

$$\|[b_{1k} \quad e_{p+1}^{\mathrm{T}} P_k^{1/2}]\|^2 \ge \frac{c}{\log r_{k-1}} \quad \forall k. \tag{66}$$

Let us denote

$$M = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e_0 & e_1 & \cdots & e_d \end{bmatrix}.$$

Then, since M is nonsingular, we have

$$\lambda_0 \triangleq \lambda_{\min}(MM^{\mathrm{T}}) > 0.$$

From this, by (64)–(66), we have

$$\begin{aligned}|\hat{b}_{1k}|^2 &= |b_{1k} + e_{p+1}^{\mathrm{T}} P_k^{1/2} e_{i_k}|^2 \\ &= \max_{0 \le i \le d} |b_{1k} + e_{p+1}^{\mathrm{T}} P_k^{1/2} e_i|^2 \\ &= \max_{0 \le i \le d} \left| [b_{1k} \quad e_{p+1}^{\mathrm{T}} P_k^{1/2}] \begin{bmatrix} 1 \\ e_i \end{bmatrix} \right|^2 \\ &\ge (1+d)^{-1} \|[b_{1k} \quad e_{p+1}^{\mathrm{T}} P_k^{1/2}] M\|^2 \\ &\ge \lambda_0 (1+d)^{-1} \|[b_{1k} \quad e_{p+1}^{\mathrm{T}} P_k^{1/2}]\|^2 \\ &\ge \frac{\lambda_0 c}{1+d} \frac{1}{\log r_{k-1}} \quad \forall k,\end{aligned}$$

which implies that H.3 holds. □

Remark 6.2 Theorem 6.3 shows that in the general case where b_1 is not available an LS-based STR can be designed so that its guaranteed convergence rate is exactly the same as that proved for the LS STR with b_1 known (see Theorem 4.1).

7 THE NONMODIFIED LS-STR

Throughout this section we consider the nonmodified LS STR defined by (11)–(15). Of course, some results on such STR have been presented in Corollary 6.1. However, we hope the results there hold with probability one (or at least with larger probability). We proceed with the following lemma (for the proof see the Appendix), which shows what will happen if the LS estimate for b_1 has a subsequence not converging to b_1.

Lemma 7.1. For the system (1), let the Conditions A.1–A.3 be satisfied, and let the control law be defined by (15). Moreover, let $\{\tau_n\}$ be a strictly increasing random sequence of integers that satisfies

$$\inf_n \sqrt{\log r_{\tau_n}}\, |b_1(\tau_n + 1) - b_1| > 0 \quad \text{a.s. on } D, \tag{67}$$

where $P(D) > 0$ and $b_1(n)$ is the LS estimate for b_1 given by θ_n. Then, by any $\epsilon > 0$

$$\sup_{k \le \tau_n} \|\varphi_k\|^2 = O(\tau_n^{\epsilon} d_{\tau_n}) \quad \text{a.s. on } D \tag{68}$$

and

$$r_{\tau_n} = O(\tau_n) \quad \text{a.s. on } D, \tag{69}$$

where r_n and d_n are defined by (36) and (37), respectively.

In the sequel, whenever the control law (15) is concerned, we always assume that

$$P(b_1(n) \ne 0 \ \forall n) = 1, \tag{70}$$

where and hereinafter we write b_{1n} appearing in (15) as $b_1(n)$ for convenience of presentation.

Equation (70) can be guaranteed by imposing additional conditions on the noise sequence. In fact, Meyn and Caines (1985) proved that if all finite-dimensional distributions of $\{w_n\}$ are absolutely continuous with respect to the Lebesgue measure then (70) must be true.

Define

$$D_1 = \left\{w : \liminf_{n\to\infty} \sqrt{\log(n + r_{n-1})}\,|b_1(n)| \neq 0\right\} \tag{71}$$

For any constant $a \in (0, |b_1|)$, define a sequence $\{\tau_n\}$ as follows:

$$\tau_n = \inf\{k > \tau_{n-1} : \sqrt{\log r_k} \times |b_1(k+1) - b_1| \geq a\}, \quad \tau_0 = 0 \tag{72}$$

Obviously, on D_1^c, the complement set of D_1, $\tau_n < \infty\ \forall n$. So, if we define

$$\sigma_n = \begin{cases} n & w \in D_1, \\ \tau_n & w \in D_1^c \end{cases}, \tag{73}$$

then $\sigma_n < \infty\ \forall n$, and $\sigma_n \to \infty$ a.s.

Theorem 7.1. Let the Conditions A.1–A.3 hold for the system (1), and let the LS STR be defined by (11)–(15). Then the following results hold:

(i) If σ_n is defined by (71)–(73) then $r_{\sigma_n} = O(\sigma_n)$ a.s., and

$$R_{\sigma_n+1} = O(\sigma_n^\epsilon d_{\sigma_n}) \quad \text{a.s.} \quad \forall \epsilon > 0, \tag{74}$$

where R_n, r_n and d_n are defined by (16), (36) and (37), respectively.

(ii) Let D_1 and τ_n be defined by (71) and (72), respectively. Denote

$$D = D_1 \cup D_2, \tag{75}$$

$$D_2 = \left\{w : \tau_n < \infty\ \forall n; \sup_{n\geq 1} \frac{\tau_{n+1}}{\tau_n} < \infty\right\} \tag{76}$$

Then

$$R_n = O(n^\delta) \quad \forall \delta \in (2/\beta, 1) \quad \text{a.s. on } D, \tag{77}$$

where β is given in the Condition A.1.

Proof (i) Since, by Corollary 6.1, the result is true on D_1, we need only consider D_1^c. By the definition of τ_n

$$\inf_n \sqrt{\log r_{\tau_n}}\,|b_1(\tau_n + 1) - b_1| \geq a > 0, \quad w \in D_1^c.$$

So, by Lemma 7.1, we know that on D_1^c

$$r_{\tau_n} = O(\tau_n), \quad \sup_{i\leq\tau_n} \|\varphi_i\|^2 = O(\tau_n^\epsilon d_{\tau_n}) \quad \forall \epsilon > 0.$$

Consequently, by (4), (14) and Corollary 3.1(i)

$$\begin{aligned} R_{\tau_n+1} &= \sum_{i=0}^{\tau_n} (\tilde{\theta}_i^{\mathrm{T}}\varphi_i)^2 \\ &= \sum_{i=0}^{\tau_n} \alpha_i(1 + \varphi_i^{\mathrm{T}} P_i \varphi_i) \\ &= O\left(\tau_n^\epsilon d_{\tau_n} \sum_{i=0}^{\tau_n} \alpha_i\right) \\ &= O(\tau_n^\epsilon d_{\tau_n} \log r_{\tau_n}) \quad \forall \epsilon > 0. \end{aligned}$$

Therefore (74) also holds on D_1^c.

(ii) First, taking d_n as defined by (38) in Corollary 6.1, we know that (77) holds on D_1. On the set D_2, by a similar argument as above, it follows that $R_{\tau_n} = O(\tau_n^\delta)\ \forall \delta \in (2/\beta, 1)$. Hence

$$\begin{aligned} \sup_n (n^{-\delta} R_n) &= \sup_k \sup_{n\in[\tau_k, \tau_{k+1}]} (n^{-\delta} R_n) \\ &\leq \sup_k \left(\frac{\tau_{k+1}}{\tau_k}\right)^\delta (\tau_{k+1}^{-\delta} R_{\tau_{k+1}}) < \infty, \end{aligned}$$

which means that (77) also holds on D_2. □

The next corollary follows directly from Theorem 7.1.

Corollary 7.1. Let the conditions A.1–A.3 be satisfied for the system (1), and let the LS STR be defined by (11)–(15). Then the following assertions hold:

(i) There always exists a random sequence $\sigma_n \to \infty$ a.s., such that the closed-loop system is optimal along this sequence, i.e. $R_{\sigma_n} = o(\sigma_n)$ a.s.

(ii) The closed-loop system is optimal, i.e. $R_n = o(n)$ a.s., if

(a) $\liminf_{n\to\infty} |b_1(n)| \neq 0$ a.s.,

(b) $\lim_{n\to\infty} |b_1(n)|$ a.s. exists, or

(c) there exists an increasing random function k_n with $k_n \to \infty$, $k_{n+1}/k_n = O(1)$, such that

$$\min_{k\in[k_n, k_{n+1}]} \{|b_1(k)|\} \xrightarrow[n\to\infty]{} 0 \quad \text{a.s.}$$

We remark that both (b) and (c) include the case where $\lim_{n\to\infty} |b_1(n)| = 0$ a.s. Next, we prove that the optimality does hold for a large class of tracking systems. We start with a lemma that can be found in Theorem 2.1 of Guo (1994).

Lemma 7.2. Consider the system (1). Let the conditions A.1–A.3 be satisfied, and let $A(z)$ and $B(z)$ be coprime with $|a_p| + |b_q| \neq 0$. Moreover, assume that there exist a nondecreasing random sequence $\{\sigma_n\}$ and a set D with positive probability such that

$$[\sigma_n(R_{\sigma_n+1} + \log\log\sigma_n)]^{1/2} = o(\lambda^*_{\min}(\sigma_n)) \quad \text{on } D \tag{78}$$

Then

$$\liminf_{n\to\infty} \frac{\lambda_{\min}\left(\sum_{i=0}^{\sigma_n} \varphi_i\varphi_i^{\mathrm{T}}\right)}{\lambda^*_{\min}(\sigma_n)} \neq 0 \quad \text{a.s. on } D, \tag{79}$$

where φ_i and R_n are defined, respectively, by (3) and (16), and

$$\lambda^*_{\min}(n) = \lambda_{\min}\left(\sum_{i=0}^{n} Y_i^* Y_i^{*\mathrm{T}}\right), \qquad Y_i^* = [y_i^* \quad \cdots \quad y^*_{i-p-q+1}]^{\mathrm{T}} \tag{80}$$

Theorem 7.2 For the system (1) let the conditions A.1–A.3 be satisfied, and let $A(z)$ and $B(z)$ be coprime with $|a_p| + |b_q| \neq 0$. Moreover, assume that the reference signal $\{y_i^*\}$ satisfies

$$n^\alpha\sqrt{d_n} = O(\lambda^*_{\min}(n)) \quad (\alpha > \tfrac{1}{2}) \tag{81}$$

If the LS STR (11)–(15) is applied then the closed-loop equation has the following properties:

(i)
$$\limsup_{n\to\infty} \frac{R_n}{\log n} \leq (p+q)\sigma^2 \quad \text{a.s.,} \tag{82}$$

$$\|\theta_n - \theta\|^2 = O\left(\frac{\log n}{\lambda^*_{\min}(n)}\right) \quad \text{a.s.}$$

(ii) If (81) is strengthened to $n = O(\lambda^*_{\min}(n))$ a.s. then the following logarithm law holds:

$$R_n \sim (p+q)\sigma^2 \log n \quad \text{a.s.}$$

and

$$\|\theta_n - \theta\|^2 = O\left(\frac{\log\log n}{n}\right) \quad \text{a.s.,}$$

where R_n, d_n and $\lambda^*_{\min}(n)$ are defined by (16), (37) and (80), respectively.

Proof. By Theorem 7.1(i) and (81), applying Lemma 7.2 to the subsequence $\{\sigma_n\}$ defined by (73), we have

$$\liminf_{n\to\infty} \frac{\lambda_{\min}\left(\sum_{i=0}^{\sigma_n} \varphi_i\varphi_i^{\mathrm{T}}\right)}{\lambda^*_{\min}(\sigma_n)} > 0 \quad \text{a.s.} \tag{83}$$

It follows from this and Corollary 3.2(i) that

$$\|\tilde{\theta}_{\sigma_n+1}\|^2 = O\left(\frac{\log r_{\sigma_n}}{\lambda^*_{\min}(\sigma_n)}\right) \quad \text{a.s.}$$

But, by Theorem 7.1(i), we know that $\log r_{\sigma_n} = O(\log\sigma_n)$. Hence, by (81) again, we have $\sqrt{\log r_{\sigma_n}}\,\|\tilde{\theta}_{\sigma_n+1}\| \to 0$; in particular

$$\sqrt{\log r_{\sigma_n}}\,|b_1(\sigma_n+1) - b_1| \xrightarrow[n\to\infty]{} 0 \quad \text{a.s.} \tag{84}$$

We now proceed to prove that the set D_1 defined by (71) satisfies $P(D_1) = 1$. If $P(D_1^c) > 0$ were true then, by the definition of σ_n, it would follow that $\sigma_n < \infty$, and on D_1^c we should have

$$\sqrt{\log r_{\sigma_n}}\,|b_1(\sigma_n+1) - b_1| \geq a > 0 \quad \forall n \geq 1.$$

This obviously contradicts (84), and hence $P(D_1) = 1$. Consequently, by Lemma 6.2 and Corollary 6.1

$$\|\varphi_n\|^2 + R_n = O(n^\epsilon d_n) \quad \text{a.s.} \quad \forall\epsilon > 0, \tag{85}$$

and $\sigma_n = n$ a.s. Therefore, by (83) and (81)

$$n^\alpha\sqrt{d_n} = O(\lambda_{\min}(P_n^{-1})) \quad (\alpha > \tfrac{1}{2}). \tag{86}$$

By (85) and (38), we know that if ϵ is taken small enough such that $\epsilon + \frac{1}{2}\delta < \alpha$, with $\delta \in (2/\beta, 1)$, then

$$\|\varphi_n\|^2 = O(n^\epsilon\sqrt{d_n}\sqrt{d_n}) = O(n^\epsilon n^{\delta/2}\sqrt{d_n}) = o(n^\alpha\sqrt{d_n}) = o(\lambda_{\min}(P_n^{-1})).$$

Consequently

$$\varphi_n^{\mathrm{T}} P_n \varphi_n \leq \frac{\|\varphi_n\|^2}{\lambda_{\min}(P_n^{-1})} \to 0,$$

and hence, by Corollary 3.3, it is evident that

$$\limsup_{n\to\infty} \frac{R_n}{\log n} = \limsup_{n\to\infty} \frac{1}{\log n}\sum_{i=0}^{n} (\varphi_i^{\mathrm{T}}\tilde{\theta}_i)^2 \leq (p+q)\sigma^2 \quad \text{a.s.}$$

This proves the first assertion of the theorem, while the second can be derived from Corollary 3.3 using a similar treatment to that for Theorem 5.1. □

Remark 7.1. Equation (85) does not require that the reference signal $\{y_i^*\}$ satisfy the usual 'persistence-of-excitation' condition. For instance, when $d_n = 1$, we need only require that $\lambda^*_{\min}(n)$ have a growth rate $O(n^\alpha)(\alpha > \frac{1}{2})$, rather than $O(n)$.

In general, for any reference signal $\{y_n^*\}$, not necessarily satisfying (81), in order to simultaneously have optimality of LS STR and consistency of LS, we may use the 'decaying excitation' method (cf. Chen and Guo, 1986) to define the control law from

$$\varphi_n^{\mathrm{T}}\theta_n = y^*_{n+1} + \frac{v_n}{n^\epsilon}, \quad 0 < \epsilon < \frac{\beta - 2}{4\beta} \tag{87}$$

instead of (14), where $\{\theta_n\}$ is defined by the standard LS (11)–(13), β is defined in (7), and $\{v_n\}$ is a bounded white noise sequence that possesses a continuous distribution and is independent of $\{y_n^*, w_n\}$, with $E(v_n^2) > 0$. The following theorem can be proved in completely the same way as Theorem 7.2.

Theorem 7.3. Let the conditions of Theorem 7.2 be satisfied except for (81). If the STR is defined by (87) then the closed-loop system has both optimality and consistency in the sense that $R_n = o(n)$ and $\theta_n \to \theta$ a.s.

The key difference between this theorem and Theorem 3 of Guo and Chen (1991) is that here no modifications are introduced to bound the estimate for the high-frequency gain b_1 from below. Certainly, the present case is more complicated to analyse.

8. CONCLUSIONS

Convergence and convergence rates of some standard LS-based STR have been studied in detail in this paper. A unified treatment for several basic theoretical problems has been presented. Some recent advances in this area have also been surveyed. Of course, much remains to be done. We remark that:

(i) many previously established results using stochastic gradient algorithms may now be established using the standard least squares;

(ii) owing to the flexibility of the various analytical ideas and techniques presented here, they are most likely to be applied to deal with other stochastic adaptive control problems, e.g. adaptive pole-assignment and robust adaptive control, by using least squares.

These issues are for further investigation.

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APPENDIX—PROOFS

Proofs of Theorem 3.1 *and Corollary* 3.3

Proof of Theorem 3.1. Consider the standard Lyapunov function $V_k = \tilde{\theta}_k^T P_k^{-1}\tilde{\theta}_k$. By (11)–(13), we have the following relationship (cf. e g. Guo and Chen, 1991, p. 808):

$$V_{k+1} = V_k - a_k(\varphi_k^T\tilde{\theta}_k)^2 - 2a_k\varphi_k^T\tilde{\theta}_k w_{k+1} + a_k\varphi_k^T P_k\varphi_k w_{k+1}^2.$$

Summing from $k = 0$ to n yields

$$V_{n+1} + \sum_{k=0}^{n} a_k(\varphi_k^T\tilde{\theta}_k)^2 = V_0 - 2\sum_{k=0}^{n} a_k\varphi_k^T\tilde{\theta}_k w_{k+1} + \sum_{k=0}^{n} a_k\varphi_k^T P_k\varphi_k w_{k+1}^2 \qquad (A.1)$$

We estimate the last two terms of (A.1) separately. By $a_k \le 1$ and $a_k\varphi_k^T\tilde{\theta}_k \in \mathscr{F}_k$, we know from Theorem 2.8 of Chen and Guo (1991) that

$$\sum_{k=0}^{n} a_k\varphi_k^T\tilde{\theta}_k w_{k+1} = O(1) + o\Big(\sum_{k=0}^{n} a_k(\varphi_k^T\tilde{\theta}_k)^2\Big) \quad \text{a.s.} \qquad (A.2)$$

We now proceed to estimate the last term in (A.1). First, following Lai and Wei (1982), by taking determinants on both sides of the identity $P_{k+1}^{-1} = P_k^{-1} + \varphi_k\varphi_k^T$, we have

$$|P_{k+1}^{-1}| = |P_k^{-1}|\,(1 + \varphi_k^T P_k\varphi_k)$$

Therefore

$$\sum_{k=0}^{n} a_k\varphi_k^T P_k\varphi_k = \sum_{k=0}^{n}\frac{|P_{k+1}^{-1}| - |P_k^{-1}|}{|P_{k+1}^{-1}|} \le \sum_{k=0}^{n}\int_{|P_k^{-1}|}^{|P_{k+1}^{-1}|}\frac{dx}{x} = \log|P_{n+1}^{-1}| + \log|P_0| \qquad (A.3)$$

By the C_r inequality and the Lyapunov inequality, it is easy to see that for any $\alpha \in [2, \min(\beta, 4)]$,

$$\sup_k E[|w_{k+1}^2 - E[w_{k+1}^2 \mid \mathscr{F}_k]|^{\alpha/2} \mid \mathscr{F}_k] \le 2\sup_k E[|w_{k+1}|^{\alpha} \mid \mathscr{F}_k] < \infty \quad \text{a.s}$$

Consequently, by Theorem 2.8 of Chen and Guo (1991), we have

$$\sum_{k=0}^{n} a_k\varphi_k^T P_k\varphi_k(w_{k+1}^2 - E[w_{k+1}^2 \mid \mathscr{F}_k]) = O(S_n(\tfrac{1}{2}\alpha)\{\log[S_n(\tfrac{1}{2}\alpha) + e]\}^{2/\alpha+2}) \quad \text{a.s} \quad \forall\eta > 0, \qquad (A.4)$$

where

$$S_n(\tfrac{1}{2}\alpha) \triangleq \Big[\sum_{k=0}^{n}(a_k\varphi_k^T P_k\varphi_k)^{\alpha/2}\Big]^{2/\alpha}$$

Note that $a_k\varphi_k^T P_k\varphi_k \le 1$ and $\frac{1}{2}\alpha > 1$. By (A.3), we know that

$$S_n(\tfrac{1}{2}\alpha) = O(1) + O((\log|P_{n+1}^{-1}|)^{2/\alpha}).$$

Thus, by (A.4), we have

$$\sum_{k=0}^{n} a_k\varphi_k^T P_k\varphi_k w_{k+1}^2 = \sigma^2\sum_{k=0}^{n} a_k\varphi_k^T P_k\varphi_k + o(\log|P_{n+1}^{-1}|) + O(1).$$

Finally, substituting this together with (A.2) into (A.1), we see that the desired result is true. This completes the proof.

Proof of Corollary 3.3. First, similarly to the proof of (A.3), we see that

$$\sum_{k=0}^{n}\varphi_k^T P_k\varphi_k = \sum_{k=0}^{n}\frac{|P_{k+1}^{-1}| - |P_k^{-1}|}{|P_k^{-1}|} \ge \sum_{k=0}^{n}\int_{|P_k^{-1}|}^{|P_{k+1}^{-1}|}\frac{dx}{x} = \log|P_{n+1}^{-1}| + \log|P_0|.$$

Then, by (24) we have

$$\sum_{k=0}^{n} a_k\varphi_k^T P_k\varphi_k = \sum_{k=0}^{n}\varphi_k^T P_k\varphi_k - \sum_{k=0}^{n} a_k(\varphi_k^T P_k\varphi_k)^2 \ge \log|P_{n+1}^{-1}| + \log|P_0| + o\Big(\sum_{k=0}^{n} a_k\varphi_k^T P_k\varphi_k\Big),$$

which, in conjunction with (A.3), yields

$$\sum_{k=0}^{n} a_k\varphi_k^T P_k\varphi_k \sim \log|P_{n+1}^{-1}|, \qquad (A.5)$$

since $r_n \to \infty$ implies $\log|P_{n+1}^{-1}| \to \infty$. Note also that

$$\sum_{k=0}^{n} a_k(\varphi_k^T\tilde{\theta}_k)^2 = \sum_{k=0}^{n}(\varphi_k^T\tilde{\theta}_k)^2 - \sum_{k=0}^{n} a_k\varphi_k^T P_k\varphi_k(\varphi_k^T\tilde{\theta}_k)^2 = \sum_{k=0}^{n}(\varphi_k^T\tilde{\theta}_k)^2 + o\Big(\sum_{k=0}^{n} a_k(\varphi_k^T\tilde{\theta}_k)^2\Big) \qquad (A.6)$$

Substituting this and (A.5) into Lemma 3.1 and noting (21), we see that (25) is true

Furthermore, if (26) and (27) hold then it is not difficult to see that

$$\log|P_{n+1}^{-1}| \sim d\log n \qquad (A.7)$$

Note that, by (56), (27) and Lemma 5.2

$$\tilde{\theta}_{n+1}^T P_{n+1}^{-1}\tilde{\theta}_{n+1} \le \lambda_{\max}(P_{n+1}^{-1})\,\|\tilde{\theta}_{n+1}\|^2 = O(n)\Big\|P_{n+1}\Big(P_0^{-1}\tilde{\theta}_0 - \sum_{i=0}^{n}\varphi_i w_{i+1}\Big)\Big\|^2 = O(\log\log n) \quad \text{a.s.}$$

Finally, substituting this together with (A.5)–(A.7) into Lemma 3.1, we derive (28). This completes the proof. □

Proofs of Lemmas 6.1, 6.2 *and* 7.1

Proof of Lemma 6.1. By (4) and (57), it is seen that

$$y_{k+1} = \varphi_k^T\tilde{\theta}_k + y_{k+1}^* + w_{k+1}, \qquad (A.8)$$

and then, by (35) and (36) (with $\tilde{\theta}_k \triangleq \theta - \hat{\theta}_k$),

$$\begin{aligned} y_{k+1}^2 &\le 2(\varphi_k^T\tilde{\theta}_k)^2 + O(d_k) \\ &\le 2\alpha_k[1 + \varphi_k^T P_{k+1}\varphi_k + \varphi_k^T(P_k - P_{k+1})\varphi_k] + O(d_k) \\ &\le 2\alpha_k(2 + \delta_k\,\|\varphi_k\|^2) + O(d_k) \\ &= 2\alpha_k\delta_k\,\|\varphi_k\|^2 + O(d_k + \log r_k) \end{aligned} \qquad (A.9)$$

By the stability of $B(z)$, it is known from (1) that there is a constant $\lambda \in (0, 1)$ such that (44) holds. Hence

$$(\|\varphi_k\|^2 - u_k^2) = O\Big(\sum_{i=0}^{k}\lambda^{k-i}y_i^2\Big) + O(d_k) \qquad (A.10)$$

Now, by the properties H.1 and H 3, it follows from (58) that

$$u_k^2 = O\Big([\log(k+r_{k-1})]^2\Big(\sum_{i=0}^{p-1} y_{k-i}^2 + \sum_{i=1}^{q-1} u_{k-i}^2\Big) + \log(k+r_{k-1})\Big)$$

Moreover, putting (44) into this, it follows that

$$u_k^2 = O\Big([\log(k+r_{k-1})]^2\Big(\sum_{i=0}^{k} \lambda^{k-i} y_i^2 + d_k\Big)\Big) \quad \text{(A 11)}$$

Hence, by (A.10) and (A.11), we have

$$\|\varphi_k\|^2 = O([\log(k+r_{k-1})]^2 L_k) + O(d_k[\log(k+r_{k-1})]^2), \quad \text{(A.12)}$$

where $L_k \triangleq \sum_{i=0}^{k} \lambda^{k-i} y_i^2$. Note also that

$$b_1 u_k = \varphi_k^{\mathrm{T}} \tilde{\theta}_k + y_{k+1}^* + (b_1 u_k - \theta^{\mathrm{T}} \varphi_k)$$

So, by (A 10), we have

$$b_1^2 u_k^2 \le 3(\varphi_k^{\mathrm{T}}\tilde{\theta}_k)^2 + O(1 + |b_1 u_k - \theta^{\mathrm{T}}\varphi_k|^2) = 3(\varphi_k^{\mathrm{T}}\tilde{\theta}_k)^2 + O(L_k + d_k) \quad \text{(A.13)}$$

Similarly to the proof of (A.9), it is known that

$$(\varphi_k^{\mathrm{T}}\tilde{\theta}_k)^2 \le \alpha_k \delta_k \|\varphi_k\|^2 + 2\alpha_k$$

Substituting this into (A 13), we see that

$$u_k^2 = O(\alpha_k\delta_k\|\varphi_k\|^2) + O(L_k + d_k + \log r_k)$$

Combining this with (A.10), we get

$$\|\varphi_k\|^2 = O(\alpha_k\delta_k\|\varphi_k\|^2) + O(L_k + d_k + \log r_k)$$

Putting (A 12) into this, we have

$$\|\varphi_k\|^2 = O(\alpha_k\delta_k[\log(k+r_{k-1})]^2 L_k) + O(L_k + d_k[\log(k+r_{k-1})]^2).$$

Finally, substituting this into (A 9), we find that there is a constant $c > 0$ such that $y_{k+1}^2 \le c f_k L_k + \xi_k$, where f_k and ξ_k are defined in Lemma 6 1. Furthermore, by this and the definition of L_k

$$L_{k+1} \le \lambda L_k + y_{k+1}^2 \le (\lambda + c f_k) L_k + \xi_k \quad \text{(A 14)}$$

Hence the lemma is true. □

Proof of Lemma 6 2. By Lemma 6 1

$$L_{n+1} \le \lambda^{n+1}\Big[\prod_{j=0}^{n}(1+\lambda^{-1} c f_j)\Big] L_0 + \sum_{i=0}^{n} \lambda^{n-i}\Big[\prod_{j=i+1}^{n}(1+\lambda^{-1} c f_j)\Big]\xi_i \quad \text{(A 15)}$$

We proceed to estimate the product $\prod_{j=i+1}^{n}(1+\lambda^{-1} c f_j)$

First, by Corollary 3.1, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\delta \sum_{j=0}^{n} \alpha_j \le \epsilon(\log r_n) \quad \forall n$$

Also, by (47), there is an integer $i_0 > 0$ sufficiently large that

$$\frac{4}{\delta}\Big(\frac{c}{\lambda}\Big)^{1/2} \sum_{j=i}^{\infty} \delta_j \le \epsilon \quad \forall i \ge i_0 \quad \text{(A 16)}$$

Then, by the inequalities

$$1 + x^2 \le e^{2x}, \quad (1+xy) \le (1+x)(1+y), \quad x \ge 0,\ y \ge 0,$$

we know that, for any $n \ge i \ge i_0$

$$\begin{aligned}
&\prod_{j=i+1}^{n}\{1 + \lambda^{-1} c[\alpha_j\delta_j \log(j+r_j)]^2\} \\
&\le \prod_{j=i+1}^{n}[1 + (\tfrac{1}{2}\delta\alpha_j)^2] \prod_{j=i}^{n}\Big\{1 + \lambda^{-1} c\Big[\frac{2}{\delta}\delta_j \log(j+r_j)\Big]^2\Big\} \\
&\le \exp\Big(\delta \sum_{j=i+1}^{n}\alpha_j\Big) \exp\Big[\frac{4}{\delta}\Big(\frac{c}{\lambda}\Big)^{1/2} \sum_{j=i+1}^{n}\delta_j \log(j+r_j)\Big] \\
&\le \exp(\epsilon \log r_n) \exp\Big\{[\log(n+r_n)]\Big[\frac{4}{\delta}\Big(\frac{c}{\lambda}\Big)^{1/2}\sum_{j=i}^{n}\delta_j\Big]\Big\} \\
&\le r_n^{\epsilon} \exp\{[\log(n+r_n)]\epsilon\} \\
&= (n+r_n)^{2\epsilon} \quad \text{a s.}
\end{aligned} \quad \text{(A 17)}$$

Furthermore, for any $n \ge i \ge i_0$

$$\prod_{j=i}^{n}(1+\lambda^{-1} c\alpha_j\delta_j) \le \exp\Big(\delta\sum_{j=i}^{n}\alpha_j\Big)\exp\Big(\frac{c}{\lambda\delta}\sum_{j=i}^{n}\delta_j\Big) = O(r_n^{\epsilon}) \quad \text{(A 18)}$$

Finally, by the definition of f_j in Lemma 6 1, it follows from (A.17) and (A 18) that

$$\prod_{j=i+1}^{n}(1 + c\lambda^{-1} f_j) \le \prod_{j=i+1}^{n}\{1 + c\lambda^{-1}[\alpha_j\delta_j \log(j+r_j)]^2\} \times \prod_{j=i+1}^{n}(1 + c\lambda^{-1}\alpha_j\delta_j) = O((n+r_n)^{3\epsilon}) \quad \text{a.s.} \quad \forall n \ge i \ge i_0.$$

Putting this into (A.15), we have, after some simple manipulations

$$L_{n+1} = O((n+r_n)^{3\epsilon} d_n \log^4(n+r_n)) \quad \forall \epsilon > 0$$

Then, by Lemma 6 1 and the arbitrariness of ϵ

$$y_{n+1}^2 \le L_{n+1} = O((n+r_n)^{\epsilon} d_n) \quad \forall \epsilon > 0$$

From this and (A 10), we know that $u_n^2 = O((n+r_n)^{\epsilon} d_n)$ $\forall \epsilon > 0$. Hence the lemma is proved. □

Proof of Lemma 7.1 From the proof of Lemma 6.1, it can easily be seen that (A.9) holds with $\tilde{\theta}_k = \theta - \theta_k$. However, since H.3 is no longer assumed, the property (A 11) cannot be directly applied. We now proceed to derive a similar upper bound for u_k^2

First, by (20) and Corollary 3.1(ii), we have

$$\sum_{i=0}^{n}(\varphi_i^{\mathrm{T}}\tilde{\theta}_{n+1})^2 = O(\log r_n) \quad \text{a.s.}, \quad \text{(A.19)}$$

which implies that

$$\max_{i \le n}(\varphi_i^{\mathrm{T}}\tilde{\theta}_{n+1})^2 = O(\log r_n) \quad \text{a s} \quad \text{(A 20)}$$

For simplicity, we shall omit the phrase 'a.s on D' in the sequel, and all relationships should be understood to be held on D with a possible exceptional set of probability zero. Denote

$$\tilde{b}_1(\tau_n+1) = b_1 - b_1(\tau_n+1).$$

Then, by (67)

$$\inf_n \sqrt{\log r_{\tau_n}}\,|\tilde{b}_1(\tau_n+1)| > 0$$

Hence by (A.10), (A 20) and the fact that $\|\tilde{\theta}_{n+1}\|^2 = O(\log r_n)$, it follows that for all $k \le \tau_n$, $n \ge 1$

$$
\begin{aligned}
u_k^2 &= \frac{1}{[\tilde{b}_1(\tau_n+1)]^2}[\tilde{b}_1(\tau_n+1)u_k]^2 \\
&= \frac{1}{[\tilde{b}_1(\tau_n+1)]^2}\{[\varphi_k^T\tilde{\theta}_{\tau_n+1} - \tilde{b}_1(\tau_n+1)u_k] \\
&\quad - \varphi_k^T\tilde{\theta}_{\tau_n+1}\}^2 \\
&\le \frac{2}{[\tilde{b}_1(\tau_n+1)]^2}\{|\varphi_k^T\tilde{\theta}_{\tau_n+1} - \tilde{b}_1(\tau_n+1)u_k|^2 \\
&\quad + (\varphi_k^T\tilde{\theta}_{\tau_n+1})^2\} \\
&= O((\log r_{\tau_n})^2(\|\varphi_k\|^2 - u_k^2)) + O(\log r_{\tau_n}) \\
&= O((\log r_{\tau_n})^2 L_k) + O(d_{\tau_n}(\log r_{\tau_n})^2),
\end{aligned}
$$

where $L_k \triangleq \sum_{i=0}^{k} \lambda^{k-i} y_i^2$. Combining this with (A 10), we obtain, for all $k \le \tau_n$

$$\|\varphi_k\|^2 = O((\log^2 r_{\tau_n})L_k) + O(d_{\tau_n}(\log r_{\tau_n})^2),$$

which is similar to (A 12) From this and the argument leading up to (A 14), we know that there is a constant $c > 0$ such that, $\forall k \le \tau_n$

$$
\begin{aligned}
L_{k+1} \le &\{\lambda + c\alpha_k\delta_k[1 + \alpha_k\delta_k(\log r_{\tau_n})^2]\}L_k \\
&+ O(d_{\tau_n}(\log r_{\tau_n})^4)
\end{aligned}
$$

Similarly to the proof of Lemma 6 2, it can be derived that

$$\sup_{k \le \tau_n} \|\varphi_k\|^2 = O(r_{\tau_n}^{\epsilon} d_{\tau_n}) \quad \forall \epsilon > 0.$$

Therefore (68) holds. Furthermore, by this and Corollary 3.1, (69) can easily be derived from (A 8) This completes the proof □