For $n \geq 0$, let $\mathcal{F}_{n+1} = \sigma\{t_{u,1}, t_{d,1}, \cdots, t_{u,k_1}, t_{d,k_1}; \cdots; t_{u,k_n+1}, t_{d,k_n+1}, \cdots, t_{u,k_{n+1}}, t_{d,k_{n+1}}\}$. Since $x(\theta_n, \zeta_n) = \theta_n$, $\xi(\zeta_n-) = 1$, $\xi(\zeta_n) = 0$; $x(\theta_n, T_{n+1}) = \theta_n$, $\xi(T_{n+1}-) = 1$, $\xi(T_{n+1}) = 0$, $[\zeta_n, T_{n+1}]$ is a regenerative cycle for $\{x(\theta_n, t), \xi(t)\}$. By Lemma 1.2 it is seen that $E[\varepsilon_{n+1}|\mathcal{F}_n] = 0$. Thus $\{\varepsilon_n, \mathcal{F}_n\}$ is a martingale difference sequence. Along the same lines as in Theorems 3.1–3.3, we arrive at the following results for the adaptive control scheme II.

Theorem 4.1: Suppose: 1) that conditions A0) and A2)-1) hold with $q_0 = 2$. Then $\lim_{n\to\infty} \theta_n = \theta^0$ a.s. and 2) that conditions A0), A2), and A3) hold with $q_0 = 2$. Then $|\theta_n - \theta^0| = o(a_n^\delta)$ a.s. for all $\delta \in [0, 1 - 1/(2\nu))$.

Theorem 4.2: Suppose that conditions A0)–A3) hold with $q_0 = 4$, $\nu > 2/3$. Then

$$\frac{\theta_n - \theta^0}{\sqrt{a_n}} \xrightarrow[n\to\infty]{d} \mathcal{N}(0, \sigma_1^2).$$

Theorem 4.3: The assertions of Theorem 3.3 remain true for the adaptive control scheme II.

It is of interest to extend our results to failure-prone manufacturing systems with multiple machine states or multiple part types.

References

[1] R. Akella and P. R. Kumar, "Optimal control of production rate in a failure prone manufacturing system," *IEEE Trans. Automat. Contr.* vol. 31, pp. 116–126, 1986.

[2] S. Asmussen, *Applied Probability and Queues.* New York: Wiley, 1987.

[3] T. Bielecki and P. R. Kumar, "Optimality of zero-inventory policy for unreliable manufacturing systems," *Oper. Res.*, vol. 36, pp. 532–541, 1988.

[4] P. Brémaud, R. P. Malhamé, and L. Massoulié, "A manufacturing system with general stationary failure process: Stability and IPA of hedging control policies," *IEEE Trans. Automat. Contr.,* vol. 42, pp. 155–170, 1997.

[5] H. F. Chen, "Stochastic approximation and its new applications," in *Proc. 1994 Hong Kong Int. Workshop on New Directions in Control and Manufacturing,* 1994, pp. 2–12.

[6] H. F. Chen and Y. M. Zhu, *Stochastic Approximation.* Shanghai Press of Science and Technology, 1996.

[7] E. K. P. Chong and P. J. Ramadge, "Stochastic optimization of regenerative systems using infinitesimal perturbation analysis," *IEEE Trans. Automat. Contr.,* vol. 39, pp. 1400–1410, 1994.

[8] Y. S. Chow and H. Teicher, *Probability Theory : Independence, Interchangeability, Martingale,* 2nd ed. New York: Springer-Verlag, 1988.

[9] P. Glasserman, *Gradient Estimation via Perturbation Analysis.* Boston, MA: Kluwer, 1991.

[10] A. Gut, *Stopped Random Walks: Limit Theorems and Applications.* New York: Springer-Verlag, 1988.

[11] J. Q. Hu and D. Xiang, "The queueing equivalence to a manufacturing system with failures," *IEEE Trans. Automat. Contr.,* pp. 38, pp. 499–502, 1993.

[12] J. G. Kimemia and S. B. Gershwin, "An algorithm for the computer control of production in flexible manufacturing systems," *IIE Trans.,* vol. 15, pp. 353–362, 1983.

[13] S. P. Sethi and Q. Zhang, *Hierarchical Decision Making in Stochastic Manufacturing Systems.* Boston, MA: Birkhäuser, 1994.

[14] A. Sharifnia, "Optimal production control of a manufacturing system with machine failures," *IEEE Trans. Automat. Contr.,* vol. 33, pp. 620–625, 1988.

[15] W. F. Stout, *Almost Sure Convergence.* New York: Academic, 1974.

[16] Q. Y. Tang and E. K. Boukas, "Adaptive control for manufacturing systems using infinitesimal perturbation analysis," *GERAD: Tech. Rep.,* 1997.

[17] Q. Y. Tang and H. F. Chen, "Convergence of perturbation analysis based optimization algorithm with fixed number of customers period," *Discrete Event Dynamic Systems: Theory and Appl.,* vol. 4, pp. 359–375, 1994.

[18] Q. Y. Tang, H. F. Chen, and Z. J. Han, "Convergence rates of perturbation-analysis-Robbins–Monro-single-run algorithms," *IEEE Trans. Automat. Contr.,* vol. 42, pp. 1442–1447, 1997.

Prediction-Based Discrete-Time Adaptive Nonlinear Stochastic Control

Chen Wei and Lei Guo

***Abstract*—Adaptive control of a class of discrete-time parametric-strict-feedback nonlinear systems with additive white noises is considered in this paper. The control law is designed based on weighted least squares (WLS) algorithms and on recursive adaptive predictors. Global stability and tracking error bounds are established for the closed-loop systems.**

***Index Terms*—Adaptive nonlinear control, discrete time, global stability, prediction, stochastic systems.**

I. Introduction

In recent years, much progress has been made for adaptive control of continuous-time nonlinear systems (cf., e.g., [6] and [8]–[12]). The counterpart result for the adaptive control of discrete-time nonlinear systems, however, has been hindered by some inherent difficulties in discrete-time models, as mentioned in [5] and [7] and as shown in [3].

The main difficulty lies in the fact that the nonlinear damping approach which is so successful for the controller design of continuous-time nonlinear systems fails in the discrete-time case. This accounts for the situation that most of the existing results deal with only systems with nonlinearities having a linear growth rate [7]. In an effort to remove the linear growth constraints, [5] analyzed the stability of a first-order discrete-time deterministic adaptive system, and [4] studied the global stability and instability of a class of least squares (LS)-based discrete-time deterministic adaptive control systems. Recently, Guo [3] found the limitations of adaptive control for discrete-time nonlinear systems by studying the critical stability of a class of nonlinear stochastic control systems. It was also shown in [3] that recursive adaptive nonlinear control schemes that have been proven to be stable in the noise-free case, may indeed lose their stability in the presence of (even) zero mean bounded white noises. Thus, it is necessary to take the noise effects into account in the study of adaptive systems.

In this paper, we shall extend the existing stability results established for deterministic systems in [7] to the stochastic case by using a prediction-based design procedure and the weighted least squares (WLS) algorithm. Under the presence of random noises, both the stability of adaptive nonlinear control systems and the output tracking error bounds are established.

II. The Main Results

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A. Problem Formulation

Consider the discrete-time parametric-strict-feedback control systems, as shown in (1) at the bottom of the page, where $y(k)$, $u(k)$, and $w_i(k)$, $1 \le i \le n$ are the system output, input and noise, respectively, $x(k) = [x_1(k) \cdots x_n(k)]^\tau$ is the measured state vector, $\theta \in R^p$ is an unknown parameter vector, and $\alpha_i(\cdot)$, $1 \le i \le n$ are known nonlinear vector-valued functions. This model may be regarded as the stochastic analogue of the deterministic one studied in [7].

The control objective is to design a feedback control sequence $\{u(k)\}$, such that the system output sequence tracks a reference signal $\{y^*(k)\}$.

In order to analyze this control problem, we introduce the following conditions.

A1) The noise sequence $\{w_i(k), \mathcal{F}_k\}\,(1 \le i \le n)$ is a martingale difference sequence (where $\{\mathcal{F}_k\}$ is a family of nondecreasing σ-algebras) with conditional variance σ_i^2, i.e., $E[w_i^2(k+1)|\mathcal{F}_k] = \sigma_i^2$ a.s. $\forall k,\ 1 \le i \le n$. We also assume that

$$\lim_{t\to\infty} \frac{1}{t}\sum_{k=1}^{t} |w_i(k)|^2 = \sigma_i^2 \quad \text{a.s.} \quad 1 \le i \le n. \tag{2}$$

A2) There exists a constant $L_1 > 0$ such that

$$|\alpha_i(\xi_1) - \alpha_i(\xi_2)| \le L_1|\xi_1 - \xi_2|, \quad \forall\, \xi_1, \xi_2 \in R^i,\quad 1 \le i \le n-1$$
$$|\alpha_n(x)| \le L_1|x|, \quad \forall\, x \in R^n.$$

A3) $\{y^*(k)\}$ is a bounded deterministic reference signal.

B. Parameter Estimation

First, similar to [7], put (1) into a compact form

$$\begin{cases} x(k+1) = Ax(k) + bu(k) + \Phi_k\theta + W_{k+1} \\ y(k) = Cx(k) \end{cases} \tag{3}$$

where

$$x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k)\end{bmatrix}, \qquad W_k = \begin{bmatrix} w_1(k) \\ w_2(k) \\ \vdots \\ w_n(k)\end{bmatrix}$$

$$\Phi_k = \begin{bmatrix} \alpha_1(x_1(k))^\tau \\ \alpha_2(x_1(k), x_2(k))^\tau \\ \vdots \\ \alpha_n(x_1(k) \cdots x_n(k))^\tau \end{bmatrix} \tag{4}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = [1 \quad 0 \quad \cdots \quad 0 \quad 0]. \tag{5}$$

Next, use the matrix version of the recursive WLS algorithm [1] to estimate θ

$$\theta_{k+1} = \theta_k + P_k\Phi_k^\tau Q_k(x(k+1) - \hat{x}(k+1)) \tag{6}$$

$$P_{k+1} = P_k - P_k\Phi_k^\tau Q_k\Phi_k P_k \tag{7}$$

$$Q_k \triangleq [\beta_k^{-1}I + \Phi_k P_k \Phi_k^\tau]^{-1} \tag{8}$$

$$\hat{x}(k+1) \triangleq Ax(k) + bu(k) + \Phi_k\theta_k \tag{9}$$

where the initial values θ_0 and $P_0 > 0$ are chosen arbitrarily and $\{\beta_k\}$ is the weighting sequence defined by

$$\beta_k = \frac{1}{\log^{1+\delta} r_k}, \qquad r_k \triangleq \|P_0^{-1}\| + \sum_{i=0}^{k} \|\Phi_i\|^2 \tag{10}$$

where δ is some positive constant.

C. Prediction and Controller Design

Iterating (1) step by step, we get the following "input–output" form:

$$y(k+n) = u(k) + \theta^\tau \sum_{i=1}^{n} \alpha_i(k+n-i) + \sum_{i=1}^{n} w_i(k+n-i+1) \tag{11}$$

where

$$\alpha_i(k) \triangleq \alpha_i(x_1(k) \cdots x_i(k)), \qquad 1 \le i \le n. \tag{12}$$

If $\alpha_i(k+n-i)\,(1 \le i \le n)$ were known at step k, we see from (11) that the "certainty equivalence" adaptive tracking control would be

$$u(k) = y^*(k+n) - \theta_k^\tau \sum_{i=1}^{n} \alpha_i(k+n-i).$$

But at step k, $x_i(k+j)$, $j > 0$ are unknown. A natural way is to use their predicted values. The minimum variance predictors for $x_i(k+j)$ and $\alpha_i(k+j)$ at time k are

$$x_i(k+j|k) \triangleq E[x_i(k+j)|\mathcal{F}_k]$$
$$\alpha_i(k+j|k) \triangleq E[\alpha_i(k+j)|\mathcal{F}_k] \tag{13}$$

where $1 \le i \le n-j$, $1 \le j \le n-1$. Clearly, the conditional expectation depends on the unknown parameter θ. Using the parameter estimate θ_k to replace θ, we can define the following optimal prediction-based adaptive predictors:

$$\overline{x}_i(k+j|k) \triangleq E[x_i(k+j)|\mathcal{F}_k]_{\theta=\theta_k}$$
$$\overline{\alpha}_i(k+j|k) \triangleq E[\alpha_i(k+j)|\mathcal{F}_k]_{\theta=\theta_k} \tag{14}$$

where $1 \le i \le n-j$, $1 \le j \le n-1$ and θ_k is defined by (6)–(9). Because of the nonlinearity involved, these adaptive predictors are hard to calculate. Instead, we use the following approximate adaptive predictor expressed in a recursive form:

$$\hat{x}_i(k+j|k) \triangleq \hat{x}_{i+1}(k+j-1|k) + \theta_k^\tau \hat{\alpha}_i(k+j-1|k) \tag{15}$$

$$\hat{\alpha}_i(k+j|k) \triangleq \alpha_i(\hat{x}_1(k+j|k) \cdots \hat{x}_i(k+j|k)) \tag{16}$$

$$\hat{x}_i(k|k) \triangleq x_i(k), \quad \hat{\alpha}_i(k|k) \triangleq \alpha_i(k) \tag{17}$$

where $1 \le i \le n-j$, $1 \le j \le n-1$.

Now, at step k, the adaptive control law can be defined as

$$u(k) = y^*(k+n) - \theta_k^\tau \sum_{i=1}^{n} \hat{\alpha}_i(k+n-i|k). \tag{18}$$

With this controller applied to (11), the closed-loop equation is

$$y(k+n) = y^*(k+n) + \theta^\tau \sum_{i=1}^{n} \alpha_i(k+n-i)$$

$$\begin{cases} x_i(k+1) = x_{i+1}(k) + \theta^\tau \alpha_i(x_1(k) \cdots x_i(k)) + w_i(k+1) \quad 1 \le i \le n-1 \\ x_n(k+1) = u(k) + \theta^\tau \alpha_n(x_1(k) \cdots x_n(k)) + w_n(k+1) \\ y(k) = x_1(k) \end{cases} \tag{1}$$

$$- \theta_k^\tau \sum_{i=1}^{n} \hat{\alpha}_i(k+n-i|k) + \sum_{i=1}^{n} w_i(k+n-i+1). \quad (19)$$

Remark 1: By transforming (1) into (11), the formula of the optimal controller can be written immediately, and the necessity to predict the states is obvious. In the noise-free case, the recursive predictors (15)–(17) are the same as the optimal prediction-based ones (14). Although (15)–(17) are not expected to perform better than (14) in the stochastic case, they have the advantage of recursive computations and do indeed give an adaptive controller that is robust with respect to stochastic disturbances, as to be shown in Theorem 1 below.

D. The Main Results

We now present the main results on stability and tracking performance of the closed-loop system (19).

Theorem 1: Consider the adaptive control system described by (6)–(12) and (15)–(19). Let Conditions A1)–A3) be satisfied. Then the closed-loop system is stable in the sense that as $T \to \infty$

$$\sum_{t=1}^{T} \left(\|x(t)\|^2 + |u(t)|^2\right) = O(T) \quad \text{a.s.}$$

and the averaged squared tracking error satisfies

$$\frac{1}{T} \sum_{t=1}^{T} |y(t+1) - y^*(t+1)|^2 = O\left(\sigma^2\right) + o(1) \quad \text{a.s.} \quad (20)$$

where $\sigma^2 \triangleq \sum_{l=1}^{n} \sigma_l^2$.

We note that in the noise-free case (i.e., $\sigma^2 = 0$), the right-hand side (RHS) of (20) reduces to $o(1)$, which means that the tracking error converges to zero in the averaging sense.

III. Proof of the Main Results

We first present some basic properties of the WLS algorithm, which can be proven in a similar way to those established for linear regression vector models (see [1] and [2]).

Lemma 1: If Condition A1) is fulfilled, then the WLS algorithm defined by (6)–(9) satisfies

1) $\sum_{k=1}^{\infty} \tilde{\theta}_k^\tau \Phi_k^\tau Q_k \Phi_k \tilde{\theta}_k < \infty$ a.s.;
2) θ_k converges to some finite random vector $\overline{\theta}$ almost surely;
3) $\sum_{k=1}^{\infty} \|\theta_{k+q} - \theta_k\|^2 < \infty$ a.s. $\forall\, q > 0$;

where $\tilde{\theta}_k \triangleq \theta - \theta_k$.

We remark that the property 1) is nothing but the matrix analogue of [2, Lemma 1, property (iii)].

Lemma 2: Under Conditions A1)–A3), we have

$$\|x(k+1)\|^2 = O(\|x(k)\|^2) + o(k). \quad (21)$$

Proof: First of all, by (2) it follows that as $k \to \infty$

$$|w_i(k)|^2 = o(k) \quad \text{a.s.} \quad 1 \le i \le n. \quad (22)$$

Next, from Condition A2) we see that

$$\|\alpha_i(x)\| \le L_1\|x\| + L_2, \qquad 1 \le i \le n \quad (23)$$

where $L_2 \triangleq \max_{1 \le i \le n-1} \|\alpha_i(0)\|$.

Then, by (4), (12), (15)–(17), (23), and 2) of Lemma 1, the following result can be derived inductively:

$$\|\hat{\alpha}_i(k+j|k)\| = O(\|x(k)\|) + O(1) \quad (24)$$

where $1 \le j \le n-1$, $1 \le i \le n-j$. Therefore, substituting (24) into (18), and applying Condition A3) and 2) of Lemma 1, we have

$$|u(k)| = O(\|x(k)\|) + O(1). \quad (25)$$

Finally, from (1), (22), (23), and (25), the desired result (21) follows immediately. □

Now, let us introduce the following notations for prediction errors:

$$\tilde{x}_i(k+j|k) = x_i(k+j) - \hat{x}_i(k+j|k) \quad (26)$$

$$\tilde{\alpha}_i(k+j|k) = \alpha_i(k+j) - \hat{\alpha}_i(k+j|k) \quad (27)$$

where $1 \le j \le n-1$, $1 \le i \le n-j$. The following two lemmas are devoted to establish upper bounds for these prediction errors.

Lemma 3: Under Conditions A1)–A2), it holds that

$$|\tilde{x}_i(k+j|k)|^2 + \|\tilde{\alpha}_i(k+j|k)\|^2 = O\left(\sum_{l=0}^{j-1} \left[\|\Phi_{k+l}\tilde{\theta}_{k+l}\|^2 + \|W_{k+l+1}\|^2 + o\left(\|x(k+l)\|^2\right)\right]\right) + o(1), \qquad \forall\, k \ge 0$$

where $1 \le j \le n-1$, $1 \le i \le n-j$.

Proof: We inductively prove the result for $1 \le j \le n-1$.

For $j = 1$, by (1), (4), (12), (15), and (17), we see that

$$\begin{aligned} |\tilde{x}_i(k+1|k)|^2 &= |x_i(k+1) - \hat{x}_i(k+1|k)|^2 \\ &= |x_{i+1}(k) + \theta^\tau \alpha_i(k) + w_i(k+1) \\ &\quad - x_{i+1}(k) - \theta_k^\tau \alpha_i(k)|^2 \\ &\le 2|\tilde{\theta}_k^\tau \alpha_i(k)|^2 + 2|w_i(k+1)|^2 \\ &\le 2\|\Phi_k \tilde{\theta}_k\|^2 + 2\|W_{k+1}\|^2. \end{aligned} \quad (28)$$

Then, by (12), (16), (28), and Condition A2), we have

$$\begin{aligned} \|\tilde{\alpha}_i(k+1|k)\|^2 &= \|\alpha_i(k+1) - \hat{\alpha}_i(k+1|k)\|^2 \\ &= \|\alpha_i(x_1(k+1), \cdots x_i(k+1)) \\ &\quad - \alpha_i(\hat{x}_1(k+1|k), \cdots \hat{x}_i(k+1|k))\|^2 \\ &\le L_1^2 \sum_{l=1}^{i} |x_l(k+1) - \hat{x}_l(k+1|k)|^2 \\ &\le O\left(\|\Phi_k \tilde{\theta}_k\|^2\right) + O\left(\|W_{k+1}\|^2\right). \end{aligned} \quad (29)$$

Hence, combining (28) and (29) we see that the lemma holds for $j = 1$.

Now, assume that the lemma holds for some $j \ge 1$. Then by (1), (4), (12), (15), and (23) we have for $1 \le i \le n-j-1$

$$\begin{aligned} &|\tilde{x}_i(k+j+1|k)|^2 \\ &= |x_i(k+j+1) - \hat{x}_i(k+j+1|k)|^2 \\ &= |x_{i+1}(k+j) + \theta^\tau \alpha_i(k+j) + w_i(k+j+1) \\ &\quad - \hat{x}_{i+1}(k+j|k) - \theta_k^\tau \hat{\alpha}_i(k+j|k)|^2 \\ &\le 5|\tilde{x}_{i+1}(k+j|k)|^2 + 5\left|\tilde{\theta}_{k+j}^\tau \alpha_i(k+j)\right|^2 \\ &\quad + 5|[\theta_{k+j} - \theta_k]^\tau \alpha_i(k+j)|^2 \\ &\quad + 5|\theta_k^\tau [\alpha_i(k+j) - \hat{\alpha}_i(k+j|k)]|^2 \\ &\quad + 5|w_i(k+j+1)|^2 \end{aligned}$$

$$
\begin{aligned}
&\leq 5|\tilde{x}_{i+1}(k+j|k)|^2 + 5\left\|\Phi_{k+j}\tilde{\theta}_{k+j}\right\|^2 \\
&\quad + 5\|\theta_{k+j}-\theta_k\|^2\cdot\left[2L_1^2\|x(k+j)\|^2+2L_2^2\right] \\
&\quad + 5\|\theta_k\|^2\cdot\|\tilde{\alpha}_i(k+j|k)\|^2 + 5\|W_{k+j+1}\|^2 \\
&\leq O\left(\sum_{l=0}^{j}\left[\|\Phi_{k+l}\tilde{\theta}_{k+l}\|^2+\|W_{k+l+1}\|^2\right.\right. \\
&\qquad \left.\left. + o(\|x(k+l)\|^2)\right]\right) + o(1), \qquad \forall\, k\geq 0 \qquad (30)
\end{aligned}
$$

where the last inequality is derived from the induction assumption and Lemma 1. Similar to the derivation of (29), by (12), (16), (30), and Condition A2) it can be shown that

$$
\begin{aligned}
&\|\tilde{\alpha}_i(k+j+1|k)\|^2 \\
&\quad = O\left(\sum_{l=0}^{j}\left[\|\Phi_{k+l}\tilde{\theta}_{k+l}\|^2+\|W_{k+l+1}\|^2\right.\right. \\
&\qquad \left.\left. + o(\|x(k+l)\|^2)\right]\right) + o(1). \qquad (31)
\end{aligned}
$$

Therefore, combining (30) and (31) we see that the lemma holds for $j+1$. Hence, the lemma holds for any $1\leq j\leq n-1$. □

The next lemma gives an estimation for the accumulated prediction errors.

Lemma 4: Under Conditions A1)–A3), we have for $1\leq i\leq n-j,\quad 1\leq j\leq n-1$

$$
\begin{aligned}
&\sum_{k=1}^{t}\left(|\tilde{x}_i(k+j|k)|^2+\|\tilde{\alpha}_i(k+j|k)\|^2\right) \\
&\quad = O(\sigma^2 t)+o(\bar{r}_t)+o(t) \qquad (32)
\end{aligned}
$$

where $\tilde{x}_i(\cdot)$ and $\tilde{\alpha}_i(\cdot)$ are defined by (26) and (27), σ^2 is defined in Theorem 1 and

$$\bar{r}_t \triangleq 1+\sum_{k=1}^{t}\|x(k)\|^2. \qquad (33)$$

Proof: By Lemma 3, we have

$$
\begin{aligned}
&\sum_{k=1}^{t}\left(|\tilde{x}_i(k+j|k)|^2+\|\tilde{\alpha}_i(k+j|k)\|^2\right) \\
&\quad = O\left(\sum_{k=1}^{t+j-1}\|\Phi_k\tilde{\theta}_k\|^2\right)+O\left(\sum_{k=1}^{t+j-1}\|W_{k+1}\|^2\right) \\
&\qquad + o\left(\sum_{k=1}^{t+j-1}\|x(k)\|^2\right)+o(t). \qquad (34)
\end{aligned}
$$

For the first term on the RHS of (34), by (8), (10), and $P_k\leq P_0$ we have

$$
\begin{aligned}
&\tilde{\theta}_k^\tau\Phi_k^\tau Q_k\Phi_k\tilde{\theta}_k \\
&\quad \geq \lambda_{\min}(Q_k)\cdot\|\Phi_k\tilde{\theta}_k\|^2 \\
&\quad \geq \left[\beta_k^{-1}+\lambda_{\max}(\Phi_k P_k\Phi_k^\tau)\right]^{-1}\cdot\|\Phi_k\tilde{\theta}_k\|^2 \\
&\quad \geq (\beta_k^{-1}+\|P_0\|\cdot\|\Phi_k\|^2)^{-1}\cdot\|\Phi_k\tilde{\theta}_k\|^2 \\
&\quad \geq [\log^{1+\delta} r_k+\|P_0\|\cdot r_k]^{-1}\cdot\|\Phi_k\tilde{\theta}_k\|^2 \qquad (35)
\end{aligned}
$$

and so by 1) of Lemma 1 and the Kronecker Lemma we have for $1\leq j\leq n$

$$\sum_{k=1}^{t+j-1}\|\Phi_k\tilde{\theta}_k\|^2 = o(r_{t+j-1})+O(1)=o(\bar{r}_t)+o(t) \qquad (36)$$

where for the last equality we have used (4), (10), (23), and Lemma 2.

For the second term on the RHS of (34), by (2) and (4) we have for $1\leq j\leq n$

$$\sum_{k=1}^{t+j-1}\|W_{k+1}\|^2 = O(\sigma^2 t) \qquad (37)$$

where σ^2 is defined in Theorem 1.

For the third term on the RHS of (34), by (33) and Lemma 2 it can be seen that for $1\leq j\leq n$

$$\sum_{k=1}^{t+j-1}\|x(k)\|^2 \leq \bar{r}_t+\sum_{l=1}^{j-1}\|x(t+l)\|^2 = O(\bar{r}_t)+o(t). \qquad (38)$$

Finally, substituting (36)–(38) into (34), we see that the lemma is true. □

We are now in a position to give the proof of Theorem 1.

Proof of Theorem 1: It follows from (4), (12), (17), (19), and (27) that

$$
\begin{aligned}
&\sum_{k=1}^{t}|y(k+n)-y^*(k+n)|^2 \\
&\quad = \sum_{k=1}^{t}\left|\sum_{j=1}^{n}\tilde{\theta}_{k+n-j}^\tau\alpha_j(k+n-j)\right. \\
&\qquad + \sum_{j=1}^{n}[\theta_{k+n-j}-\theta_k]^\tau\alpha_j(k+n-j) \\
&\qquad + \sum_{j=1}^{n}\theta_k^\tau[\alpha_j(k+n-j)-\hat{\alpha}_j(k+n-j|k)] \\
&\qquad \left. + \sum_{j=1}^{n}w_j(k+n-j+1)\right|^2 \\
&\quad \leq 4n\sum_{k=1}^{t}\sum_{j=1}^{n}\left\|\Phi_{k+n-j}\tilde{\theta}_{k+n-j}\right\|^2 \\
&\qquad + 4n\sum_{k=1}^{t}\sum_{j=1}^{n}\|\theta_{k+n-j}-\theta_k\|^2 \\
&\qquad \cdot\|\alpha_j(k+n-j)\|^2 \\
&\qquad + 4n\sum_{k=1}^{t}\sum_{j=1}^{n-1}\|\theta_k\|^2\|\tilde{\alpha}_j(k+n-j|k)\|^2 \\
&\qquad + 4n\sum_{k=1}^{t}\sum_{j=1}^{n}\|W_{k+n-j+1}\|^2. \qquad (39)
\end{aligned}
$$

As to the second term on the RHS of (39), by (4), (12), (23), (38), and 3) of Lemma 1, it is clear that

$$
\begin{aligned}
&\sum_{k=1}^{t}\sum_{j=1}^{n}\|\theta_{k+n-j}-\theta_k\|^2\|\alpha_j(k+n-j)\|^2 \\
&\quad = O\left(\sum_{k=1}^{t}\sum_{j=1}^{n}\|\theta_{k+n-j}-\theta_k\|^2\cdot[\|x(k+n-j)\|^2+1]\right) \\
&\quad = o\left(\sum_{k=1}^{t+n-1}\|x(k)\|^2\right)+O(1) \\
&\quad = o(\bar{r}_t)+o(t). \qquad (40)
\end{aligned}
$$

For the third term on the RHS of (39), by 2) of Lemmas 1 and 4 we have

$$\begin{aligned}\sum_{k=1}^{t}\sum_{j=1}^{n-1}\|\theta_k\|^2\|\tilde{\alpha}_j(k+n-j|k)\|^2 &= O\left(\sum_{k=1}^{t}\sum_{j=1}^{n-1}\|\tilde{\alpha}_j(k+n-j|k)\|^2\right)\\ &= O(\sigma^2 t)+o(\bar{r}_t)+o(t) \end{aligned}\tag{41}$$

where σ^2 is given in Theorem 1.

Therefore, by substituting (36), (37), (40), and (41) into (39), we have

$$\sum_{k=1}^{t}|y(k+n)-y^*(k+n)|^2 = O(\sigma^2 t)+o(\bar{r}_t)+o(t). \tag{42}$$

From (1), (42), and Condition A3), we then have

$$\sum_{k=1}^{t}|x_1(k+n)|^2=\sum_{k=1}^{t}|y(k+n)|^2=O(t)+o(\bar{r}_t). \tag{43}$$

Now, starting from (43) and repeatedly using (1), (2), and (23) we have

$$\sum_{k=1}^{t}|x_j(k+n-j+1)|^2=O(t)+o(\bar{r}_t), \qquad 1\le j\le n$$

which implies that

$$\sum_{k=1}^{t}|x_j(k)|^2=O(t)+o(\bar{r}_t), \qquad 1\le j\le n. \tag{44}$$

Finally, it follows from (4), (33), and (44) that $\bar{r}_t=O(t)+o(\bar{r}_t)$. Therefore,

$$\bar{r}_t=O(t). \tag{45}$$

Hence the desired stability follows from (44), (45), and (25), and (20) follows from (42) and (45). This completes the proof of Theorem 1. □

IV. Concluding Remarks

In this paper, we have studied the adaptive control of a class of discrete-time stochastic nonlinear systems whose nonlinearities satisfy the linear growth condition. (This condition cannot be essentially relaxed in general as recently shown in [13].) The prediction/WLS-based adaptive control is shown to be globally stable in the presence of random noise. Of course, there are many problems which still remain open. For example, 1) it would be of considerable interest to find a simple recursive procedure for calculating the optimal prediction-based adaptive predictors (14) and 2) it is not clear if the WLS algorithm used in the controller design can be replaced by the standard LS. These belong to a further investigation.

References

[1] B. Bercu, "Weighted estimation and tracking for ARMAX models," *SIAM J. Contr. Optimiz.*, vol. 33, no. 1, pp. 89–106, 1995.

[2] L. Guo, "Self-convergence of weighted least-squares with applications to stochastic adaptive control," *IEEE Trans. Automat. Contr.*, vol. 41, pp. 79–89, Jan. 1996.

[3] ——, "On critical stability of discrete-time adaptive nonlinear control," *IEEE Trans. Automat. Contr.*, vol. 42, pp. 1488–1499, Nov. 1997.

[4] L. Guo and C. Wei, "Global stability/instability of LS-based discrete-time adaptive nonlinear control," in *Proc. 13th IFAC World Congr.*, 1996, vol. K, pp. 277–282.

[5] I. Kanellakopoulos, "A discrete-time adaptive nonlinear system," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 2362–2365, Nov. 1994.

[6] M. Krstic, I. Kanellakopoulos, and P. V. Kokotovic, *Nonlinear and Adaptive Control Design.* New York: Wiley, 1995.

[7] P.-C. Yeh and P. V. Kokotovic, "Adaptive control of a class of nonlinear discrete-time systems," *Int. J. Contr.*, vol. 62, no. 2, pp. 303–324, 1995.

[8] I. Kanellakopoulos, P. V. Kokotović, and A. S. Morse, "Systematic design of adaptive controllers for feedback linearizable systems," *IEEE Trans. Automat. Contr.*, vol. 36, pp. 1241–1253, 1991.

[9] R. Marino and P. Tomei, *Nonlinear Control Design.* Englewood Cliffs, NJ: Prentice-Hall, 1995.

[10] S. S. Sastry and A. Isidori, "Adaptive control of linearizable systems," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 1123–1131, Nov. 1989.

[11] G. Didinsky and T. Başar, "Minimax adaptive control of uncertain plants," *ARI*, vol. 50, pp. 3–20, Mar. 1997.

[12] Z. Pan and T. Başar, "Adaptive controller design for tracking and disturbance attenuation in parameter-strict-feedback nonlinear systems," *IEEE Trans. Automat. Contr.*, vol. 41, July 1998.

[13] L. L. Xie and L. Guo, "Fundamental limitations of discrete-time adaptive nonlinear control," *IEEE Trans. Automat. Contr.*, this issue, pp. 1777–1782.

Discrete-Time Approximated Linearization of SISO Systems Under Output Feedback

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***Abstract*— This paper deals with higher order approximation for discrete-time systems. It is shown that approximated feedback linearization at the second order can always be achieved under feedback compensation based on an approximated observer. An example is given in order to illustrate the control design and the efficiency of the proposed method.**

***Index Terms*—Dynamic state feedback, nonlinear discrete-time systems, nonlinear observer, quadratic approximation.**

I. Introduction

The paper deals with output feedback control for achieving an approximated feedback linearization of a given nonlinear single-input/single-output (SISO) discrete-time system. The control scheme is the usual one based on an approximated observer coupled with

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