ON LIMITATIONS OF THE SAMPLED-DATA FEEDBACK FOR NONPARAMETRIC DYNAMICAL SYSTEMS*

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Abstract. In this paper, we study a basic class of first order sampled-data control systems with unknown nonlinear structure and with sampling rate not necessarily fast enough, aiming at understanding the capability and limitations of the sampled-data feedback. We show that if the unknown nonlinear function has a linear growth rate with its "slope" (denoted by L) being a measure of the "size" of uncertainty, then the sampling rate should not exceed $1/L$ multiplied by a constant (≈ 7.53) for the system to be globally stabilizable by the sampled-data feedback. If, however, the unknown nonlinear function has a growth rate faster than linear, and if the system is disturbed by noises modeled as the standard Brownian motion, then an example is given, showing that the corresponding sampled-data system is not stabilizable by the sampled-data feedback in general, no matter how fast the sampling rate is.

Key words. Sampled-data control, adaptive control, uncertain nonlinear systems, Brownian motion, stability.

1 Introduction

Sampled-data control systems are prevalent in practice due to the wide use of digital computers. However, unlike the linear case (see e.g. [1], [2]), there are only a few papers devoted to the theoretical investigation of sampled-data nonlinear control systems (see e.g. [3]–[8]), and most of them are only concerned with the case where the sampling rate is fast enough.

The main difficulties with the theoretical investigation of sampled-data control of nonlinear systems lie not only in the impossibility of obtaining explicit solutions of nonlinear equations, but also in the structure complexity of the closed-loop system—a hybrid system consisting of both continuous-and discrete-time signals. Moreover, just as there is a fundamental difference between adaptive stabilizability of continuous-and discrete-time nonlinear models (see, [9]), a sampled-data nonlinear controller derived from a standard continuous-time stabilizing controller may indeed lose its stability[10], unless the sampling rate is sufficiently fast[3]. However, due

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to physical constraints, sufficiently fast sampling rate is usually not feasible in practice. Thus, a central problem in sampled-data control is how to properly chose the sampling rate, and the most difficult part is to quantitatively understand the capability and limitations of the sampled-data feedback in the case where the sampling period is prescribed.

We will in this paper initiate a study of the above mentioned problem for a typical class of first order nonlinear control systems with uncertain structure. We will treat deterministic and stochastic systems separately, and will prove the following main results:

(i) The proper choice of the sampling rate h is closely related to the "slope" L of the unknown nonlinear function in consideration. To be precise, if h is larger than L^{-1} multiplied by a constant ($\approx$7.53), then there exists no sampled-data control which can globally stabilize the prescribed class of uncertain nonlinear systems; if, however, h is less than L^{-1} multiplied by $\log 4$, then a stabilizing sampled-data feedback for the whole class of uncertain systems can be constructed.

(ii) In the stochastic case where the random noise is described by the standard Brownian motion, the unknown system is globally stabilizable whenever the nonlinear function has a linear growth rate and the sampling rate satisfies $h < 0.15L^{-1}$. If, however, the unknown nonlinear function has a growth rate faster than linear, then we give an example showing that even though the continuous-time stabilizing controller exists, there is no stabilizing sampled-data feedback, no matter how fast the sampling rate is.

In Section 2, we will formulate the problem considered in the paper and present the main results. Sections 3 and 4 are devoted to the proofs of the main theorems in deterministic and stochastic cases respectively. Some concluding remarks will be given in Section 5.

2 The Main Results

2.1 Deterministic Systems

Consider the following basic control system:

$$\dot{x}_t = f(x_t) + u_t, \quad t \geq 0, x_0 \in R^1. \tag{1}$$

The system signals are assumed to be sampled at a constant rate $h > 0$, and the input is assumed to be implemented via the familiar zero-order hold device(piecewise constant function):

$$u_t = u_{kh}, \quad kh \leq t < (k+1)h \tag{2}$$

where u_{kh} depends on $\{x_0, x_h, ..., x_{kh}\}$.

Definition 2.1 $\{u_t, t \geq 0\}$ is called *a sampled-data feedback control* if at each step k, u_{kh} is a causal function of the past and present sampled data $\{x_0, x_h, \cdots, x_{kh}\}$, i.e., there exists a function $g_k(\cdot) : R^{k+1} \to R^1$ such that $u_{kh} = g_k(x_0, x_h, \cdots, x_{kh})$.

The nonlinear function f in (1) is assumed to be unknown but belongs to the following class of functions:

$$G_c^L = \{f | f \text{ is locally Lipschitz and satisfies } |f(x)| \leq L|x| + c, \forall x \in R^1\} \tag{3}$$

where $c > 0$ and $L > 0$ are constants. A function f is called locally Lipschitz if, for any $M > 0$, there exists a constant K such that $|f(x) - f(y)| \leq K|x - y|, \forall(x, y); |x| \leq M, |y| \leq M$.

In the above definition, L is (the upper bound of) the "slope" of the function $f \in G_c^L$, which may be regard as a measure of the size of the uncertainty and plays a crucial role in the determination of the sampling rate h as will be shown by the following theorems.

Theorem 2.1 *Let $b > 0$ be the unique positive solution of the following equation*

$$\frac{1}{b}\left\{\frac{10}{3} + \log\left(\frac{b+2}{b-2}\right) + \log(3 + 2b^{-1})\right\} = \frac{2}{3}. \tag{4}$$

If $Lh > b$, then for any $c > 0$ and any sampled-data control $\{u_{kh}, k \geq 0\}$ there always exists a function $f^* \in G_c^L$, such that the state signal of (1)–(2) corresponding to f^* with initial point $x_0 = 0$ satisfies ($k \geq 1$)

$$|x_{kh}| \geq \left(\frac{Lh}{2}\right)^{k-1} \cdot ch \underset{k\to\infty}{\longrightarrow} \infty.$$

Remark 2.1

a) Theorem 2.1 is a main result of this paper. Since this result is of "negative" character, it is obviously valid also for a more general class of uncertain systems as long as the basic model class (1) is included as a subclass. Also, the value of b determined by (4) can be shown to be approximately 7.53.

b) If in (1) the function $f(\cdot)$ is known *a priori*, then the stabilization problem by sampled-data feedback is trivial. In fact, in this case, we can simply take $u_{kh} = -f(x_0)$, $\forall k \geq 0$, for any given sampling period $h > 0$. This will result in $x_t \equiv x_0$, $\forall t \geq 0$ by the uniqueness of the solution to the closed-loop equation.

c) In the case where $f(\cdot)$ is not known *a priori*, but is still contained in the class G_c^L, it is also a trivial problem to stabilize the system by continuous-time feedback laws, for example, by $u_t = -2Lx_t$. Our Theorem 2.1 shows that for sampled-data feedback laws, however, there exists a limit to the stabilizability of the class of uncertain systems when the sampling period h is larger than b/L. The mechanism causing such a limit may be explained as follows: At any step k, any given sampled-data feedback which is constructed on the basis of the information $\{y_0, y_h, \ldots, y_{kh}\}$ will no longer incorporate new information about the system during the sampling interval $[kh, (k+1)h)$ (in contrast to the continuous-time case where one can continuously use the information flow to adjust the feedback). Therefore, if the system uncertainty measured by L is large (say, larger than b/h), the information contained in $\{y_0, y_h, \ldots, y_{kh}\}$ will turn out to be insufficient for any sampled-data feedback to be able to cope with the class of uncertain systems described by G_c^L.

The following theorem is easy to prove, which shows that once Lh is suitably small, a stabilizing sampled-data feedback can indeed be constructed.

Theorem 2.2 *Let $Lh < \log 4$, $L > 0$ and $c > 0$. Then the following sampled-data control* ($a \triangleq e^{Lh/2}$)

$$u_{kh} = -(c + L|x_{kh}|)\text{sgn}(x_{kh}) - \frac{(2-a)L}{2a} \cdot x_{kh} \tag{5}$$

is globally stabilizing for the system (1) *with any* $f \in G_c^L$. *Moreover,*

$$\lim_{t\to\infty} |x_t| \le \frac{(3a-2)(a^2-1)}{(2-a)a} \cdot \frac{2c}{L}, \quad \forall f \in G_c^L.$$

The proofs of the above two theorems are given in the next section.

2.2 Stochastic Systems

We now consider the following stochastic control system

$$dx_t = f(x_t)dt + u_t dt + \sigma dw_t, \tag{6}$$

where the unknown nonlinear function f belongs to G_c^L defined by (3), and where $\{w_t\}$ is the standard Brownian motion, and $\sigma > 0$.

Theorem 2.3 *For any* $f \in G_c^L$, *let the sampled-data control* (2) *be defined by*

$$u_{kh} = -(1+\lambda)Lx_{kh}, \tag{7}$$

where $\lambda > 0$ *is a constant. If the sampling rate* h *satisfies*

$$Lh < \frac{\lambda}{(1+\lambda)(\sqrt{2}+1+\lambda)}, \tag{8}$$

then the closed-loop system (6)–(7) *is globally stable, i.e. for any* $f \in G_c^L$,

$$\overline{\lim_{t\to\infty}}\, Ex_t^2 < \infty, \quad \forall x_0 \in R^1.$$

Remark 2.2 The right-hand-side(RHS) of (8) takes its maximum value ($\approx$ 0.15) at $\lambda = \sqrt{\sqrt{2}+1}$. We remark that the ideas used in Theorems 2.2 and 2.3 can be extended to higher order systems with no essential difficulty.

In the above, we have constrained ourselves to the case where the nonlinear function has a linear growth rate. A natural question is: Can we find a stabilizing sampled-data control for systems where the unknown nonlinear function has a nonlinear growth rate? The following theorem gives us a negative answer for a class of nonlinear stochastic systems, even in the case where the nonlinear function is known *a priori*, and the sampling period is arbitrarily small.

Theorem 2.4 *Consider the stochastic control system* (6). *Assume that*

(i) u_t *is continuous on* $[kh,(k+1)h), \forall k \ge 0$, *and*

$$|u_t - u_{kh}| \le M, \quad \forall t \in [kh,(k+1)h), \quad \forall k \ge 0,$$

for some constant $M > 0$. *Also,*

$$\sigma\{u_t, t \in [kh,(k+1)h)\} \subseteq \sigma\{w_t, t \le kh\},$$

where $\sigma\{x\}$ *denotes the* σ*-algebra generated by* x.

(ii) *The function* $f(x)$ *is locally Lipschitz and there exist two positive constants* R_0 *and* δ *such that*

$$xf(x) \geq |x|^{2+\delta}, \quad \forall x : |x| \geq R_0.$$

Then for any $h > 0$ *and any feedback control satisfying* (i), *the closed-loop system is unstable in the sense that*

$$Ex_T^2 = \infty, \quad \forall T > 0.$$

Remark 2.3

a) The class of sampled-data control defined by condition (i) includes the standard zero-order hold device (2) as a special case. It also includes other familiar hold devices such as the first order hold device, etc.

b) The condition (ii) in Theorem 2.4 is obviously satisfied if we take $f(x) = |x|^{1+\delta}\text{sgn}(x)$. In this case, it is easy to show that the simple state feedback control $u_t = -|x_t|^{1+\delta}\text{sgn}(x_t) - x_t$ will globally stabilize the system in the sense that $\sup_{T>0} Ex_T^2 < \infty, \forall x_0$. This example in conjunction with Theorem 2.4 demonstrates the fundamental differences between continuous-time control and sampled-data control for stochastic system described by the standard stochastic differential equation, even in the case where the nonlinear function $f(\cdot)$ is known *a priori*.

3 Proofs of Theorems 2.1 and 2.2

First, we introduce a definition.

Definition 3.1 Consider the following two sampled-data control systems:

$$\Sigma_f : \quad \begin{cases} \dot{x} = f(x) + u_t, \quad t \geq 0, \quad x(t_0) = a, \\ u_t = u_{kh}, \quad kh \leq t < (k+1)h; \end{cases} \tag{9}$$

$$\Sigma_g : \quad \begin{cases} \dot{z} = g(z) + u_t, \quad t \geq 0, \quad z(t_0) = a, \\ u_t = u_{kh}, \quad kh \leq t < (k+1)h. \end{cases} \tag{10}$$

Under the same sampled-data control sequence $\{u_t\}$, the above two systems Σ_f and Σ_g are called N-step equivalent starting from the same initial point $a \in R^1$, if the sampled signals or observations of the two systems are equal, i.e., $x_{t_0+kh} = z_{t_0+kh}$, $k = 0, 1, \cdots N$.

Such an equivalent relationship will be denoted by

$$\Sigma_f \overset{a}{\underset{N}{\Longleftrightarrow}} \Sigma_g, \quad \text{s.t.}\{u_t\}.$$

If $N = 1$, we will simply use the notation $\Sigma_f \overset{a}{\Longleftrightarrow} \Sigma_g$, s.t. u to denote that $x(t_0 + h) = z(t_0 + h)$ when $x(t_0) = z(t_0) = a$ and $u_t \equiv u, t_0 \leq t < t_0 + h$.

Now, we proceed to present the proof of Theorem 2.1. This proof is prefaced with the following four lemmas whose proofs are given in Appendix A.

Lemma 3.1 *Consider the one dimensional autonomous system:*

$$\begin{cases} \dot{x} = \phi(x), \quad t \geq 0 \\ x(0) = x_0 \end{cases} \tag{11}$$

where $\phi(\cdot)$ *is locally Lipschitz. Then*

(i) *The trajectory* $x(t)$ *is a monotonous function of* t;

(ii) *For any* $T > 0$, *and* $x_T \neq x_0$, *the necessary and sufficient condition for* $x(T) = x_T$ *is* $\int_{x_0}^{x_T} \frac{dx}{\phi(x)} = T$ *together with* $\phi(x) \neq 0$ *on* $[\min(x_T, x_0), \max(x_T, x_0)]$.

Remark 3.1 The state signal $x(t)$ defined by (1)–(2) is monotonous in any fixed sampling interval. With the help of (ii), we can calculate how much time it will take for the state $x(t)$ to travel from one point to another or how far $x(t)$ will travel in a certain time period. The first part (i) of Lemma 3.1 can be found in [10], but for the sake of easy reference, we still give a simple proof here in Appendix A.

Lemma 3.2 *Let the function* $\hat{g} \in G_c^L$ *satisfy* $\hat{g}(z) \equiv L|z_0| + c$, *for* $z \geq |z_0|$, *such that the state signal of the system*

$$\Sigma_{\hat{g}} : \begin{cases} \dot{z} = \hat{g}(z) + u_0, & t \geq 0 \\ z(0) = z_0 \end{cases} \tag{12}$$

satisfies $z(1) = z_1 > |z_0| > 0$. *Then there exists a function* $g_1 \in G_c^L$ *satisfying* $g_1(z_1) = Lz_1 + c$, *and* $g_1[z_0, |z_0|] = \hat{g}$, $g_1[|z_0|, z_1] \geq 0$, *such that the state signal of the following system:*

$$\Sigma_{g_1} : \begin{cases} \dot{x} = g_1(x) + u_0, & t \geq 0 \\ x(0) = z_0 \end{cases} \tag{13}$$

satisfies $x(1) = z_1$, *where by definition* $f_1[\alpha, \beta] = f_2$ *means* $f_1(x) = f_2(x)$, $\forall x \in [\min(\alpha, \beta), \max(\alpha, \beta)]$.

Remark 3.2 Lemma 3.2 shows that although $\sum_{\hat{g}}$ and $\sum_{g_1}$ are one-step equivalent by Definition 3.1, the terminal values $\hat{g}(z_1)$ and $g_1(z_1)$ can be quite different. This key fact makes it possible for us to construct the nonstabilizable system in the proof of Theorem 2.1 later on.

Like Lemma 3.2, we have the following "adjoint" lemma.

Lemma 3.2′ *Let the function* $\hat{g} \in G_c^L$ *satisfy* $\hat{g}(z) \equiv -L|z_0| - c, z \leq -|z_0|$ *such that the state signal of the system:*

$$\begin{cases} \dot{z} = \hat{g}(z) + u_0, & t \geq 0 \\ z(0) = z_0 \end{cases} \tag{14}$$

satisfies $z(1) = z_1 < -|z_0| < 0$. *Then there exists a function* $g_1 \in G_c^L$ *satisfying* $g_1(z_1) = Lz_1 - c$, *and* $g_1[-|z_0|, z_0] = \hat{g}$, $g_1[z_1, -|z_0|] \leq 0$, *such that the state signal of the following system:*

$$\begin{cases} \dot{x} = g_1(x) + u_0, & t \geq 0 \\ x(0) = z_0 \end{cases} \tag{15}$$

satisfies $x(1) = z_1$.

Lemma 3.3 *If we explicitly denote the system* (1)–(2) *as* $Sys(f, x_0, h, \{u_{kh}\})$, *then for any positive constant* λ, *there is a "linear time-transforming" relationship between the state signal* $x(t)$ *of the system* (1)–(2) *and the state signal* $z(t)$ *of the system* $Sys(\lambda f, x_0, \frac{1}{\lambda} h, \{\lambda u_{kh}\})$, i.e.,

$$z(t) = x(\lambda t), \quad \forall t \geq 0.$$

This lemma will make it possible to transfer the general sampling rate case $h > 0$ to the special case $h = 1$ in the proofs of the main theorem to be given in the sequel.

On The basis of the above lemmas, we are now in a position to present the detailed proof of Theorem 2.1. The key idea behind the proof is as follows: Given any sampled-data feedback $\{u_{kh}\}$, we try to find a "worst case" function $f^* \in G_c^L$ such that the corresponding system is not stabilized.

Proof of Theorem 2.1

We first consider the case where $h = 1$.

It is easy to verify that the left-hand-side of (4) decreases with b when b is greater than 2. Since $L > b_0 > 2$, where b_0 is the solution of (4), we can select a constant $\delta \in (0,1)$ such that

$$\frac{1}{L}\Big(\frac{10}{3} + \log\frac{L+2}{L-2} + \log(3 + 2L^{-1}) + \frac{4\delta}{2+L}\Big) \le \frac{2}{3}. \tag{16}$$

We will construct the function f^* step by step to deal with the possible control effects of any given feedback sequences $\{u_{kh}\}$.

Throughout the rest of this paper, we use $f[\alpha,\beta)$ to represent the function f on the interval $[\alpha,\beta)$, and $f_1[\alpha_1,\beta_1]\bigoplus f_2[\alpha_2,\beta_2]$ to denote a function f satisfying

$$f(x) = \begin{cases} f_1(x), & x \in [\alpha_1,\beta_1], \\ f_2(x), & x \in [\alpha_2,\beta_2], \end{cases}$$

where we assume $[\alpha_1,\beta_1]\bigcap[\alpha_2,\beta_2] = \phi$.

The remaining proofs are divided into three steps.

Step 1 $t = 0$.

Given the initial input u_0 and $x_0 = 0$, we consider two cases seperately.

Case (i) $u_0 \ge 0$.

Denote $a_1^+ = u_0 + c > c$ and define f^* on $[-a_1^+, 0]$ to be

$$f^*[-a_1^+,0] = \begin{cases} (L + 2\delta^{-1})x + c, & x \in [-\delta c, 0]; \\ Lx - c, & x \in [-a_1^+, -\delta c). \end{cases} \tag{17}$$

On the interval $[0, a_1^+]$, let us define $g_0^+[0,a_1^+] \equiv c$. Then it is easy to verify that the system $\Sigma_{g_0^+}: \ \dot{x} = g_0^+(x) + u_0, \ \ t \ge 0, x_0 = 0$ satisfies $x(1) = a_1^+$. By Lemma 3.2 with $z_0 = 0$ and $z_1 = a_1^+$, we know that there exists a $\phi_0^+ \in G_c^L$ satisfying $\Sigma_{\phi_0^+} \overset{0}{\Longleftrightarrow} \Sigma_{g_0^+}$, s.t. u_0, $\phi_0^+(a_1^+) = La_1^+ + c$ and $\phi_0^+[0,a_1^+] \ge 0$.

Let G_0^+ be the set consisting of only two functions defined on $[-a_1, a_1]$, i.e.,

$$G_0^+ \triangleq \big\{\ f^*[-a_1^+,0]\bigoplus g_0^+(0,a_1^+],\ f^*[-a_1^+,0]\bigoplus \phi_0^+(0,a_1^+]\big\}\ \subseteq G_c^L,$$

where and hereafter $G_0^+ \subseteq G_c^L$ simply signifies that any function f in G_0^+ is locally Lipschitz and satisfies $|f(x)| \le L|x| + c$ on its defined interval.

Then the state $x(t)$ of the uncertain system (1)–(2) may be produced by a system corresponding to any function in G_0^+. But it is easily seen that for whichever function in G_0^+, we will always get $x_1 = a_1^+$ under u_0.

Case (ii) $u_0 < 0$.

Denote $a_1^- = -u_0 + c \geq c$ and define f^* on $[0, a_1^-]$ to be

$$f^*[0,a_1^-] = \begin{cases} (L+2\delta^{-1})x - c, & x \in [0, \delta c]; \\ Lx + c, & x \in [\delta c, a_1^-). \end{cases} \tag{18}$$

On the interval $[-a_1^-, 0)$ let us define $g_0^-[-a_1^-, 0) \equiv -c$. Then it is easy to verify that the system $\Sigma_{g_0^-} : \dot{x} = g_0^-(x) + u_0, t \geq 0, x_0 = 0$ satisfies $x(1) = -a_1^-$. By Lemma 3.2' with $z_0 = 0$ and $z_1 = -a_1^-$, there exists a $\phi_0^- \in G_c^L$ satisfying: $\Sigma_{\phi_0^-} \overset{0}{\Longleftrightarrow} \Sigma_{g_0^-}$, s.t. u_0, $\phi_0^-(-a_1^-) = -La_1^- - c$, and $\phi_0^-[-a_1^-, 0] \leq 0$.

Similarly to the previous case, let us denote

$$G_0^- \triangleq \{\ g_0^-[-a_1^-, 0) \bigoplus f^*[0, a_1^-],\ \phi_0^-[-a_1^+, 0) \bigoplus f^*[0, a_1^-]\}\ \subseteq G_c^L.$$

Then the state $x(t)$ of the uncertain system (1)–(2) has the possibility to evolve as the state of a system corresponding to any function in G_0^-. But we can see that for whichever function in G_0^-, we are bound to have $x_1 = -a_1^-$ under u_0.

Step 2 $t = 1$.

We are now given the control u_1 and the observation x_1.

The following discussion is divided into four cases according to the values of (x_1, u_1).

Case (i)

$$x_1 > 0, \quad u_1 \geq -(Lx_1 + c) + \frac{L}{2}(|x_1| - |x_0|). \tag{19}$$

In view of (19), we define $f^*[0, x_1] = \phi_0^+$, where ϕ_0^+ is defined in Case (i) of the previous step. And consequently, we have $f^*(x_1) = Lx_1 + c$.

Next, denote

$$a_2^{++} \triangleq x_1 + (u_1 + Lx_1 + c) > x_1, \tag{20}$$

and extend the function f^* already defined on $[-x_1, x_1]$ to $[-a_2^{++}, -x_1)$ as

$$f^*[-a_2^{++}, -x_1) = \begin{cases} \dfrac{Lx_1 + f^*(-x_1) + (1+\delta L)c}{\delta c}(x + x_1) + f^*(-x_1), \\ \qquad x \in [-x_1 - \delta c, -x_1); \\ Lx - c, \quad x \in [-a_2^{++}, -x_1 - \delta c). \end{cases} \tag{21}$$

On the interval $[x_1, a_2^{++}]$, we define a function $g_1^{++}[x_1, a_2^{++}] \equiv Lx_1 + c$. Then it is easy to verify that the system $\Sigma_{g_1^{++}} : \dot{x} = g_1^{++}(x) + u_1, t \geq 1$ travels from $x(1) = x_1$ to $x(2) = a_2^{++}$.

By Lemma 3.2 with $z_0 = x_1$ and $z_1 = a_2^{++}$, there exists a $\phi_1^{++} \in G_c^L$ satisfying $\Sigma_{\phi_1^{++}} \overset{x_1}{\Longleftrightarrow} \Sigma_{g_1^{++}}$, s.t. u_1, $\phi_1^{++}(a_2^{++}) = La_2^{++} + c$ and $\phi_1^{++}[x_1, a_2^{++}] \geq 0$.

Now, denote

$$G_1^{++} \triangleq \{\ f^*[-a_2^{++}, x_1] \bigoplus g_1^{++}(x_1, a_2^{++}],\ f^*[-a_2^{++}, x_1] \bigoplus \phi_1^{++}(x_1, a_2^{++}]\}\ \subseteq G_c^L.$$

It is clear that the state $x(t)$ of the uncertain system (1)–(2) may be produced by a system corresponding to any function in G_1^{++}. Obviously, for $f_1, f_2 \in G_1^{++}$, we have $f_1 \underset{2}{\overset{0}{\Longleftrightarrow}} f_2$, s.t. $\{u_0, u_1\}$. In particular, for whichever function in G_1^{++}, we will always have $x_2 = a_2^{++}$ under u_1.

Hence, it follows from (19)–(20) that $|x_2| \geq \frac{L}{2}|x_1|$.

Case (ii)

$$x_1 > 0, \quad u_1 < -(Lx_1 + c) + \frac{L}{2}(|x_1| - |x_0|). \tag{22}$$

In view of (22), we define $f^*[0, x_1] = g_0^+$, and hence we have $f^*[0, x_1] \equiv c \equiv Lx_0 + c$.

Let $g_1^{+-}(-\infty, -x_1) \equiv f^*(-x_1)$ and $\hat{f}_1^+(-\infty, x_1] \triangleq g_1^{+-}(-\infty, -x_1) \oplus f^*[-x_1, x_1]$.

It can be shown that the system: $\dot{z} = \hat{f}_1^+(z) + u_1, t \geq 1, z(1) = x_1$ satisfies

$$z(2) \leq -\frac{L}{2}x_1. \tag{23}$$

The proof of this inequality is put in Appendix A for simplicity of presentation.

Next, we denote $a_2^{+-} \triangleq -z(2) > x_1$.

By Lemma 3.2$'$ with $z_0 = x_1$ and $z_1 = -a_2^{+-}$, there exists a $\phi_1^{+-} \in G_c^L$ satisfying: $\Sigma_{\phi_1^{+-}} \overset{x_1}{\Longleftrightarrow} \Sigma_{\hat{f}_1^+(-\infty, x_1]}$, s.t. u_1; $\phi_1^{+-}[-x_1, x_1] = f^*[-x_1, x_1]$, $\phi_1^{+-}(-a_2^{+-}) = -La_2^{+-} - c$, and $\phi_1^{+-}[-a_2^{+-}, -x_1] \leq 0$.

Let

$$f^*(x_1, a_2^{+-}] = \begin{cases} \dfrac{Lx_1 - f^*(x_1) + (1 + \delta L)c}{\delta c}(x - x_1) + f^*(x_1), & x \in (x_1, x_1 + \delta c]; \\ Lx + c, & x \in (x_1 + \delta c, a_2^{+-}]. \end{cases} \tag{24}$$

and denote

$$G_1^{+-} \triangleq \left\{ g_1^{+-}[-a_2^{+-}, -x_1) \bigoplus f^*[-x_1, a_2^{+-}],\ \phi_1^{+-}[-a_2^{+-}, -x_1) \bigoplus f^*[-x_1, a_2^{+-}] \right\} \subseteq G_c^L.$$

Then the state $x(t)$ of the uncertain system (1)–(2) may be produced by a system corresponding to any function in G_1^{+-}. But it can be easily seen that, for whichever function in G_1^{+-}, we will be bound to get $x_2 = -a_2^{+-}$ under u_1.

Obviously, we get from (23) that $|x_2| \geq \frac{L}{2}|x_1|$.

Case (iii)

$$x_1 < 0, \quad u_1 \geq -(Lx_1 - c) - \frac{L}{2}(|x_1| - |x_0|). \tag{25}$$

The conditions in this case are "symmetric" to those in Case (ii), so the proof ideas are similar.

In view of (25), we define $f^*[x_1, 0] = g_0^-$, where g_0^- is defined in Case (ii) of Step 0. And hence we get $f^*[x_1, 0] \equiv Lx_0 - c$.

Let $g_1^{-+}(-x_1, \infty) \equiv f^*(-x_1)$, and let $\hat{f}_1^-[x_1, , \infty) \triangleq f^*[x_1, -x_1] \oplus g_1^{-+}(-x_1, \infty)$.

Similarly to Case (ii), it is easy to show that the system $\dot{z} = \hat{f}_1^-(z) + u_1, t \geq 1, z(1) = x_1$ satisfies

$$z(2) \geq -\frac{L}{2}x_1. \tag{26}$$

Next, we denote $a_2^{-+} \triangleq z(2) > 0$.

By Lemma 3.2 with $z_0 = x_1$ and $z_1 = a_2^{-+}$, there exists a $\phi_1^{-+} \in G_c^L$ satisfying: $\Sigma_{\phi_1^{-+}} \overset{x_1}{\Longleftrightarrow} \Sigma_{\hat{f}_1^-[x_1, \infty)}$, s.t. u_1; $\phi_1^{-+}[x_1, -x_1] = f^*[x_1, -x_1]$, $\phi_1^{-+}(a_2^{-+}) = La_2^{-+} + c$, and $\phi_1^{-+}[-x_1, a_2^{-+}] \geq 0$.

Let

$$f^*[-a_2^{-+}, x_1) = \begin{cases} \dfrac{-Lx_1 + f^*(x_1) + (1 + \delta L)c}{\delta c}(x - x_1) + f^*(x_1), & x \in [x_1 - \delta c, x_1); \\ Lx - c, & x \in [-a_2^{-+}, x_1 - \delta c), \end{cases} \tag{27}$$

and denote

$$G_1^{-+} \triangleq \{\ f^*[-a_2^{-+}, -x_1] \bigoplus g_1^{-+}(-x_1, a_2^{-+}], f^*[-a_2^{-+}, -x_1] \bigoplus \phi_1^{-+}(-x_1, a_2^{-+}]\} \ \subseteq G_c^L.$$

Then the state $x(t)$ of the uncertain system (1)–(2) may be produced by a system corresponding to any function in G_1^{-+}. But it can be easily seen that, for whichever function in G_1^{-+}, we will get $x(2) = a_2^{-+}$ under u_1.

Obviously, it follows from (26) that $|x_2| \geq \frac{L}{2}|x_1|$.

Case (iv)

$$x_1 < 0, \quad u_1 < -(Lx_1 - c) - \frac{L}{2}(|x_1| - |x_0|). \tag{28}$$

In view of (28), we define $f^*[x_1, 0] = \phi_0^-$, and get $f^*(x_1) = Lx_1 - c$.

Next, denote

$$a_2^{--} \triangleq -(x_1 + (u_1 + Lx_1 - c)) > -x_1, \tag{29}$$

and extend the definition of f^* to $(-x_1, a_2^{--}]$ as

$$f^*(-x_1, a_2^{--}] = \begin{cases} \dfrac{-Lx_1 - f^*(-x_1) + (1+\delta L)c}{\delta c}(x + x_1) + f^*(-x_1), \\ \qquad x \in (-x_1, -x_1 + \delta c]; \\ Lx + c, \quad x \in (-x_1 + \delta c, a_2^{--}]. \end{cases} \tag{30}$$

On the interval $[-a_2^{--}, x_1]$, we define a function $g_1^{--}[-a_2^{--}, x_1] \equiv Lx_1 - c$. Then it is easy to verify that the system $\Sigma_{g_1^{--}} : \dot{x} = g_1^{--}(x) + u_1, t \geq 1$ travels from $x(1) = x_1$ to $x(2) = -a_2^{--}$.

By Lemma 3.2$'$ with $z_0 = x_1$ and $z_1 = -a_2^{--}$, there exists a $\phi_1^{--} \in G_c^L$ satisfying: $\Sigma_{\phi_1^{--}} \overset{x_1}{\Longleftrightarrow} \Sigma_{g_1^{--}}$, s.t. u_1, $\phi_1^{--}(-a_2^{--}) = -La_2^{--} - c$, and $\phi_1^{--}[-a_2^{--}, x_1] \leq 0$.

Now, denote

$$G_1^{--} \triangleq \{\ g_1^{--}[-a_2^{--}, x_1) \bigoplus f^*[x_1, a_2^{--}],\ \phi_1^{--}[-a_2^{--}, x_1) \bigoplus f^*[x_1, a_2^{--}]\} \ \subseteq G_c^L.$$

Then the state $x(t)$ of the uncertain system (1)–(2) may be produced by a system corresponding to any function in G_1^{--}(obviously $\forall f_1, f_2 \in G_1^{--}$, we have $f_1 \underset{2}{\overset{0}{\Longleftrightarrow}} f_2$, s.t. $\{u_0, u_1\}$). But it is easy to see that, for whichever function in G_1^{--}, we will get $x_2 = -a_2^{--}$ under u_1.

Obviously, we get from (28)–(29) that $|x_2| \geq \frac{L}{2}|x_1|$.

To proceed further, we denote

$$g_1^+ \triangleq \begin{cases} g_1^{++}, \text{ in Case (i);} \\ g_1^{-+}, \text{ in Case (iii).} \end{cases} \qquad g_1^- \triangleq \begin{cases} g_1^{+-}, \text{ in Case (ii);} \\ g_1^{--}, \text{ in Case (iv).} \end{cases}$$

$$\phi_1^+ \triangleq \begin{cases} \phi_1^{++}, \text{ in Case (i);} \\ \phi_1^{-+}, \text{ in Case (iii).} \end{cases} \qquad \phi_1^- \triangleq \begin{cases} \phi_1^{+-}, \text{ in Case (ii);} \\ \phi_1^{--}, \text{ in Case (iv).} \end{cases}$$

Step 3 $t = k$.

We now use the induction argument. Suppose that at some time k, for the given feedback sequence $\{u_0, u_1, \cdots, u_k\}$ we have found a trajectory $\{x_1, x_2, \cdots, x_{k+1}\}$ together with the corresponding nonlinear system or function f^*, which have the following properties:

a) $|x_{k+1}| \geq \frac{L}{2}|x_k|$, $|x_1| \geq c$.

b) If $x_{k+1} > 0$, then f^* is defined on the interval $[-|x_{k+1}|, |x_k|]$, together with g_k^+ and ϕ_k^+ defined on $(|x_k|, x_{k+1}]$, such that

$$\Sigma_{f^*[x_k,|x_k|] \bigoplus g_k^+(|x_k|, x_{k+1}]} \overset{0}{\underset{k+1}{\Longleftrightarrow}} \Sigma_{f^*[x_k,|x_k|] \bigoplus \phi_k^+(|x_k|, x_{k+1}]},$$

$$\text{s.t.} \quad \{u_t, t = 0, 1, \cdots, k\}$$

c) If $x_{k+1} < 0$, then f^* is defined on the interval $[-|x_k|, |x_{k+1}|]$, together with g_k^- and ϕ_k^- defined on $[x_{k+1}, -|x_k|)$, such that

$$\Sigma_{g_k^+[x_{k+1}, -|x_k|) \bigoplus f^*[-|x_k|, x_k]} \overset{0}{\underset{k+1}{\Longleftrightarrow}} \Sigma_{\phi_k^+[x_{k+1}, -|x_k|) \bigoplus f^*[-|x_k|, x_k]},$$

$$\text{s.t.} \quad \{u_t, t = 0, 1, \cdots, k\}.$$

Our objective now is to show that for any given u_{k+1} at time $k+1$, the above properties can also be made true with k replaced by $k+1$. Since the proof ideas are similar to those used as above, we put the proof in Appendix B.

Therefore, according to the induction principle, for any given feedback sequence $\{u_i, i \geq 0\}$ we can define a nonlinear function $f^* \in G_c^L$ such that the corresponding closed-loop system with initial point $x_0 = 0$ is unstable in the sense that $|x_k| \geq (\frac{L}{2})^{k-1} \cdot c$.

Hence the proof of Theorem 2.1 is completed for the case $h = 1$. The general $h > 0$ case can be easily proven by applying Lemma 3.3 as follows.

By Lemma 3.3, the stabilizability of $Sys(f, x_0 = 0, h, \{\hat{u}_{kh}\})$ is equivalent to that of $Sys(hf, x_0 = 0, 1, \{h\hat{u}_{kh}\})$. If $hL > b$ where b is defined by (4), then according to the results established above, there exists a function hf^* in G_{hc}^{hL} which makes the state of $Sys(hf^*, 0, 1, \{h\hat{u}_{kh}\})$ satisfy

$$|z(k)| \geq \left(\frac{Lh}{2}\right)^{k-1} \cdot ch, \quad k = 1, 2, \cdots.$$

Thus by Lemma 3.3, f^* is the desired function such that the state of $Sys(f, x_0 = 0, h, \{\hat{u}_{kh}\})$ satisfies

$$|x(kh)| \geq \left(\frac{Lh}{2}\right)^{k-1} \cdot ch, \quad k = 1, 2, \cdots,$$

and hence the proof is completed. ▮

Before presenting the proof of Theorem 2.2, we first introduce the following lemma whose proof is placed in Appendix A.

Lemma 3.4 *Let the system* $\dot{x} = g(x) + u_0, x(0) = x_0$ *satisfy*

(i) $g \in G_c^L, c \geq 0$ *and* $0 < L < \log 4$;

(ii) $|x_0| > 2 \cdot \frac{2c}{L} \cdot \frac{a-1}{2-a}$, *where* $a \stackrel{\triangle}{=} e^{\frac{L}{2}}$.

If $u_0 = -(c + L|x_0|)\text{sgn}(x_0) - \frac{2-a}{2a}Lx_0$, *then we have* $|x(1)| \leq \mu \cdot |x_0|$, *where* $\mu \in (0, 1)$ *is a constant.*

Proof of Theorem 2.2

We first consider the case where $h = 1$.

By Lemma 3.4 , if the initial point x_0 satisfies $|x_0| > 2 \cdot \frac{2c}{L} \cdot \frac{a-1}{2-a}$, and the system is under the control (5), then there must exist a finite number N such that

$$|x_N| \leq 2 \cdot \frac{2c}{L} \cdot \frac{a-1}{2-a}. \tag{31}$$

Without loss of generality, we assume $x_N \geq 0$.

Since $g(x_N) + u_N < 0$, and $x(t)$ is monotonous in $t \in [N, N+1]$, it follows from Lemma 3.1 that $x_{N+1} < x_N$. The conclusion is obviously true if $x_{N+1} \geq 0$, we hence assume $x_{N+1} < 0$ in the remainder of the proof.

Taking into account of the structure of u_N, we have

$$u_N \geq -c - L\left[1 + \frac{(2-a)}{2a}\right] \cdot \frac{4c}{L} \cdot \frac{a-1}{2-a} = -\frac{a^2+4a-4}{a(2-a)} \cdot c. \tag{32}$$

We proceed to show that

$$x_{N+1} \geq -\frac{(3a-2)(a^2-1)}{a(2-a)} \cdot \frac{2c}{L} \triangleq -b.$$

By Lemma 3.1, it suffices to show

$$t \triangleq \int_0^{-b} \frac{dx}{g(x) + u_N} \geq 1. \tag{33}$$

Now, by (32) and the constraint that $g(x) \geq Lx - c, x \leq 0$, we have

$$t \geq \int_0^{-b} \frac{dx}{-c + Lx - \dfrac{a^2+4a-4}{a(2-a)} \cdot c} = 1.$$

Hence, (33) is true. Consequently, since $x(t)$ is monotonous on $[N, N+1]$, we get

$$|x(t)| \leq \max\left(2 \cdot \frac{2c}{L} \cdot \frac{a-1}{2-a}, b\right) = b, \quad t \in (N, N+1]. \tag{34}$$

Now, we proceed to show that the above inequality is also true for $t \in [N+1, N+2]$.

First, if $|x_{N+1}| > 2 \cdot \frac{2c}{L} \cdot \frac{a-1}{2-a}$, then by Lemma 3.4 and the monotonicity of the trajectory, we see that $|x_t| \leq |x_{N+1}|$, $t \in [N+1, N+2]$. On the other hand, if $|x_{N+1}| \leq 2 \cdot \frac{2c}{L} \cdot \frac{a-1}{2-a}$, then applying the same argument as that used above for x_N and starting from (31), we see that (34) is also valid on $t \in [N+1, N+2]$.

Continuing this argument, we see that $|x(t)|$ is bounded by the same upper bound as in (34) when $t \geq N+1$. So the proof is completed for the case $h = 1$.

In the general case, we know from Lemma 3.3 that there is an explicit relationship between the states of $Sys(f, x_0, h, \{u_{kh}\})$ and $Sys(hf, x_0, 1, \{hu_{kh}\})$.

Suppose that $f \in G_c^L$ and $hL < \log 4$. Then we can apply the result just established for the case $h = 1$ to $Sys(hf, x_0, 1, \{hu_{kh}\})$. Moreover, since the controller is constructed by just replacing the constants L and c in (5) by hL and hc, therefore the desired stability result for $Sys(hf, x_0, 1, \{hu_{kh}\})$ is true, and the upper bound for the state of $Sys(f, x_0, h, \{u_{kh}\})$ is also valid. Hence, the proof is completed. ∎

4 Proofs of Theorems 2.3 and 2.4

The proof of Theorem 2.3 is prefaced with the following lemma whose proof is given in Appendix C.

Lemma 4.1 *Let us denote* $e_t \triangleq Ex_t^2$. *Then for the system* (6)–(7), *we have for any* $t \in [kh, (k+1)h)$,

$$\begin{aligned}\frac{de_t}{dt} \le\ & 2c \cdot \sqrt{e_t} + 2Le_t - 2L \cdot (1+\lambda)e_{kh} + 2L^2(1+\lambda)^2 e_{kh} \cdot (t-kh) \\ & +2L(1+\lambda)\sqrt{e_{kh}} \cdot \int_{kh}^{t} \sqrt{2c^2 + 2L^2 e_s}\, ds + \sigma^2.\end{aligned} \tag{35}$$

Proof of Theorem 2.3

(i) We first prove that there exists a constant $M_1 > 0$ such that if at some kh, the state of the system (6)–(7) satisfies $Ex_{kh}^2 \ge M_1$, then

$$\frac{de_t}{dt} < 0, \quad t \in [kh, (k+1)h). \tag{36}$$

Now, by (8), we know that $\delta \triangleq \lambda - Lh(1+\lambda)(\sqrt{2}+1+\lambda) > 0$. Hence, we have

$$\begin{aligned}\psi(e_{kh}) \triangleq\ & 2c\sqrt{e_{kh}} + 2Le_{kh} - 2L(1+\lambda)e_{kh} + 2L^2(1+\lambda)^2 e_{kh} \cdot h \\ & +2L(1+\lambda)\sqrt{e_{kh}} \cdot \int_{kh}^{(k+1)h} \sqrt{2c^2 + 2L^2 e_{kh}}\, ds + \sigma^2 \\ =\ & 2e_{kh}L\Big[-\lambda + L(1+\lambda)(1+\sqrt{2}+\lambda)h + O\Big(\frac{1}{\sqrt{e_{kh}}}\Big)\Big] \\ =\ & 2e_{kh}L\Big[-\delta + O\Big(\frac{1}{\sqrt{e_{kh}}}\Big)\Big] \quad \longrightarrow -\infty, \quad \text{as} \quad e_{kh} \longrightarrow \infty.\end{aligned}$$

So, there must exist a constant M_1 such that $\psi(e_{kh}) < 0, \forall e_{kh} \ge M_1$. Hence, when $e_{kh} \ge M_1$, it is easy to see from Lemma 4.1 that $\frac{de_t}{dt}|_{t=kh} < 0$.

If (36) were not true, then there would exist a $t_1 < (k+1)h$ such that $\frac{de_t}{dt}|_{t=t_1} \ge 0$. Let us denote $t^* \triangleq \inf\{t \ge kh : \frac{de_t}{dt} = 0\}$.

Now, since $\frac{de_t}{dt}$ is continuous on $[kh, kh+h)$, we have $kh < t^* \le t_1$, $\frac{de_t}{dt}|_{t^*} = 0$, and $\frac{de_t}{dt} < 0, \forall t < t^*$. Therefore, we have $e_s \le e_{kh}$, $s \in [kh, t^*]$.

Let $h^* \triangleq t^* - kh < h$. By Lemma 4.1, we see that

$$\begin{aligned}\frac{de_t}{dt}|_{t=t^*} \le\ & 2c\sqrt{e_{kh}} + 2Le_{kh} - 2L(1+\lambda)e_{kh} + 2L^2(1+\lambda)^2 e_{kh} h^* \\ & +2L(1+\lambda)\sqrt{e_{kh}} \cdot \int_{kh}^{kh+h^*} \sqrt{2c^2 + 2L^2 e_{kh}}\, ds + \sigma^2 \\ \le\ & \psi(e_{kh}) < 0, \quad \text{for} \quad e_{kh} \ge M_1.\end{aligned}$$

This contradicts $\frac{de_t}{dt}|_{t^*} = 0$. Hence, (36) is valid.

(ii) Next, we prove that if for some $k \ge 0$, $Ex_{kh}^2 \le M_1$, then there exists a positive constant M_2 which depends only on M_1, h, c, L, and λ such that the state of the system (6)–(7) satisfies

$$Ex_t^2 \le M_2, \quad t \in [kh, (k+1)h). \tag{37}$$

By Lemma 4.1 and the elementary inequality $\sqrt{x} \leq (x+1)/2, x \geq 0$, we know there exist positive constants a_1, a_2, a_3, a_4 which depend only on c, L, h and λ such that

$$\frac{de_t}{dt} \leq a_1 + a_2 e_t + a_3(t-kh) + a_4 \int_{kh}^t e_s ds, \quad t \in [kh, (k+1)h).$$

Let z_t be the solution of the following equation:

$$\begin{cases} \dfrac{dz_t}{dt} = a_1 + a_2 z_t + a_3(t-kh) + a_4 \displaystyle\int_{kh}^t z_s ds, \\ z_{kh} = e_{kh}. \end{cases} \tag{38}$$

Then, by the comparison principle for differential equations, we have $z_t \geq e_t$.

Since (38) is a linear ordinary differential equation, there must exist a constant M_2 such that $Ex_t^2 \leq z_t \leq M_2, \quad t \in [kh, (k+1)h)$. Hence, (37) is true.

Finally, combining (36) and (37) we get $\max_{t \geq 0} Ex_t^2 \leq \max\{Ex_0^2, M_2\}$. ∎

To prove Theorem 4, we need the following property on the standard Brownian motion[11].

Property 1 *Let w_t be the standard Brownian motion, and let η be a Markov time defined by*

$$\eta \triangleq \inf\{t \geq 0 : w_t = -a + bt\},$$

where $a > 0$, $0 \leq b < \infty$. Then the probability density of η is $p_\eta(t) = \frac{a}{\sqrt{2\pi} t^{3/2}} \exp\{-(bt-a)^2/2t\}$.

By Property 1, it is clear that the following lemma is true.

Lemma 4.2 *For any T and $c_1 > 0$, we have*

$$P\{\sigma w_t > c_1 t - 1, \quad \forall t \in [0, T]\} > 0,$$

where w_t is the standard Brownian motion.

The proofs of the following two lemmas are presented in Appendix C.

Lemma 4.3 *Let a function $f(x) \in C^1(R^1)$ satisfy $f'(x) > 0, \forall x > a-1$, and $f(a-1)+b > 0$, where $a, b \in R^1$ are two constants. Also, let $x(t)$ and $y(t)$ be two continuous functions of t, which satisfy*

$$\begin{cases} x(t) \geq a + \displaystyle\int_0^t f(x_s)ds + bt - 1, \quad t \geq 0; \\ x(0) = a; \end{cases}$$

and

$$y(t) = a + \int_0^t f(y_s)ds + bt - 1.$$

Then

$$x(t) > y(t) \geq a - 1, \quad \forall t \geq 0.$$

Lemma 4.4 *For any positive constants T, ν, g_0 and any $x_0 \in R^1$, there exists a constant $b > 0$ such that the solution of the following integral equation*

$$z_t = x_0 + \int_0^t (|z_s - x_0 + 1|^{1+\nu} - g_0)ds + bt - 1 \tag{39}$$

blows up at time T, i.e., $z(T) = \infty$.

Proof of Theorem 2.4

To prove the theorem, we need only to show that:

$$P\{x_T = \infty\} > 0, \quad \forall 0 < T < h. \tag{40}$$

Let the initial point be x_0. By Assumption (ii), we see that $f(x) \geq x^{1+\delta}, x \geq R_0$. Hence, there must exist constants $\nu \in (0, \delta)$ and $g_0 > 0$ such that there is a locally Lipschitz function g^* satisfying $g(x) > g^*(x), \forall x \in R^1$, and

$$g^*(x) = |x - x_0 + 1|^{1+\nu} - g_0, \quad \forall x \geq x_0 - 1. \tag{41}$$

Now, consider the following set

$$W_{c_1}^+ \triangleq \{\sigma w_t > c_1 t - 1, \ \forall t \in [0, T]\},$$

where $c_1 > 0$. By (6) and Assumption (i), we have a.s. on $W_{c_1}^+$:

$$\begin{aligned} x_t &= x_0 + \int_0^t g(x_s)ds + \int_0^t u_s ds + \sigma w_t \\ &\geq x_0 + \int_0^t g(x_s)ds + (u_0 - M)t + c_1 t - 1 \\ &\geq x_0 + \int_0^t g^*(x_s)ds + (c_1 + u_0 - M)t - 1, \quad \forall t \geq 0. \end{aligned}$$

So, if we define $y(t)$ to be the solution of

$$y(t) = x_0 + \int_0^t g^*(y_s)ds + (c_1 + u_0 - M)t - 1, \tag{42}$$

then by Lemma 4.3 with $a = x_0$ and $b = c_1 + u_0 - M$, we have for c_1 sufficiently large,

$$x(t) > y(t), \quad \forall t \geq 0, \quad \text{a.s. on } W_{c_1}^+.$$

Applying Lemma 4.4 to the system (42), we know that there exists a $c_1 > 0$ large enough such that $y(T) = \infty$. So we have $x(T) = \infty$ a.s. on $W_{c_1}^+$. Furthermore, by Lemma 4.2, we know that $P\{W_{c_1}^+\} > 0$. Hence the proof is completed. ▮

5 Concluding Remarks

In this paper, we have tried to understand how the capability of the sampled-data feedback for systems with uncertain nonlinear structure depends upon the (not necessarily small) value of the sampling period and upon the "size" of the uncertainty. As a starting point, we have considered a typical class of first-order sampled-data control systems with unknown nonlinearity, and have obtained several concrete theoretical results. These results show, among other things, that for the uncertain system to be stabilizable by sampled-data feedback, the choice of the sampling rate h should be of the magnitude $O(\frac{1}{L})$ with a suitable "O" constant, where L is the "slope" of the unknown nonlinear function. For further investigation, it is desirable to bridge the gap between the bounds for Lh in Theorems 2.1 and 2.2 (as has been done for the pure discrete-time case in [12]) , and to study more general systems. Finally, it is worthwhile to remark that, since our main result—Theorem 2.1 is of "negative" character, it is valid also for a more general class of uncertain systems which include the basic model (1) as a special case.

Appendix A

Proof of Lemma 3.1

(i) Since $\phi(x)$ satisfies the Lipschitz condition, the trajectory of the system is unique. If $x(t)$ is not monotonous, then there must exist $t_2 > t_1 \geq 0$ such that $\dot{x}(t_1) \cdot \dot{x}(t_2) < 0$. By the continuity of the time derivative of the trajectory, there exists a $t^* \in (t_1, t_2)$ such that $\dot{x}(t^*) = 0$. Hence, by (11) we have $\phi(x(t^*)) = 0$.

Now, let

$$y(t) = \begin{cases} x(t), & t \leq t^*; \\ x(t^*), & t > t^*. \end{cases}$$

Then $y(t)$ is also a trajectory of the system. By the uniqueness of the trajectory we have $\dot{x}(t_2) = \dot{y}(t_2) = 0$. This is a contradiction! Hence, $x(t)$ must be monotonous.

(ii) Without loss of generality, suppose that $x(T) = x_T > x_0$. We first prove that $\phi(x) > 0$ on $[x_0, x_T]$.

Since $x_T > x_0$, by the continuity there must exist a $x^+ \in (x_0, x_T)$ such that $\phi(x^+) > 0$.

If there is an $x^- \in (x_0, x_T)$ such that $\phi(x^-) \leq 0$, then there must exist an $x^* \in (x_0, x_T)$ such that $\phi(x^*) = 0$. Let

$$y(t) = \begin{cases} x(t), & x(t) \leq x^*; \\ x^*, & x(t) > x^*. \end{cases}$$

Then $y(t)$ is also a trajectory. By the monotonicity we have $x(t) \leq x^*$, $\forall t > 0$. This contradicts $x(T) = x_T$! Hence we get $\phi(x) \neq 0, \forall x \in [x_0, x_T]$.

By the continuity of the trajectory, we have $\phi(x) > 0, x \in [x_0, x_T]$, and so

$$\frac{dx}{dt} = \phi(x) \Longleftrightarrow \frac{dx}{\phi(x)} = dt.$$

Integrating both sides, we have

$$\int_{x_0}^{x_T} \frac{dx}{\phi(x)} = \int_0^T dt = T.$$

On the other hand, if $\phi(x) \neq 0$ on the interval $[\min(x_T, x_0), \max(x_T, x_0)]$, we then have $\frac{dx}{\phi(x)} = dt$ on the interval (without loss of generality, we assume $\phi(x) > 0$). So, $x(T) = x_T$ only if

$$\int_{x_0}^{x_T} \frac{dx}{\phi(x)} = T.$$

∎

Proof of Lemma 3.2

For convenience of presentation, we denote $\alpha \triangleq L|z_0|+c+u_0$ and $\beta \triangleq z_1-|z_0|$ in the sequel. Obviously, we have $\alpha > 0$, $\beta > 0$.

First, by Lemma 3.1 it can be calculated that

$$t_{|z_0|\to z_1} = \int_{|z_0|}^{z_1} \frac{dx}{\hat{g}(x)+u_0} = \int_{|z_0|}^{z_1} \frac{dx}{L|z_0|+c+u_0} = \frac{\beta}{\alpha}.$$

where and hereafter $t_{|z_0|\to z_1}$ denotes the time needed for $z(t)$ to travel from $|z_0|$ to z_1.

By the assumption and Lemma 3.1, we see that if we can construct a locally Lipschitz function g^* on $[|z_0|, z_1]$ to satisfy

a) $|g^*(x)| \le L|x|+c$, $x \in [|z_0|, z_1]$;

b) $g^*(|z_0|) = \hat{g}(|z_0|)$, $g^*(z_1) = Lz_1 + c$;

c) $g^*[|z_0|, z_1] \ge 0$;

d) $\int_{|z_0|}^{z_1} \frac{dx}{g^*(x)+u_0} = \frac{\beta}{\alpha}$,

then $\hat{g}[z_0, |z_0|] \bigoplus g^*[|z_0|, z_1]$ is just the desired function g_1.

Let s and l be two small positive constants, and let $\eta > 0$ satisfy $\alpha-\eta > 0$ and $L|z_0|+c-\eta > 0$. We define a function $g_{s,l}$ on the interval $[|z_0|, z_1]$:

$$g_{s,l}(x) = \begin{cases} L|z_0|+c-\dfrac{\eta}{s}(x-|z_0|), & x \in [|z_0|, |z_0|+s]; \\ L|z_0|+c-\eta+\dfrac{\eta}{s}(x-|z_0|-s), & x \in [|z_0|+s, |z_0|+2s]; \\ L|z_0|+c, & x \in [|z_0|+2s, z_1-l]; \\ \dfrac{L(z_1-|z_0|)}{l}(x-z_1+l)+L|z_0|+c, & x \in [z_1-l, z_1]. \end{cases}$$

It is easy to verify that $g_{s,l}$ is locally Lipschitz and satisfies a), b) and c) required above when s and l are small enough.

Next, it is easy to calculate that

$$\begin{aligned} \int_{|z_0|}^{z_1} \frac{dx}{g_{s,l}(x)+u_0} &= \Big(\int_{|z_0|}^{|z_0|+s} + \int_{|z_0|+s}^{z_0+2s} + \int_{|z_0|+2s}^{z_1-l} + \int_{z_1-l}^{z_1}\Big)\frac{dx}{g_{s,l}(x)+u_0} \\ &= 2\frac{s}{\eta}\log\frac{\alpha}{\alpha-\eta} + \frac{\beta-2s-l}{\alpha} + \frac{l}{L\beta}\log\frac{L\beta+\alpha}{\alpha}. \end{aligned}$$

Now, to make $g_{s,l}$ satisfy d), let

$$2\frac{s}{\eta}\log\frac{\alpha}{\alpha-\eta} + \frac{\beta-2s-l}{\alpha} + \frac{l}{L\beta}\log\frac{L\beta+\alpha}{\alpha} = \frac{\alpha}{\beta}.$$

We have

$$0 = \frac{2s}{\eta}\log\frac{\alpha}{\alpha-\eta} - \frac{2s+l}{\alpha} + \frac{l}{L\beta}\log\frac{L\beta+\alpha}{\alpha}.$$

So,

$$-\Big[1+\frac{\alpha}{\eta}\log\Big(1-\frac{\eta}{\alpha}\Big)\Big]2s = \Big[1-\frac{\alpha}{L\beta}\log\Big(1+\frac{L\beta}{\alpha}\Big)\Big]l.$$

Since $\log(1-x) < -x, \forall 0 < x < 1$, and $\log(1+x) < x, \forall x > 0$, we know that both sides are positive. So, we can select $s > 0$ and $l > 0$ small enough to make the requirement d) hold. Finally, the $g_{s,l}$ is just the desired function g^*. ∎

Proof of Lemma 3.3

Applying the time transform $t = \lambda\tau$ to the system (1)–(2), and denoting $y(\tau) \triangleq x(\lambda\tau) = x(t)$, we have

$$\begin{aligned}\frac{dy(\tau)}{d\tau} &= \frac{dx(t)}{dt}\cdot\frac{dt}{d\tau}\\ &= [f(x(t)) + u_{kh}]\cdot\lambda = \lambda f(y(\tau)) + \lambda u_{kh}, \quad (k-1)h \le \lambda\tau < kh,\end{aligned}$$

i.e.,

$$\begin{cases}\dfrac{dy(\tau)}{d\tau} = \lambda f(y) + \lambda u_{kh}, \quad (k-1)\cdot\dfrac{1}{\lambda}h \le \tau < k\cdot\dfrac{1}{\lambda}h;\\ y(0) = x_0.\end{cases}$$

By the uniqueness of the trajectory of the above differential equation, we have $y(t) = z(t), t \ge 0$, i.e., $x(\lambda t) = z(t), t \ge 0$, where $z(t)$ is the trajectory of $Sys(\lambda f, x_0, \frac{1}{\lambda}h, \{\lambda u_{kh}\})$. ∎

Proof of Lemma 3.4

We are going to prove that $\mu = \max(1 - \frac{2-a}{2a}, 1 - \frac{(a^2-1)(2-a)}{2a})$. But, before pursuing further, let us first verify that for μ defined as this, we have $\mu \in (0,1)$.

Obviously, we have $a \in (1,2)$, and so $1 - \frac{2-a}{2a}$ is in $(0,1)$. Moreover,

$$0 < \frac{(a^2-1)(2-a)}{2a} \triangleq d < 1.$$

So, we have $\mu \in (0,1)$.

Now, without loss of generality, we assume $x_0 > 0$ in the sequel.

Since $g(x) + u_0 \le Lx + c + u_0 < 0$ on the interval $[(1-d)x_0, x_0]$, by Lemma 3.1 we know that the time needed for $x(t)$ to travel from x_0 to $(1-d)x_0$ is

$$t_{x_0 \to (1-d)x_0} = \int_{x_0}^{(1-d)x_0} \frac{dx}{g(x) + u_0} \le \int_{x_0}^{(1-d)x_0} \frac{dx}{Lx + c + u_0} = 1.$$

Hence, we have

$$x(1) \le (1-d)x_0. \tag{43}$$

Next, since $g(x)+u_0 < 0, \forall x \le x_0$, by Lemma 3.1 and $g(x) \ge -Lx-c, \forall x \ge 0$, $g(x) \ge Lx-c$, $\forall x < 0$, we have to denote $a_1 = \frac{2-a}{2a}$

$$\begin{aligned}t_{x_0 \to -(1-a_1)x_0} &= \int_{x_0}^{0} \frac{dx}{g(x)+u_0} + \int_0^{-(1-a_1)x_0} \frac{dx}{g(x)+u_0}\\ &\ge \int_{x_0}^{0} \frac{dx}{-Lx - c + u_0} + \int_0^{-(1-a_1)x_0} \frac{dx}{Lx - c + u_0}\\ &= \frac{1}{L}\left[\log\frac{2\cdot\frac{c}{L} + (2+a_1)x_0}{2\cdot\frac{c}{L} + (1+a_1)x_0} + \log\frac{2\cdot\frac{c}{L} + 2x_0}{2\cdot\frac{c}{L} + (1+a_1)x_0}\right].\end{aligned}$$

Since the last part increases with x_0 when $x_0 > 0$, and since by (ii) $x_0 \ge \frac{4c}{L}\cdot\frac{a-1}{2-a}$, it is easy to show that $t_{x_0 \to -(1-a_1)x_0} \ge 1$. Hence, by Lemma 3.1, we have

$$x(1) \ge -\Big(1 - \frac{2-a}{2a}\Big)x_0. \tag{44}$$

Finally, combining (43) and (44), we get the desired result. ∎

Proof of (23)

By the construction of $\hat{f}_1^+(-\infty, x_1]$, it follows that

$$\hat{f}_1^+(x) = \begin{cases} c, \quad x \in [0, x_1]; \\ \left(L + \dfrac{2}{\delta}\right)x + c, \quad x \in [-\delta c, 0]; \\ Lx - c, \quad x \in [-x_1, -\delta c]; \\ -Lx_1 - c, \quad x \leq -x_1. \end{cases} \tag{45}$$

Also, by (22), the control satisfies $u_1 \leq -(\frac{1}{2}Lx_1 + c)$.

It is easy to verify that $\hat{f}_1^+(x) + u_1 < 0, \forall x \leq x_1$, so we can apply Lemma 3.1 here. By Lemma 3.1, it is clear that to verify the desired result (23) we need only to show that

$$t \triangleq \int_{x_1}^{-\frac{L}{2}x_1} \frac{dx}{\hat{f}_1^+(x) + u_1} \leq 1.$$

For simplicity, we will continue to use $t_{\alpha\to\beta}$ to denote the time that the trajectory needs to travel from α to β in the sequel.

By (45), we know that $\hat{f}_1^+(x) \leq c$ on $[-\delta c, x_1]$. And since $x_1 \geq c$, we have

$$t_{x_1\to-\delta c} \leq \int_{x_1}^{-\delta c} \frac{dx}{c - \left(\dfrac{1}{2}Lx_1 + c\right)} \leq \frac{1 + \frac{\delta c}{c}}{\frac{L}{2}}.$$

Also, we have

$$\begin{aligned} t_{-\delta c\to-x_1} &= \int_{-\delta c}^{-x_1} \frac{dx}{Lx - c + u_1} \leq \int_{-\delta c}^{-x_1} \frac{dx}{Lx - c - \dfrac{L}{2}x_1 - c} \\ &= \frac{1}{L}\log\frac{\dfrac{3L}{2}x_1 + 2c}{\dfrac{L}{2}x_1 + (2 + L\delta)c} < \frac{1}{L}\log 3 \end{aligned}$$

and

$$\begin{aligned} t_{-x_1\to-\frac{L}{2}x_1} &= \int_{-x_1}^{-\frac{L}{2}x_1} \frac{dx}{-Lx_1 - c + u_1} \\ &\leq \frac{1}{3} - \frac{2}{3}\cdot\frac{1}{L}. \end{aligned}$$

So, we have

$$\begin{aligned} t &= t_{x_1\to-\delta c} + t_{-\delta c\to-x_1} + t_{-x_1\to-\frac{L}{2}x_1} \\ &\leq \frac{2(1+\delta)}{L} + \frac{1}{L}\log 3 + \frac{1}{3} - \frac{2}{3}\cdot\frac{1}{L} \\ &\leq \frac{1}{L}\left[\frac{4}{3} + \log(3 + 2L^{-1}) + 2 + \log\frac{L+2}{L-2} + \frac{4\delta}{2+L}\right] + \frac{1}{3}. \end{aligned}$$

Hence, by (16), we have $t \leq \frac{2}{3} + \frac{1}{3} = 1$. This completes the proof. ∎

Appendix B

To complete Step 3 *in the proof of Theorem* 2.1

Given the control u_{k+1} and the observation x_{k+1}, we need to show that a)–c) still hold with k replaced by $k+1$. Similarly to Step 2, we consider four cases separately.

Case (i)

$$x_{k+1} > 0, \quad u_{k+1} \geq -(Lx_{k+1} + c) + \frac{L}{2}(|x_{k+1}| - |x_k|). \tag{46}$$

In view of (46), we define $f^*[|x_k|, x_{k+1}] = \phi_k^+$, and consequently we have $f^*(x_{k+1}) = Lx_{k+1} + c$.

Next, denote

$$a_{k+2}^{++} \triangleq x_{k+1} + (u_{k+1} + Lx_{k+1} + c) > x_{k+1}, \tag{47}$$

and extend the definition of f^* already defined on $[-x_{k+1}, x_{k+1}]$ to $[-a_{k+2}^{++}, -x_{k+1})$ by

$$f^*[-a_{k+2}^{++}, -x_{k+1}) = \begin{cases} \dfrac{Lx_{k+1} + f^*(-x_{k+1}) + (1+\delta L)c}{\delta c}(x + x_{k+1}) + f^*(-x_{k+1}), \\ \qquad x \in [-x_{k+1} - \delta c, -x_{k+1}); \\ Lx - c, \quad x \in [-a_{k+2}^{++}, -x_{k+1} - \delta c). \end{cases} \tag{48}$$

On the interval $[x_{k+1}, a_{k+2}^{++}]$, we define a function $g_{k+1}^{++}[x_{k+1}, a_{k+2}^{++}] \equiv Lx_{k+1} + c$. Then it is easy to verify that the system $\Sigma_{g_{k+1}^{++}} : \dot{x} = g_{k+1}^{++}(x) + u_{k+1}, t \geq k+1$, travels from $x(k+1) = x_{k+1}$ to $x(k+2) = a_{k+2}^{++}$.

By Lemma 3.2 with $z_0 = x_{k+1}$ and $z_1 = a_{k+2}^{++}$, there exists a ϕ_{k+1}^{++} satisfying: $\Sigma_{\phi_{k+1}^{++}} \overset{x_{k+1}}{\Longleftrightarrow} \Sigma_{g_{k+1}^{++}}$, s.t. u_{k+1} and $\phi_{k+1}^{++}(a_{k+2}^{++}) = La_{k+2}^{++} + c$, $\phi_{k+1}^{++}[x_{k+1}, a_{k+2}^{++}] \geq 0$.

Now, denote

$$G_{k+1}^{++} \triangleq \left\{ f^*[-a_{k+2}^{++}, x_{k+1}] \bigoplus g_{k+1}^{++}(x_{k+1}, a_{k+2}^{++}], f^*[-a_{k+2}^{++}, x_{k+1}] \bigoplus \phi_{k+1}^{++}(x_{k+1}, a_{k+2}^{++}] \right\} \subseteq G_c^L.$$

Then the state $x(t)$ of the uncertain system (1)–(2) may be produced by a system corresponding to any function in G_{k+1}^{++} (obviously $\forall f_1, f_2 \in G_{k+1}^{++}$, we have $\Sigma_{f_1} \underset{k+2}{\overset{0}{\Longleftrightarrow}} \Sigma_{f_2}$, s.t. $\{u_t, t = 0, 1, \cdots, k+1\}$). But it is easily seen that, for whichever function in G_{k+1}^{++}, we will get $x_{k+2} = a_{k+2}^{++}$ under u_{k+1}.

From (46) and (47) it is obvious that $|x_{k+2}| \geq \frac{L}{2}|x_{k+1}|$.

Case (ii)

$$x_{k+1} > 0, \quad u_{k+1} < -(Lx_{k+1} + c) + \frac{L}{2}(|x_{k+1}| - |x_k|). \tag{49}$$

In view of (49), we define $f^*[|x_k|, x_{k+1}] = g_k^+$, and hence we get $f^*[|x_k|, x_{k+1}] \equiv L|x_k| + c$.

Let $g_{k+1}^{+-}(-\infty, -x_{k+1}) \equiv f^*(-x_{k+1})(=-Lx_{k+1} - c\ )$, and let

$$\hat{f}_{k+1}^{+}(-\infty, x_{k+1}] \triangleq g_{k+1}^{+-}(-\infty, -x_{k+1}) \bigoplus f^*[-x_{k+1}, x_{k+1}].$$

It can be shown that the system: $\dot{z} = \hat{f}_{k+1}^{+}(z) + u_{k+1}, t \geq k+1, z(k+1) = x_{k+1}$ satisfies

$$z(k+2) \leq -\frac{L}{2}x_{k+1}. \tag{50}$$

For the smooth flow of presentation, we put this proof at the end of this appendix.

Next, denote $a_{k+2}^{+-} \triangleq -z(k+2) > 0$.

By Lemma 3.2′ with $z_0 = x_{k+1}$ and $z_1 = -a_{k+2}^{+-}$, there exists a $\phi_{k+1}^{+-} \in G_c^L$ satisfying: $\Sigma_{\phi_{k+1}^{+-}} \overset{x_{k+1}}{\Longleftrightarrow} \Sigma_{\hat{f}_{k+1}^{+}(-\infty, x_{k+1}]}$, s.t. u_{k+1}; $\phi_{k+1}^{+-}[-x_{k+1}, x_{k+1}] = f^*[-x_{k+1}, x_{k+1}]$, $\phi_{k+1}^{+-}(-a_{k+2}^{+-}) = -La_{k+2}^{+-} - c$ and $\phi_{k+1}^{+-}[-a_{k+2}^{+-}, -x_{k+1}] \le 0$.

Now, let

$$f^*(x_{k+1}, a_{k+2}^{+-}] = \begin{cases} \dfrac{Lx_{k+1} - f^*(x_{k+1}) + (1+\delta L)c}{\delta c}(x - x_{k+1}) + f^*(x_{k+1}), \\ \qquad\qquad x \in (x_{k+1}, x_{k+1} + \delta c]; \\ Lx + c, \quad x \in (x_{k+1} + \delta c, a_{k+2}^{+-}], \end{cases} \tag{51}$$

and denote

$$\begin{aligned} G_{k+1}^{+-} &\triangleq \left\{ g_{k+1}^{+-}[-a_{k+2}^{+-}, -x_{k+1}) \bigoplus f^*[-x_{k+1}, a_{k+2}^{+-}], \phi_{k+1}^{+-}[-a_{k+2}^{+-}, -x_{k+1}) \bigoplus f^*[-x_{k+1}, a_{k+2}^{+-}] \right\} \\ &\subseteq G_c^L. \end{aligned}$$

Then the state $x(t)$ of the uncertain system (1)–(2) may be produced by a system corresponding to any function in G_{k+1}^{+-}. But it is easily seen that, for whichever function in G_{k+2}^{+-}, we will get $x_{k+2} = -a_{k+2}^{+-}$ under u_{k+1}.

Obviously, we get from (50) that $|x_{k+2}| \ge \frac{L}{2}|x_{k+1}|$.

Case (iii)

$$x_{k+1} < 0, u_{k+1} \ge -(Lx_{k+1} - c) - \frac{L}{2}(|x_{k+1}| - |x_k|). \tag{52}$$

All the conditions in this case are "symmetric" to those in Case (ii), so the proof ideas are similar.

In view of (52), we define $f^*[x_{k+1}, -|x_k|] = g_k^-$, and hence we get $f^*[x_{k+1}, -|x_k|] \equiv -L|x_k| - c$.

Let $g_{k+1}^{-+}(-x_{k+1}, \infty) \equiv f^*(-x_{k+1}) (= L(-x_{k+1}) + c)$, and let

$$\hat{f}_{k+1}^{-}[x_{k+1}, \infty) \triangleq f^*[x_{k+1}, -x_{k+1}] \bigoplus g_{k+1}^{-+}(-x_{k+1}, \infty).$$

Similarly to the proof in Case (ii), we can prove that the system $\dot{z} = \hat{f}_{k+1}^{-}(z) + u_{k+1}, t \ge k+1, z(k+1) = x_{k+1}$ satisfies

$$z(k+2) \ge -\frac{L}{2}x_{k+1}. \tag{53}$$

Now, let us denote $a_{k+2}^{-+} \triangleq z(k+2) > 0$.

By Lemma 3.2 with $z_0 = x_{k+1}$ and $z_1 = a_{k+2}^{-+}$, there exists a $\phi_{k+1}^{-+} \in G_c^L$ satisfying: $\Sigma_{\phi_{k+1}^{-+}} \overset{x_{k+1}}{\Longleftrightarrow} \Sigma_{\hat{f}_{k+1}^{-}[x_{k+1}, \infty)}$, s.t. u_{k+1}; $\phi_{k+1}^{-+}[x_{k+1}, -x_{k+1}] = f^*[x_{k+1}, -x_{k+1}]$, $\phi_{k+1}^{-+}(a_{k+2}^{-+}) = La_{k+2}^{-+} + c$, and $\phi_{k+1}^{-+}[-x_{k+1}, a_{k+2}^{-+}] \ge 0$.

Let

$$f^*[-a_{k+2}^{-+}, x_{k+1}) = \begin{cases} \dfrac{-Lx_{k+1} + f^*(x_{k+1}) + (1+\delta L)c}{\delta c}(x - x_{k+1}) + f^*(x_{k+1}), \\ \qquad\qquad x \in [x_{k+1} - \delta c, x_{k+1}); \\ Lx - c, \quad x \in [-a_{k+2}^{-+}, x_{k+1} - \delta c), \end{cases} \tag{54}$$

and denote

$$G_{k+1}^{-+} \triangleq \{f^*[-a_{k+2}^{-+}, -x_{k+1}] \bigoplus g_{k+1}^{-+}(-x_{k+1}, a_{k+2}^{-+}], f^*[-a_{k+2}^{-+}, -x_{k+1}] \bigoplus \phi_{k+1}^{-+}(-x_{k+1}, a_{k+2}^{-+}]\} \subseteq G_c^L.$$

Then the state $x(t)$ of the uncertain system (1)–(2) may be produced by a system corresponding to any function in G_{k+1}^{-+}. But it can be easily seen that, for whichever function in G_{k+1}^{-+}, we will get $x(k+2) = a_{k+2}^{-+}$ under u_{k+1}.

Obviously, we get from (53) that $|x_{k+2}| \geq \frac{L}{2}|x_{k+1}|$.

Case (iv)

$$x_{k+1} < 0, \quad u_{k+1} < -(Lx_{k+1} - c) - \frac{L}{2}(|x_{k+1}| - |x_k|). \tag{55}$$

In view of (55), we define $f^*[x_{k+1}, -|x_k|] = \phi_k^-$, and get $f^*(x_{k+1}) = Lx_{k+1} - c$.

Next, denote

$$a_{k+2}^{--} \triangleq -(x_{k+1} + (u_{k+1} + Lx_{k+1} - c)) > -x_{k+1}, \tag{56}$$

and extend the definition of f^* to $(-x_{k+1}, a_{k+2}^{--}]$ by

$$f^*(-x_{k+1}, a_{k+2}^{--}] = \begin{cases} \dfrac{-Lx_{k+1} - f^*(-x_{k+1}) + (1+\delta L)c}{\delta c}(x + x_{k+1}) + f^*(-x_{k+1}), \\ \qquad x \in (-x_{k+1}, -x_{k+1} + \delta c]; \\ Lx + c, \quad x \in (-x_{k+1} + \delta c, a_{k+2}^{--}]. \end{cases} \tag{57}$$

On the interval $[-a_{k+2}^{--}, x_{k+1}]$, we define a function $g_{k+1}^{--}[-a_{k+2}^{--}, x_{k+1}] \equiv Lx_{k+1} - c$. Then it is easy to verify that the system $\Sigma_{g_{k+1}^{--}} : \dot{x} = g_{k+1}^{--}(x) + u_{k+1}, t \geq k+1$, travels from $x(k+1) = x_{k+1}$ to $x(k+2) = -a_{k+2}^{++}$.

By Lemma 3.2$'$ with $z_0 = x_{k+1}$ and $z_1 = -a_{k+2}^{--}$, there exists a $\phi_{k+1}^{--} \in G_c^L$ satisfying: $\Sigma_{\phi_{k+1}^{--}} \overset{x_{k+1}}{\Longleftrightarrow} \Sigma_{g_{k+1}^{--}}$, s.t. u_{k+1}, $\phi_{k+1}^{--}(-a_{k+2}^{--}) = -La_{k+2}^{--} - c$, and $\phi_{k+1}^{--}[-a_{k+2}^{--}, x_{k+1}] \leq 0$.

Now, denote

$$G_{k+1}^{--} \triangleq \{ g_{k+1}^{--}[-a_{k+2}^{--}, x_{k+1}) \bigoplus f^*[x_{k+1}, a_{k+2}^{--}], \phi_{k+1}^{--}[-a_{k+2}^{--}, x_{k+1}) \bigoplus f^*[x_{k+1}, a_{k+2}^{--}]\} \subseteq G_c^L.$$

Then the state $x(t)$ of the uncertain system (1)–(2) may be produced by a system corresponding to any function in G_{k+1}^{--} (obviously $\forall f_1, f_2 \in G_{k+1}^{--}$, we have $\Sigma_{f_1} \underset{k+2}{\overset{0}{\Longleftrightarrow}} \Sigma_{f_2}$, s.t. $\{u_t, t = 0, 1, \cdots, k+1\}$). But it is easily seen that, for whichever function in G_{k+1}^{--}, we will get $x_{k+2} = -a_{k+2}^{--}$ under u_{k+1}.

Obviously, we get from (55)–(56) that $|x_{k+2}| \geq \frac{L}{2}|x_{k+1}|$.

Finally, denote

$$g_{k+1}^{+} \triangleq \begin{cases} g_{k+1}^{++}, \text{ in Case (i)}; \\ g_{k+1}^{-+}, \text{ in Case (iii)}. \end{cases} \qquad g_{k+1}^{-} \triangleq \begin{cases} g_{k+1}^{+-}, \text{ in Case (ii)}; \\ g_{k+1}^{--}, \text{ in Case (iv)}. \end{cases}$$

$$\phi_{k+1}^{+} \triangleq \begin{cases} \phi_{k+1}^{++}, \text{ in Case (i)}; \\ \phi_{k+1}^{-+}, \text{ in Case (iii)}. \end{cases} \qquad \phi_{k+1}^{-} \triangleq \begin{cases} \phi_{k+1}^{+-}, \text{ in Case (ii)}; \\ \phi_{k+1}^{--}, \text{ in Case (iv)}. \end{cases}$$

We get the following desired results:

a) $|x_{k+2}| \geq \frac{L}{2}|x_{k+1}|$, $|x_1| \geq c$.

b) If $x_{k+2} > 0$, then a Lipschitz function f^* on the interval $[-|x_{k+2}|, |x_{k+1}|]$ together with its accompanying functions g^+_{k+1} and ϕ^+_{k+1} on $(|x_{k+1}|, x_{k+2}]$ can be defined. Furthermore

$$\Sigma_{f^*[x_{k+1},|x_{k+1}|] \bigoplus g^+_{k+1}(|x_{k+1}|, x_{k+2}]} \overset{0}{\underset{k+2}{\Longleftrightarrow}} \Sigma_{f^*[x_{k+1},|x_{k+1}|] \bigoplus \phi^+_{k+1}(|x_{k+1}|, x_{k+2}]},$$

$$\text{s.t.} \quad \{u_t, t = 0, 1, \cdots, k+1\}.$$

c) If $x_{k+2} < 0$, then a Lipschitz function f^* on the interval $[-|x_{k+1}|, |x_{k+2}|]$ together with its accompanying functions g^-_{k+1} and ϕ^-_{k+1} on $[x_{k+2}, -|x_{k+1}|)$ can be defined. Moreover,

$$\Sigma_{g^+_{k+1}[x_{k+2}, -|x_{k+1}|) \bigoplus f^*[-|x_{k+1}|, x_{k+1}]} \overset{0}{\underset{k+2}{\Longleftrightarrow}} \Sigma_{\phi^+_{k+1}[x_{k+2}, -|x_{k+1}|) \bigoplus f^*[-|x_{k+1}|, x_{k+1}]},$$

$$\text{s.t.} \quad \{u_t, t = 0, 1, \cdots, k+1\}.$$

∎

Proof of (50)

By the construction of $\hat{f}^+_{k+1}$, we see that

$$\hat{f}^+_{k+1}(z) \leq M(z), \quad \forall z \leq x_{k+1},$$

where $M(\cdot)$ is defined by

$$M(z) = \begin{cases} L|x_k| + c, & z \in [|x_k|, x_{k+1}]; \\ Lz + c, & z \in [0, |x_k|]; \\ c, & z \in [-\delta c, 0]; \\ 0, & z \in [-\delta c - |x_k|, -\delta c]; \\ Lz - c, & z \in [-x_{k+1}, -\delta c - |x_k|]; \\ -(Lx_{k+1} + c), & z \leq -x_{k+1}. \end{cases}$$

Also, by induction we have

$$|x_k| \geq c, \quad x_{k+1} \geq \frac{L}{2}|x_k| > 0, \tag{58}$$

and by (49)

$$u_{k+1} < -\Big(\frac{L}{2}(x_{k+1} + |x_k|) + c\Big). \tag{59}$$

Now, we define $y(t)$ to satisfy

$$\begin{cases} \dot{y} = M(y) - \Big(\dfrac{L}{2}(x_{k+1} + |x_k|) + c\Big); \\ y(k+1) = x_{k+1}. \end{cases} \tag{60}$$

Since $M(y(k+1)) - (\frac{L}{2}(x_{k+1} + |x_k|) + c) < 0$, by Lemma 3.1 we know that $y(t)$ is monotonically decreasing. By the comparison principle for differential equations, we have $z(t) \leq y(t)$, $t \geq k+1$. So, to prove the desired result we need only to show that $y(k+2) \leq -\frac{L}{2}x_{k+1}$.

Now, by the definition of $M(z)$ and Lemma 3.1, and with the help of (58) and (59), it is clear that the time needed for the system (60) to travel from x_{k+1} to $-\frac{L}{2}x_{k+1}$ via $|x_k|$, 0, $-\delta c$, $-|x_k|-\delta c$, and $-x_{k+1}$ can be calculated as follows:

$$
\begin{aligned}
t_{x_{k+1}\to|x_k|} &= \frac{2}{L},\\
t_{|x_k|\to 0} &= \frac{1}{L}\int_{|x_k|}^{0}\frac{dy}{-\frac{1}{2}(x_{k+1}+|x_k|)+y} = \frac{1}{L}\log\frac{1+\dfrac{|x_k|}{x_{k+1}}}{1-\dfrac{|x_k|}{x_{k+1}}} \le \frac{1}{L}\log\frac{1+\dfrac{2}{L}}{1-\dfrac{2}{L}},\\
t_{0\to-\delta c} &= \frac{\delta c}{\frac{1}{2}L(x_{k+1}+|x_k|)} \le \frac{\delta}{\frac{1}{2}L\Big(1+\dfrac{L}{2}\Big)},\\
t_{-\delta c\to-|x_k|-\delta c} &= \frac{|x_k|}{\frac{1}{2}L(x_{k+1}+|x_k|)+c} < \frac{2}{L},\\
t_{-|x_k|-\delta c\to -x_{k+1}} &= \int_{-|x_k|-\delta c}^{-x_{k+1}}\frac{dy}{Ly-c-\Big[\frac{1}{2}L(x_{k+1}+|x_k|)+c\Big]}\\
&= \frac{1}{L}\log\frac{\frac{1}{2}(3x_{k+1}+|x_k|)+\frac{2}{L}c}{\frac{1}{2}(x_{k+1}+3|x_k|)+\Big(\frac{2}{L}+\delta\Big)c}\\
&\le \frac{1}{L}\log\frac{3x_{k+1}+|x_k|}{x_{k+1}+3|x_k|} \le \frac{1}{L}\log\Big(3+\frac{2}{L}\Big),\\
t_{-x_{k+1}\to-\frac{L}{2}x_{k+1}} &= \frac{(L-2)x_{k+1}}{3Lx_{k+1}+L|x_k|+4c} \le \frac{L-2}{3L} = \frac{1}{3}-\frac{2}{3}\cdot\frac{1}{L}.
\end{aligned}
$$

Hence, by the calculations made above and (16), we have

$$
\begin{aligned}
t_{x_{k+1}\to-\frac{L}{2}x_{k+1}} &\le \frac{2}{L}+\frac{1}{L}\log\frac{1+\frac{2}{L}}{1-\frac{2}{L}}+\frac{4\delta}{L(2+L)}+\frac{2}{L}+\frac{1}{L}\log\Big(3+\frac{2}{L}\Big)+\frac{1}{3}-\frac{2}{3}\cdot\frac{1}{L}\\
&= \frac{1}{L}\Big(\frac{10}{3}+\log\frac{1+\frac{2}{L}}{1-\frac{2}{L}}+\log\Big(3+\frac{2}{L}\Big)+\frac{4\delta}{2+L}\Big)+\frac{1}{3}\le 1.
\end{aligned}
$$

So, by Lemma 3.1, we know that $y(k+2)\le-\frac{L}{2}x_{k+1}$. This completes the proof of (50). ▮

Appendix C

Proof of Lemma 4.1

By Ito's formula and (6)–(7), we have

$$dx_t^2 = 2x_t(g(x_t)-(1+\lambda)Lx_{kh})dt+\sigma^2dt+2x_t\sigma dw_t,\quad t\in[kh,kh+h),$$

and so,

$$dEx_t^2 = 2Ex_tg(x_t)dt-2(1+\lambda)L\cdot Ex_tx_{kh}\cdot dt+\sigma^2dt. \tag{61}$$

Since $x_t = x_{kh}+\int_{kh}^t g(x_s)ds-(1+\lambda)Lx_{kh}(t-kh)+\sigma(w_t-w_{kh})$, we have

$$Ex_tx_{kh} = Ex_{kh}^2+\int_{kh}^t Eg(x_s)x_{kh}ds-L(1+\lambda)Ex_{kh}^2(t-kh).$$

Substituting this into (61), we have

$$\frac{de_t}{dt} = 2Ex_t g(x_t) - 2L(1+\lambda)e_{kh} + 2L^2(1+\lambda)^2 e_{kh}(t-kh) - 2L(1+\lambda)\int_{kh}^t Eg(x_s)x_{kh}ds + \sigma^2.$$

Furthermore, by $|g(x)| \le c + L|x|$ and $|Ex_t| \le \sqrt{e_t}$, we have

$$\begin{aligned}\frac{de_t}{dt} \le\ & 2c\cdot\sqrt{e_t} + 2Le_t - 2L\cdot(1+\lambda)e_{kh} + 2L^2(1+\lambda)^2 e_{kh}\cdot(t-kh) \\ & +2L(1+\lambda)\sqrt{e_{kh}}\cdot\int_{kh}^t \sqrt{2c^2+2L^2e_s}ds + \sigma^2.\end{aligned}$$

∎

Proof of Lemma 4.3

Obviously, we have $x(0) > y(0)$. Hence, if the conclusion is not true, there must exist a $t_1 > 0$ such that $x(t_1) = y(t_1)$. This ensures the existence of the intersection time $t^* \triangleq \inf_{t\ge 0}\{t : x(t) = y(t)\}$. By this definition and the assumptions, we have

$$y(t^*) = a + \int_0^{t^*} f(y_s)ds + bt^* - 1 = x(t^*) \ge a + \int_0^{t^*} f(x_s)ds + bt^* - 1,$$

so,

$$\int_0^{t^*} f(y_s)ds \ge \int_0^{t^*} f(x_s)ds. \tag{62}$$

Now, since $y(t)$ satisfies $\dot{y}(t) = f(y_t) + b$, we have $\dot{y}(0) = f(a-1) + b > 0$. Also, by the assumption $f'(x) > 0, x > a-1$, it is easy to verify that $y(t)$ is monotonically increasing, hence we get $y(t) > a-1, t > 0$.

By the definition of t^* and the continuity of $x(t)$ and $y(t)$, we have $x(t) > y(t) > a-1$, $\forall 0 < t < t^*$. Consequently, we have $f'(\xi_t) > 0, \forall \xi_t \in [y(t), x(t)], t \in (0, t^*)$.

Therefore, we have

$$\begin{aligned}\int_0^{t^*} f(x_s)ds &= \int_0^{t^*}(f(x_s) - f(y_s))ds + \int_0^{t^*} f(y_s)ds \\ &= \int_0^{t^*} f'(\xi_s)(x_s - y_s)ds + \int_0^{t^*} f(y_s)ds \\ &> 0 + \int_0^{t^*} f(y_s)ds.\end{aligned}$$

This contradicts (62), so the lemma is true. ∎

Proof of Lemma 4.4

Taking the time derivative on both sides of (39), we get

$$\frac{dz}{dt} = |z - x_0 + 1|^{1+\nu} - g_0 + b.$$

Hence,

$$\frac{dz}{|z - x_0 + 1|^{1+\nu} - g_0 + b} = dt.$$

Integrating both sides(z from x_0-1 to z_T while t from 0 to T), we have

$$\int_{x_0-1}^{z_T}\frac{dz}{|z-x_0+1|^{1+\nu}-g_0+b}=\int_0^{z_T-x_0+1}\frac{dz}{|z|^{1+\nu}-g_0+b}=\int_0^T dt=T.$$

Obviously, when $b-g_0>0$,

$$t_b\triangleq\int_0^\infty\frac{dz}{|z|^{1+\nu}-g_0+b}<\infty.$$

Hence, by the dominated convergence theorem, $\lim\limits_{b\longrightarrow\infty}t_b=0$. Therefore, we can choose a $b_T>0$ large enough such that $t_{b_T}\leq T$, which means $z_T=\infty$. ▮

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