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Exploring the maximum capability of adaptive feedback

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SUMMARY

The main purpose of adaptive feedback is to deal with dynamical systems with internal and/or external uncertainties, by using the on-line observed information. Thus, a fundamental problem in adaptive control is to understand the maximum capability (and limits) of adaptive feedback. This paper gives a survey of some basic ideas and results developed recently in this direction, for several typical classes of uncertain dynamical systems including parametric and non-parametric non-linear systems, sampled-data systems and time-varying systems. Copyright © 2002 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Feedback is ubiquitous, and exists in almost all goal-directed behaviours [1]. It is indispensible to the human intelligence, and is important in learning, adaptation, organization and evolution, etc. In the area of control, feedback is a concept which is as basic as the causal law in physics.

Feedback is necessary in dealing with uncertainties in complex systems whose global behaviours are the results of complicated interactions of subsystems. The uncertainties are usually classified into two types: internal (structure) and external (disturbance) uncertainties, depending on the specific dynamical systems to be controlled. In the ideal case where the mathematical model can exactly describe the true system, the feedback law that are designed based on the full knowledge of the model may be referred to as traditional feedback. In the case where the true system model is not exactly known but lies in a ball centred at a known nominal model with reliable model error bounds, the feedback laws designed based on the nominal model (and the prior knowledge) may be called robust feedbacks (cf. e.g. References [2–5]).

By adaptive feedback we mean the (non-linear) feedback which captures the uncertain (structure or parameter) information of the underlying system by properly utilizing the measured on-line data. The well-known certainty-equivalence principle in adaptive control is an

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example of such philosophy. Since an on-line learning mechanism is usually embedded in the structure of adaptive feedback, it is conceivable that adaptive feedback can deal with larger uncertainties than other forms of feedback can do.

Note that, as mentioned by Åström and Wittenmark [6], it is practically difficult, in general, to distinguish adaptive feedback from ordinary non-linear feedback by looking at either the software or the hardware of a controller, we will define adaptive feedback as any causal functions of the observed output process (see, e.g. Definition 1 in the next section). This will prevent us from restricting the capability of adaptive feedback at the outset by (artificial) definitions. Thus, the capability of adaptive feedback that we are going to explore is also the capability of the generally defined feedback.

Over the past several decades, much progress has been made in the area of adaptive control (cf. e.g. References [6–12]). For linear finite dimensional systems with uncertain parameters, a well-developed theory of adaptive control exists today, both for stochastic systems (cf. References [10–13]) and for deterministic systems with small unmodelled dynamics (cf. Reference [8]). This theory can be generalized to non-linear systems with linear unknown parameters and with linearly growing non-linearities ([14, 15]). However, fundamental difficulties may emerge in the design of stabilizing adaptive controls when these structural conditions are removed. This has motivated a series of studies on the maximum capability (and limits) of adaptive feedback starting from Reference [16].

To explore the maximum capability of adaptive feedback, we have to place ourselves in a framework that is somewhat different from the traditional robust control and adaptive control. First, the system structure uncertainty may be non-linear and/or non-parametric, and a known or reliable ball containing the true system and centred at a known nominal model may not be available *a priori*. Second, we need to study the maximum capability of the whole feedback mechanism (not only a special class of feedback laws), not only answering what the adaptive feedback can do, but also answering, the more difficult and important question, what the adaptive feedback cannot do. We shall also work with discrete-time feedback laws, which can reflect the basic causal law as well as the limitations of actuator and sensor in a certain sense, when implemented with digital computers.

In this paper, we will give a survey of some basic results on the maximum capability of adaptive feedback, which were discovered and established in the recent few years ([15–22]). To be specific, we will study some basic classes of discrete-time parametric and non-parametric non-linear control systems in Sections 2 and 3, respectively. Section 4 will turn to sampled-data systems, and Section 5 will focus on time-varying linear systems with hidden Markovian jumps. Some conclusions will be given in Section 6.

2. PARAMETRIC NON-LINEAR SYSTEMS

Consider the following basic discrete-time parametric non-linear system:

$$y_{t+1} = \theta f(y_t) + u_t + w_{t+1} \tag{1}$$

where y_t, u_t and w_t are the scalar system output, input and noise processes, respectively. We assume that

(A1) $\{w_t\}$ is a Gaussian noise process;

(A2) θ is an unknown non-degenerate Gaussian random variable;

(A3) The function $f(\cdot)$ is known and has the following growth rate:

$$f(x) \sim Mx^b \quad \text{as } x \to \infty$$

where $b \geqslant 0$, $M > 0$ are constants. Obviously, if $b \leqslant 1$, then the non-linear function $f(\cdot)$ has a growth rate which is bounded by linear growth. This case can be easily dealt with by the existing theory in adaptive control (see, e.g. Reference [15]). Our prime concern here is to know whether or not the system can be globally stabilized by adaptive feedback for any $b > 1$? To answer this question rigorously, let us introduce the precise definition of adaptive feedback first.

Definition 1

An input sequence $\{u_t, t \geqslant 0\}$ is called adaptive feedback, if at each time $t \geqslant 0$, u_t belongs to $\sigma(y_0, \ldots, y_t)$, the σ-algebra generated by $\{y_0, \ldots, y_t\}$, or, in other words, if there exists a Lebesgue measurable function $f_t(\cdot)$ such that $u_t = f_t(y_0, y_2, \ldots, y_t)$. Furthermore, system (1) is said to be globally stabilizable if there exists a sequence of adaptive feedback $\{u_t\}$ such that the output process is bounded in the mean square sense, i.e.

$$\overline{\lim_{t \to \infty}} E y_t^2 < \infty, \quad \forall y_0 \in R^1$$

The following theorem gives a critical value of b, which characterizes the maximum capability of adaptive feedback.

Theorem 1

Consider system (1) with Assumptions (A1) to (A3). Then $b = 4$ is a critical case for adaptive stabilizability. In other words,

(i) If $b \geqslant 4$, then for any adaptive feedback $\{u_t\}$, there always exist a set D (in the basic probability space) with positive probability such that

$$|y_t| \to \infty \quad \text{on } D$$

at a rate faster than exponential.

(ii) If $b < 4$, then the least-squares-based adaptive minimum variance control $u_t = -\theta_t f(y_t)$ where θ_t is the least-squares estimate for θ at time t, can render the system to be globally stable and optimal, with the best rate of convergence ([13]):

$$\sum_{t=1}^{T} (y_t - w_t)^2 = O(\log T) \text{ a.s.} \quad \text{as } T \to \infty$$

Remark 1

This result is somewhat surprising since the assumptions in our problem formulation have no explicit relationships with the value $b = 4$. So, a natural question that one may ask is: why $b = 4$ is a critical value for stabilizability?

Exploring the answer to this question may be interesting, but we remark that many other phenomena in science and nature are also related to the number 4, e.g. the maximum degree of algebraic equations whose general solutions can be expressed by explicit formulas (Abel Theorem), and the dimension of the space–time structure in Einstein's relativity theory. We remark that the related results were first found and established in a somewhat general framework in Reference [16]. In particular, the first part (i) was contained in Remark 2.2 in Reference [16], and was later extended to general unknown parameter case in Reference [15] by using a conditional Cramer–Rao inequality. The second part (ii) is a special case of Theorem 2.2 in Reference [16].

Remark 2

There are many implications of Theorem 1. For example,

(i) The limitation of adaptive feedback given in Theorem 1 (i) is applicable also to general class of systems of the form

$$y_{t+1} = f_t(y_t, \ldots, y_{t-p}, u_t, \ldots, u_{t-q}) + w_{t+1}$$

as long as it contains the basic class (1) as a subclass.

(ii) There are fundamental differences between continuous-time and discrete-time non-linear adaptive control. To see this, consider the following continuous-time analogue of the stochastic model (1):

$$\mathrm{d}y_t = [\theta f(y_t)\mathrm{d}t + u_t]\,\mathrm{d}t + \mathrm{d}w_t$$

where θ is an unknown parameter, w_t is the standard Brownian motion, and $f(\cdot)$ has a growth rate $f(x) = O(x^b)$ with $b > 0$. Then the non-linear damping control

$$u_t = -cy_t + y_t|y_t|^{2b}, \quad c > 0$$

can globally stabilize the system regardless of the value of b (see Reference [15]). However, Theorem 1 shows that the value of b plays a crucial role in stabilizability of discrete-time systems. This fact presages the limitations of sampled-data feedback to be studied in Section 4.

(iii) There are also fundamental differences between deterministic and stochastic adaptive control. To see this, consider the following noise-free system:

$$y_{t+1} = \theta f(y_t) + u_t$$

Then, one can identify the unknown parameter θ in the first step by $\theta = |y_1 - u_0|/f(y_0)$. Consequently, for $t \geqslant 1$, if we take $u_t = -\theta f(y_t)$, then it is obviously stabilizing, regardless of the growth rate of the non-linear function $f(\cdot)$. Hence, by Theorem 1, the fundamental difference between deterministic and stochastic adaptive control is clearly seen.

From the above analyses, it is seen that the noise effect in (1) plays an essential role in the non-stabilizability result of Theorem 1(i): the noise effect gives estimation errors to even the best estimates, which are then amplified step by step by the non-linearity of the system, leading to the final instability of the closed-loop systems, despite of the strong consistency of the estimates [16].

Theorem 1 concerns with the case where the unknown parameter θ is a scalar. To see what happens when the number of the unknown parameters increases, let us consider the following polynomial non-linear regression:

$$y_{t+1} = \theta_1 y_t^{b_1} + \theta_2 y_t^{b_2} + \cdots + \theta_p y_t^{b_p} + u_t + w_{t+1} \tag{2}$$

Assume that

(A1)′ $b_1 > b_2 > \cdots > b_p > 0$;
(A2)′ $\{w_t\}$ is a sequence of independent random variable with a common distribution $N(0, 1)$;
(A3)′ $\theta \triangleq [\theta_1 \cdots \theta_p]^T$ is a random parameter with distribution $N(\bar{\theta}, I_p)$ (extensions to non-Gaussian parameter case can be found in Reference [15]).

Denote

$$f(x) = [x^{b_1}, x^{b_2}, \ldots, x^{b_p}]^T$$

Then (2) can be written as

$$y_{t+1} = \theta^T f(y_t) + u_t + w_{t+1}$$

Note that

$$\|f(x)\| \sim x^{b_1}, \quad x \to \infty$$

we see that b_1 may be regarded as the degree of non-linearity of the non-linear function $f(\cdot)$. As explained before, we only consider the non-trivial case $b_1 > 1$ here.

Now, introduce a characteristic polynomial

$$P(z) = z^{p+1} - b_1 z^p + (b_1 - b_2) z^{p-1} + \cdots + (b_{p-1} - b_p) z + b_p$$

which plays a crucial role in characterizing the limitations of adaptive feedback as shown in the following theorem.

Theorem 2

If there exists a real number $z \in (1, b_1)$ such that $P(z) < 0$, then the above system (2) is not stabilizable by adaptive feedback. In fact, for any adaptive feedback $\{u_t\}$ and any initial condition $y_0 \in \mathscr{R}^1$, it is always true that

$$E|y_t|^2 \to \infty \quad \text{as } t \to \infty$$

at a rate faster than exponential.

Remark 3

The proof the Theorem 2 can be found in References [15, 17]. An important consequence of this theorem is that the following system:

$$y_{t+1} = \theta_1 y_t^{b_1} + \theta_2 y_t^{b_2} + \cdots + \theta_p y_t^{b_p} + u_t + w_{t+1}$$

is not stabilizable by adaptive feedback in general, whenever $b_1 > 1$ and the number of unknown parameters p is large (see Reference [17] for details). This fact implies that the non-linear system

$$y_{t+1} = f(y_t) + u_t + w_t$$

with $f(\cdot)$ unknown and satisfying

$$\|f(x)\| \leqslant c_1 + c_2 \|x\|^b, \quad b > 1$$

may not be stabilizable by adaptive feedback in general. This may further imply that the linear growth condition on $f(\cdot)$ cannot be essentially relaxed in a certain sense. This gives us another fundamental limitation on adaptive feedback in the presence of parametric uncertainties in non-linear systems, and motivates the study of non-parametric control systems with linear growth conditions in the next section.

3. NON-PARAMETRIC NON-LINEAR SYSTEMS

Consider the following first-order non-parametric control system:

$$y_{t+1} = f(y_t) + u_t + w_{t+1}, \quad t \geqslant 0, \quad y_0 \in \mathscr{R}^1 \tag{3}$$

where $\{y_t\}$ and $\{u_t\}$ are the output and input, and $\{w_t\}$ is an 'unknown but bounded' noise sequence, i.e. $|w_t| \leqslant w$, $\forall t$, for some constant $w > 0$. The non-linear function $f(\cdot) : \mathscr{R}^1 \to \mathscr{R}^1$ is completely unknown. We are interested in understanding how much uncertainty in $f(\cdot)$ can be dealt with by adaptive feedback? In order to do this, we need to introduce a proper measure of uncertainty first.

Now, define

$$\mathscr{F} \triangleq \{f : \mathscr{R}^1 \to \mathscr{R}^1\}$$

and introduce a quasi-norm on $\mathscr{F}$ as follows:

$$\|f\| \triangleq \lim_{\alpha \to \infty} \sup_{(x,y) \in \mathscr{R}^2} \frac{|f(x) - f(y)|}{|x - y| + \alpha}, \quad \forall f \in \mathscr{F}$$

where the limit exists by the monotonicity of the function concerned.

It is not difficult to see [19] that this norm is closely related to the following generalized Lipschitz condition:

$$|f(x) - f(y)| \leqslant L|x - y| + c$$

In fact, $\|f\|$ is the infimum of all possible Lipschitz constants L for $f(\cdot)$.

Having introduced the norm $\|\cdot\|$, we can then define a ball in the space $(\mathscr{F}, \|\cdot\|)$ centred at its: 'zero' θ_F with radius L:

$$\mathscr{F}(L) \triangleq \{f \in \mathscr{F} : \|f\| \leqslant L\}$$

where $\theta_F \triangleq \{f \in \mathscr{F} : \|f\| = 0\}$. It is obvious that the size of $\mathscr{F}(L)$ depends on the radius L, which will be regarded as the measure of the size of uncertainty in our study to follow.

The following theorem establishes a quantitative relationship between the capability of feedback and the size of uncertainty.

Theorem 3

Consider the non-parametric control system (3). Then the maximum uncertainty that can be dealt with by adaptive feedback is a ball with radius $L = \frac{3}{2} + \sqrt{2}$ in the normed function space $(\mathscr{F}, \|\cdot\|)$, centred at the zero θ_F. To be precise,

(i) If $L < \frac{3}{2} + \sqrt{2}$, then there exists an adaptive feedback $\{u_t\}$ such that for any $f \in \mathscr{F}(L)$, the corresponding closed-loop control system (3) is globally stable in the sense that

$$\sup_{t \geqslant 0} |y_t| + |u_t| < \infty, \quad \forall y_0 \in \mathscr{R}^1$$

(ii) If $L \geqslant \frac{3}{2} + \sqrt{2}$, then for any adaptive feedback $\{u_t\}$ and any $y_0 \in \mathscr{R}^1$, there always exists some $f \in \mathscr{F}(L)$ such that the corresponding closed-loop system (3) is unstable, i.e.

$$\sup_{t \geqslant 0} |y_t| = \infty$$

Remark 4

The proof of the above theorem is given in Reference [19], where it is also shown that once the stability of the closed-loop system is established, it is a relatively easy task to evaluate the control performance.

Similar to Theorem 1, a natural question that one may ask for Theorem 3 is why $L = \frac{3}{2} + \sqrt{2}$ is a critical value? Analytically, what we can say is that this result has some connections with oscillation theory. In fact, it can be seen from [19] that the adaptive stabilizability of system (3) is closely related to the behaviour of the trajectory of the following difference equation:

$$a_{n+1} = (L + \tfrac{1}{2})a_n - La_{n-1}$$

It can be shown that the necessary and sufficient condition for any solution of this equation either converges to zero or oscillates about zero is that $L < \frac{3}{2} + \sqrt{2}$. Geometrically, Theorem 3 also has connections with the following 'geometric inequality':

$$|a_{t+1} - (\text{centre})_t| \leqslant L|a_t - (\text{neighbour})_t| + 1$$

where $(\text{centre})_t$ is the centre of the data observed up to t, defined by $\frac{1}{2}(\min_{0<i\leqslant t} a_i + \max_{0\leqslant i\leqslant t} a_i)$, while the $(\text{neighbour})_t$ is defined as the point in $\{a_i, 0 \leqslant i \leqslant t-1\}$ which is closest to a_t. It can be shown that the necessary and sufficient condition for the boundedness of any non-negative sequence $\{a_t\}$ satisfying the above inequality is $L < \frac{3}{2} + \sqrt{2}$.

Remark 5

The adaptive feedback used in Theorem 3 (i) is constructive, it is constructed as follows (see Reference [19]):

Step 1: Define the nearest neighbour (NN) estimator by

$$\hat{f}_t(y_t) \triangleq y_{i_t+1} - u_{i_t}$$

where

$$i_t \triangleq \arg\min_{0 \leqslant i \leqslant t-1} |y_t - y_i|$$

which means that

$$|y_t - y_{i_t}| = \min_{0 < i \leqslant t-1} |y_t - y_i|$$

Step 2: Define u_t as a switching feedback based on a stabilizing feedback u'_t and a tracking feedback u''_t, i.e.

$$u_t = \begin{cases} u'_t & \text{if } |y_t - y_{i_t}| > \varepsilon \\ u''_t & \text{if } |y_t - y_{i_t}| \leqslant \varepsilon \end{cases}$$

where $\varepsilon > 0$ is any given threshhold, and

$$u_t' = -\hat{f}_t(y_t) + \tfrac{1}{2}(\underline{b}_t + \bar{b}_t)$$

where

$$\underline{b}_t = \min_{0 \leqslant i \leqslant t} y_i, \quad \bar{b}_t = \max_{0 \leqslant i \leqslant t} y_i$$

and

$$u_t'' = -\hat{f}_t(y_t) + y_{t+1}^*$$

where $\{y_t^*\}$ is a bounded reference sequence. It is obvious that u_t depends only on the observations $\{y_0, y_1, \ldots, y_t\}$.

Remark 6

One may try to generalize Theorem 3 to the following high-order non-linear systems ($p \geqslant 1$):

$$y_{t+1} = f(y_t, y_{t-1}, \ldots, y_{t-p+1}) + u_t + w_{t+1} \tag{4}$$

where $f(\cdot): R^p \to R^1$ is assumed to be completely unknown, but belongs to the following class of Lipschitz functions:

$$\mathscr{F}(L) = \{f(\cdot) : |f(x) - f(y)| \leqslant L\|x - y\|, \ \forall x, y \in R^p\}$$

where $L > 0$, $\|x\| = \sum_{i=1}^p |x_i|$, $x = (a_1, \ldots, x_p)^\tau \in R^p$. Again, $\{w_t\}$ is a sequence of 'unknown but bounded' noises. It can be shown that (see Reference [22]) if L and p satisfy

$$L + \frac{1}{2} \geqslant \left(1 + \frac{1}{p}\right)(pL)^{1/(p+1)} \tag{5}$$

then there does not exist any globally stabilizing adaptive feedback for the class of uncertain systems (4) with all $f \in \mathscr{F}(L)$.

It is easy to see that if $p = 1$ then the above inequality (5) reduces to $L \geqslant \frac{3}{2} + \sqrt{2}$, which we know to be the critical case for feedback capability by Theorem 3.

However, when $p > 1$ and (5) does not hold, whether or not there exists a stabilizing adaptive feedback for (4) with $f \in \mathscr{F}(L)$ still remains as an open question.

4. SAMPLED-DATA SYSTEMS

Consider the following basic control system:

$$\dot{x}_t = f(x_t) + u_t, \quad t \geqslant 0, \, x_0 \in R^1 \tag{6}$$

The system signals are assumed to be sampled at a constant rate $h > 0$, and the input is assumed to be implemented via the familiar zero-order hold device (i.e. piecewise constant functions):

$$u_t = u_{kh}, \quad kh \leqslant t < (k+1)h \tag{7}$$

where u_{kh} depends on $\{x_0, x_h, \ldots, x_{kh}\}$.

Similar to Definition 1, we need a precise definition for sampled-data feedback.

Definition 2

$\{u_t, t \geqslant 0\}$ is called a *sampled-data feedback control* if at each step k, u_{kh} is a causal function of the past and present sampled data $\{x_0, x_h, \ldots, x_{kh}\}$, i.e. for any $k \geqslant 0$ there exists a function $g_k(\cdot) : R^{k+1} \to R^1$ such that $u_{kh} = g_k(x_0, x_h, \ldots, x_{kh})$.

The non-linear function f in (6) is assumed to be unknown but belongs to the following class of local Lipschiz (LL) functions:

$$G_c^L = \{f | f \text{ is LL and satifies } |f(x)| \leqslant L|x| + c, \ \forall x \in R^1\} \tag{8}$$

where $c > 0$ and $L > 0$ are constants. A function f is called LL if, for any $R > 0$, there exists a constant L such that $|f(x) - f(y)| \leqslant L|x - y|$, $\forall (x, y) : |x| \leqslant R, |y| \leqslant R$.

According to the definition, L upper bounds the 'slope' of the unknown non-linear function $f \in G_c^L$, which may be regarded as a measure of the size of the uncertainty. Similar to the discrete-time case in Theorem 3, L plays a crucial role in the determination of the capability and limits of the sample-data feedback.

Theorem 4

Consider the sampled-data control system (6) and (7). If $Lh > 7.53$, then for any $c > 0$ and any sampled-data control $\{u_{kh}, k \geqslant 0\}$ there always exists a function $f^* \in G_c^L$, such that the state signal of (6) and (7) corresponding to f^* with initial point $x_0 = 0$ satisfies $(k \geqslant 1)$

$$|x_{kh}| \geqslant \left(\frac{Lh}{2}\right)^{k-1} ch \underset{k \to \infty}{\to} \infty$$

Remark 7

It is easy to show (see Reference [18]) that if $Lh < \log 4$, then a globally stabilizing sampled-data feedback control can be constructed. Theorem 7 shows that if Lh is larger than a certain value, then there will exist no stabilizing sampled-data feedback. As obvious open question here is how to bridge the gap between $\log 4$ and 7.53. Needless to say, Theorem 4 gives us some useful quantitative guidelines in properly choosing the sampling rate in practical applications.

Next, we consider the following stochastic control system:

$$\mathrm{d}x_t = f(x_t)\,\mathrm{d}t + u_t\,\mathrm{d}t + \sigma\,\mathrm{d}w_t \tag{9}$$

where f is a non-linear function, and where $\{w_t\}$ is the standard Brownian motion, and $\sigma > 0$.

As in the deterministic case, if $f \in G_c^L$ and if Lh is suitably small, then a stabilizing sampled-data feedback can be constructed (see Reference [18]). A natural question is: Can we find a stabilizing sampled-data control for (9) when the unknown non-linear function has a non-linear growth rate? The following theorem in Reference [18] gives us a negative answer for a class of non-linear stochastic systems, even in the case where the sampling rate is arbitrarily high and where the non-linear function is known *a priori*.

Theorem 5

Consider the stochastic control system (9). Assume that the function $f(x)$ is locally Lipschitz and there exist two positive constants R_0 and δ such that

$$x f(x) \geqslant |x|^{2+\delta}, \quad \forall x : |x| \geqslant R_0$$

Then for *any* $h > 0$ and *any* sampled-data feedback, the closed-loop system is unstable in the sense that:

$$Ex_T^2 = \infty, \quad \forall T > 0.$$

Remark 8

This theorem can be used to demonstrate the fundamental differences between continuous-time control and sampled-data control. For example, if we take $f(x) = |x|^{1+\delta}\,\mathrm{sgn}(x)$ then the condition on f in Theorem 5 is obviously satisfied, and so, there is no stabilizing sampled-data feedback for any $h > 0$. However, if we simply take the continuous-time state feedback control $u_t = -|x_t|^{1+\delta}\,\mathrm{sgn}(x_t) - x_t$, it is then easy to show that this continuous-time feedback will globally stabilize the system in the sense that $\sup_{T>0} Ex_T^2 < \infty$, $\forall x_0$. In Theorem 5, the random noise described by the Brownian motion $\{w_t\}$ plays an essential role. In the noise free case, it can be shown by examples that some standard continuous-time stabilizing feedback laws may indeed lose their stabilizability if their corresponding sampled-data versions are implemented (see Reference [15]).

5. TIME-VARYING STOCHASTIC SYSTEMS

Consider the following linear time-varying stochastic model:

$$x_{t+1} = A(\theta_t)x_t + B(\theta_t)u_t + w_{t+1}, \quad t \geqslant 1 \tag{10}$$

where $x_t \in R^n$, $u_t \in R^m$ and $w_{t+1} \in R^n$ are the state, input and noise vectors respectively. We assume that

(H1) $\{\theta_t\}$ is an unobservable Markov chain which is homogeneous, inreducible and aperiodic, and which takes values in a finite set $\{1, 2, \ldots, N\}$ with transition matrix denoted by $P = (p_{ij})_{\mathrm{NN}}$, where by definition $p_{ij} = P\{\theta_t = j | \theta_{t-1} = i\}$.

(H2) There exists some mn matrix L such that

$$\det[(A_i - A_j) - (B_i - B_j)L] \neq 0, \quad \forall i \neq j$$

where $1 \leqslant i, j \leqslant N$, and $A_i \triangleq A(i) \in R^{nn}$, $B_i \triangleq B(i) \in R^{nm}$ are the system matrices.

(H3) $\{w_t\}$ is a martingale difference sequence which is independent of $\{\theta_t\}$, and satisfies

$$\sigma I \leqslant Ew_t w_t', \quad Ew_t' w_t \leqslant \sigma_w, \ \forall t$$

where σ and σ_w are two positive constants, and the prime superscript represents matrix transpose.

We remark that Condition (H1) implies that each state in $\{1, 2, \ldots, N\}$ can be visited by $\{\theta_t\}$ with positive probability when t is suitably large (this is what we really need in Theorem 6 below), while Condition (H2) is a sort of identifiability condition useful in the construction of stabilizing feedback laws (it may be further weakened). Moreover, the lower bound to the noise covariance in Condition (H3) is assumed for simplicity of derivations, and the case where $w_t = 0, \ \forall t$ can be treated analogously.

For simplicity of presentation, we will denote $S \triangleq \{1, 2, \ldots, N\}$ throughout the sequel. We will continue to adopt the general definition for adaptive feedback as in Definition 1, but with y_t replaced by x_t for the current model (10). The following theorem gives a complete characterization of adaptive stabilizability for the hidden Markovian model (10).

Theorem 6

Let the above Assumptions (H1)–(H3) hold for the control system (10). Then the system is stabilizable by adaptive feedback if and only if the following coupled algebraic Riccati-like equations have a solution consisting of N positive definite matrices $\{M_i > 0,\ i \in S\}$:

$$\sum_j A_j' p_{ij} M_j A_j - \left(\sum_j A_j' p_{ij} M_j B_j\right)\left(\sum_j B_j' p_{ij} M_j B_j\right)^+ \left(\sum_j B_j' p_{ij} M_j A_j\right) - M_i = -I \qquad (11)$$

where $i \in S$ and $(\cdot)^+$ denotes the Moore–Penrose generalized-inverse of the corresponding matrix.

Remark 9

In contrast to most of the previous publications in the literature of adaptive control, we have neither restricted ourselves to the class of linear feedback laws, nor imposed any conditions on the rate of parameter changes. Hence, Theorem 6 enables us to explore the full capability and limitations of the adaptive feedback mechanism.

Also, as shown in Reference [21], the stabilizing feedback in Theorem 6 can be adaptively constructed, and three other conditions equivalent to (11) can be found.

Remark 10

It was shown in Reference [23] that for the case where the Markov chain $\{\theta_t\}$ is observable, System (9) is stabilizable by linear feedback if the following Lyapunov-like equation has a solution $\{M_i, i \in S\}$:

$$(A_i - B_i L_i)' \left[\sum_{j=1}^N p_{ij} M_j\right] (A_i - B_i L_i) - M_i = -I, \quad i \in S \qquad (12)$$

Obviously, the existence of the solution of Equation (11) derived in the present (adaptive) case should imply the existence of the solution of (12), but the converse assertion is not true. (see Reference [21] for detailed analyses).

To given an explicit comparison between (11) and (12), we denote

$$L_i^* \triangleq L^*(M) \triangleq \left(\sum_{j=1}^N B_j' p_{ij} M_j B_j\right)^+ \left(\sum_{j=1}^N B_j' p_{ij} M_j A_j\right), \quad i \in S$$

where $M \triangleq [M_1, \ldots, M_N]^i$. Then by properties of generalized-inverse, it is easy to see that (11) can be written in the following form:

$$\sum_{j=1}^N (A_j - B_j L_i^*) p_{ij} M_j (A_j - B_j L_i^*) - M_i = -I, \quad i \in S \qquad (13)$$

Remark 11

Theorem 6 shows that the capability of adaptive feedback depends on both the structure complexity measured by $\{A_j, B_j, 1 \leqslant j \leqslant N\}$ and the information uncertainty measured by $\{p_{ij}, 1 \leqslant i, j \leqslant N\}$. To make it more clear in understanding how the capability of adaptive feedback depends on both the complexity and uncertainty of the system, we consider the following example.

Example 1

Consider the following simple scalar model

$$x_{t+1} = a(\theta_t)x_t + u_t + w_{t+1} \tag{14}$$

Assume that the Markov chain $\{\theta_t\}$ have two states $\{1, 2\}$ only and let $p_{12} = p_{21}$. Now by the fact that $N = 2$, $p_{12} = p_{21}$, and $p_{i1} + p_{i2} = 1$, $i = 1, 2$, it can be shown by Theorem 6 and (13) that the system is stabilizable if and only if

$$(a_1 - a_2)^2(1 - p_{12})p_{12} < 1$$

(see, also Reference [20]). Let us denote $I(p_{12})$ as

$$I(p_{12}) \triangleq 1 - (a_2 - a_1)^2(1 - p_{12})p_{12}$$

which may be regarded as a measure of the capability of adaptive feedback and may be further represented by

$$I(p_{12}) = 1 - CU \tag{15}$$

where $C \triangleq (a_2 - a_1)^2$ and $U \triangleq (1 - p_{12})p_{12}$ can be interpreted as measures of the structure complexity (degree of dispersion) and the information uncertainty of the system, respectively. Obviously, the system is stabilizable $\Leftrightarrow I(p_{12}) > 0$. We now have the following interesting observations (cf. Reference [20]):

(i) The capability of adaptive feedback defined by $I(p_{12})$ is a monotonic function of the uncertainty U but is not monotonic in the rate of transition p_{12}. Furthermore, there does not exist a critical rate of transition $p_{12}^* \in (0, 1)$ such that the system is stabilizable or $I(p_{12}) > 0 \Leftrightarrow p_{12} \in [0, p_{12}^*)$. Moreover, the capability of adaptation $I(p_{12})$ achieves its maximum when the uncertainty U reaches its minimum.

(ii) The uncertainty measure U is closely related to the well-known Shannon information entropy, which is a measure of information uncertainty defined by

$$H = -\sum_{i=1}^{2} p_{1i} \log p_{1i}$$

in the current case (see, e.g. Reference [24]). Note that H can be rewritten as

$$H = -(1 - p_{12})\log(1 - p_{12}) - p_{12} \log p_{12}$$

It is not difficult to see that there exists a monotonically increasing function $m(\cdot)$ such that $U = m(H)$. This fact justifies why we refer to U as the measure of information uncertainty here.

Now, by (15) we have

$$I(p_{12}) = 1 - Cm(H)$$

which implies that the capability of adaptive feedback is also a monotonically decreasing function of the Shannon information entropy H.

6. CONCLUSIONS

Adaptive feedback is essential in dealing with uncertainties that always exist in the modelling of complex systems. In this paper, we have presented a survey of some basic ideas and results towards understanding the maximum capability (and limits) of the discrete-time (or sampled-data) adaptive feedback in dealing with both structure and disturbance uncertainties. In particular, we have presented several critical values or equations characterizing the maximum capability of the adaptive feedback. The basic findings may be briefly summarized as follows:

(i) For non-linear systems with parametric uncertainties, the capability of adaptive feedback depends on both the growth rate of the non-linear functions and the number of unknown parameters. In the scalar parameter case, the growth rate $O(x^4)$ is a critical case for adaptive stabilizability.
(ii) For non-linear systems with non-parametric uncertainties, the capability of feedback depends on the minimum Lipschitz constant L of the non-parametric function and on the order of the systems. In the first-order case, the value $L = \frac{3}{2} + \sqrt{2}$ is a critical case for adaptive stabilizability.
(iii) For continuous-time non-linear systems with non-parametric uncertainties, the capability of the sampled-data feedback depends both on the Lipschitz constant (or the slope) L of the non-parametric function and on the sampling period h of the feedback. If Lh is not suitably small, then no stabilizing sampled-data feedback exists.
(iv) For time-varying linear systems with hidden Markovian jump parametres, the capability of adaptive feedback is characterized by the existence of the solution of a set of coupled Riccati-like equations. Roughly speaking, the capability of adaptive feedback depends on both the structure complexity determined by the state matrices (A_i, B_i), $1 \leqslant i \leqslant N$, and the information uncertainty described by the transition probability $\{p_{ij}, 1 \leqslant i \leqslant N\}$. The rate of parameter changes is not a proper measure for characterizing the feedback capability.

It is obvious that many important problems still remain open, and more research efforts are called for in this new and challenging direction.

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