

# A Note on Overshoot Estimation in Pole Placements

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## Abstract

In this note we show that for a given controllable pair  $(A, B)$  and any  $\lambda > 0$ , a gain matrix  $K$  can be chosen so that the transition matrix  $e^{(A+BK)t}$  of the system  $\dot{x} = (A + BK)x$  decays at the exponential rate  $e^{-\lambda t}$  and the overshoot of the transition matrix can be bounded by  $M\lambda^L$  for some constants  $M$  and  $L$  that are independent of  $\lambda$ . As a consequence, for any  $h > 0$ , a gain matrix  $K$  can be chosen so that the magnitude of the transition matrix  $e^{(A+BK)t}$  can be reduced by  $\frac{1}{2}$  (or by any given portion) over  $[0, h]$ . An interesting application of the result is in the stabilization of switched linear systems with any given switching rate (see [1]).

*Key words:* Linear system, transition matrix, Squashing Lemma.

## 1 Introduction

Consider a linear system

$$\dot{x} = Ax + Bu, \quad (1)$$

where  $x(\cdot)$  takes values in  $\mathbb{R}^n$ ,  $u(\cdot)$  takes values in  $\mathbb{R}^m$ , and where  $A$  and  $B$  are matrices of appropriate dimensions. Suppose  $(A, B)$  is a controllable pair. It is a well known fact that for any  $\lambda > 0$ , a gain matrix  $K$  can be chosen so that the transition matrix of the system  $\dot{x} = (A + BK)x$  decays exponentially at the rate of  $e^{-\lambda t}$ , that is, for some  $R > 0$ ,

$$\|e^{(A+BK)t}\| \leq Re^{-\lambda t},$$

where and hereafter  $\|\cdot\|$  denotes the operator norm induced by the Euclidean norm on  $\mathbb{R}^n$ . To get a faster decay rate, it is natural to consider a “higher gain” matrix  $K_1$ . However, such a gain matrix in general results in a bigger overshoot for the transition matrix  $e^{(A+BK_1)t}$ . In this note, we show that in the pole placement practice, a gain matrix  $K$  can be chosen so that the overshoot of the transition matrix  $e^{(A+BK)t}$  can be bounded by  $M\lambda^L$

for some constants  $M$  and  $L$  independent of  $\lambda$ . As a consequence, one sees that for any  $h > 0$ , a gain matrix  $K$  can be chosen so that the magnitude of the transition matrix  $e^{(A+BK)t}$  can be reduced by  $\frac{1}{2}$  (or by any given portion) over  $[0, h]$ . Note that this is a stronger requirement than merely requiring  $e^{(A+BK)t}$  to decay at an exponential rate. An interesting application of the result is in the stabilization of switched linear systems with a given switching rate (see [1]).

The estimate of the overshoots of transition matrices in the practice of pole assignments has been studied widely (see e.g. [5], [9] and [7]). Our main result in this note can be considered an enhancement of the Squashing Lemma (see [7], [6] and [4]) which says the following: for any  $\tau_0 > 0$ ,  $\delta > 0$ , any  $\lambda > 0$ , it is possible to find  $K$  such that

$$\|e^{(A+BK)t}\| \leq \delta e^{-\lambda(t-\tau_0)}. \quad (2)$$

In the current note, we show that  $K$  can be chosen so that the estimate in (2) can be strengthened to

$$\|e^{(A+BK)t}\| \leq M\lambda^L e^{-\lambda t}$$

for some constants  $M$  and  $L$  which are independent of  $\lambda$ . Our proof is constructive that shows explicitly how  $M$  and  $L$  are chosen.

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## 2 Main Result

In this section we present our main result.

**Proposition 2.1** Let  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^{n \times m}$  be two matrices such that the pair  $(A, B)$  is controllable. Then for any  $\lambda > 0$ , there exists a matrix  $K \in \mathbb{R}^{m \times n}$  such that

$$\left\| e^{(A+BK)t} \right\| \leq M \lambda^L e^{-\lambda t}, \quad \forall t \geq 0, \quad (3)$$

where  $L = (n-1)(n+2)/2$  and  $M > 0$  is a constant, which is independent of  $\lambda$  and can be estimated precisely in terms of  $A, B$  and  $n$ .

Comparing with the Squashing Lemma obtained in [7], Proposition 2.1 has two improvements: (i). In (2), the estimate on the transient overshoot is exponentially proportional to the decay rate  $\lambda$ , which resulted in an estimation of the transition matrix in terms of  $e^{-\lambda(t-\tau_0)}$  instead of  $e^{-\lambda t}$ . In (3), the estimate on the transient overshoot is proportional to  $\lambda^L$  instead of  $e^{\lambda \tau_0}$  as in (2). This distinction between the two types of estimations may be significant for some possible extensions of our results to systems with external inputs. (ii). The value of the constant  $M$  in estimate (3) can be precisely calculated by using our constructive proof (see equation (10) in the sequel). This is certainly a very desirable feature for practical purposes. See Example 3.1 for some illustrations.

Proposition 2.1 was primarily presented and applied to a stabilization problem of switched linear systems in [2]. It was found later that a recent paper [3] also provides a similar result with similar proofs. The difference is that [3] only considered the single input case and the upper bound  $M \lambda^L$  in (3) was found to be a polynomial  $p(\lambda)$  in [3] without an explicit expression. Hence, our result has obvious merits in control design.

**Proof of Proposition 2.1.** First we consider a linear system  $(A, b)$  of a single input. Without loss of generality, we assume that  $(A, b)$  is in the Brunovsky canonical form:

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_n \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

Let  $\lambda_1, \dots, \lambda_n$  be  $n$  distinct, negative real numbers. There exists some  $k \in \mathbb{R}^{1 \times n}$  such that the characteristic equation of the closed-loop system  $A + bk$  is  $p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$ . Note that the closed-loop system is given by

$$\begin{aligned} \dot{x}_1 &= x_2, & \dot{x}_2 &= x_3, & \dots, & \dot{x}_{n-1} &= x_n, \\ \dot{x}_n &= \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n \end{aligned}$$

for some  $\beta_1, \beta_2, \dots, \beta_n \in \mathbb{R}$ . Hence,  $x_1$  satisfies the equa-

tion

$$x_1^{(n)} = \beta_1 x_1 + \beta_2 \dot{x}_1 + \cdots + \beta_n x_1^{(n-1)}, \quad (4)$$

whose characteristic equation is the same as  $p(\lambda)$ . Hence, the general solution of (4) is

$$x_1(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \cdots + c_n e^{\lambda_n t},$$

where  $c_1, c_2, \dots, c_n$  are constants. From the equations  $x_2 = \dot{x}_1, x_3 = \dot{x}_2, \dots, x_n = \dot{x}_{n-1}$ , we have  $x(t) = \Lambda_0 e^{Dt} c$ , where

$$\Lambda_0 = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix},$$

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

and where  $c = (c_1 \ c_2 \ \cdots \ c_n)^T$ . Now, observe that  $x(0) = \Lambda_0 c$ , that is,  $c = \Lambda_0^{-1} x(0)$  (note that  $\Lambda_0$  is an invertible Vandermonde matrix). Comparing this with the transition matrix of the system, one sees that

$$e^{(A+bk)t} = \Lambda_0 e^{Dt} \Lambda_0^{-1}. \quad (5)$$

Let  $\lambda_{\max} = \max\{|\lambda_1|, \dots, |\lambda_n|\}$ . Without loss of generality, assume that  $\lambda_{\max} \geq 1$ . To get an estimate on  $\|\Lambda_0\|$  and  $\|\Lambda_0^{-1}\|$ , we need the following simple fact: for an  $n \times n$  matrix  $C$ , let  $c_{\max} = \max_{1 \leq i, j \leq n} |c_{ij}|$ . It is not hard to see that

$$\|C\| \leq n c_{\max}.$$

Hence, we have

$$\|\Lambda_0\| \leq n \lambda_{\max}^{n-1}. \quad (6)$$

To get an estimate on  $\Lambda_0^{-1}$ , first note that

$$\Lambda_0^{-1} = \frac{1}{\det \Lambda_0} \text{adj } \Lambda_0, \quad (7)$$

where  $\text{adj } \Lambda_0$  denotes the adjoint matrix of  $\Lambda_0$ , and that

$$\det \Lambda_0 = \prod_{j>i} (\lambda_j - \lambda_i).$$

Hence, if we choose  $\lambda_1, \dots, \lambda_n$  in such a way that  $\lambda_{i+1} \leq \lambda_i - 1$  with  $\lambda_1 < 0$ , we get  $|\det \Lambda_0| \geq 1$ .

Taking the structure of  $\text{adj}\Lambda_0$  into account, it is easy to see that for  $C = \text{adj}\Lambda_0$ ,

$$\begin{aligned} c_{\max} &\leq (n-1)! \lambda_{\max}^{1+2+\dots+(n-1)} \\ &= (n-1)! \lambda_{\max}^{\frac{n(n-1)}{2}}. \end{aligned}$$

Hence, by (7), we have

$$\|\Lambda_0^{-1}\| \leq \|\text{adj}\Lambda_0\| \leq n(n-1)! \lambda_{\max}^{\frac{n(n-1)}{2}}. \quad (8)$$

Consequently, (6) and (8) yield that

$$\begin{aligned} \|\Lambda_0 e^{Dt} \Lambda_0^{-1}\| &\leq n \lambda_{\max}^{n-1} \|e^{Dt}\| n(n-1)! \lambda_{\max}^{n(n-1)/2} \\ &\leq nn! \lambda_{\max}^{(n-1)(n+2)/2} e^{-\lambda_{\min} t}, \end{aligned}$$

where  $\lambda_{\min} = \min\{|\lambda_1|, \dots, |\lambda_n|\}$ .

Suppose for some  $\rho > 1$ ,  $\lambda_{\max} \leq \rho \lambda_{\min}$ . Then, it follows that

$$\|\Lambda_0 e^{Dt} \Lambda_0^{-1}\| \leq M \lambda_{\min}^{(n-1)(n+2)/2} e^{-\lambda_{\min} t}, \quad (9)$$

where

$$M = nn! \rho^{(n-1)(n+2)/2}. \quad (10)$$

In summary, we need the following conditions on the  $\lambda_i$ 's:

- $\lambda_1, \lambda_2, \dots, \lambda_n$  are distinct, real, and negative;
- $\lambda_{i+1} \leq \lambda_i - 1$  for  $1 \leq i \leq n-1$ , and hence,  $\lambda_{\max} = |\lambda_n|$ ,  $\lambda_{\min} = |\lambda_1|$ ;
- $|\lambda_n| \leq \rho |\lambda_1|$ , for some constant  $\rho > 1$ .

Obviously, for any given  $\lambda > 0$ , it is easy to choose  $\lambda_1, \dots, \lambda_n$  to satisfy all the above conditions together with the condition that  $\lambda_1 \leq -\lambda$ . For example, one can choose  $\lambda_1 < \min\{-1, -\lambda\}$ , and let  $\lambda_{i+1} = \lambda_i - 1$  for  $1 \leq i \leq n-1$ . Since  $|\lambda_n| = |\lambda_1 - (n-1)| \leq n |\lambda_1|$ , we see that  $\rho$  can be set as  $\rho = n$ .

With such choices of  $\lambda_1, \lambda_2, \dots, \lambda_n$ , we see from (5) and (9) that the desired result hold.

Now we consider the case when  $(A, b)$  is not in the Brunovsky canonical form. In this case, find an invertible  $T \in \mathbb{R}^{n \times n}$  such that  $(T^{-1}AT, T^{-1}b)$  is in the Brunovsky canonical form.

For any given  $\lambda > 0$ , the above proof has shown that for  $A_1 = T^{-1}AT$ ,  $b_1 = T^{-1}b$ , one can find  $k_0 \in \mathbb{R}^{1 \times n}$  such that

$$e^{(A_1 + b_1 k_0)t} \leq M \lambda^L e^{-\lambda t},$$

where  $M$  is given by (10) for some chosen  $\rho$ , and  $L = (n-1)(n+1)/2$ . Clearly, with  $k = k_0 T^{-1}$ , one has

$$e^{(A+bk)t} = T(e^{(A_1 + b_1 k_0)t})T^{-1} \leq M_1 \lambda^L e^{-\lambda t}, \quad (11)$$

where  $M_1 = M \|T\| \|T^{-1}\|$ .

Finally, we consider the multi-input system

$$\dot{x} = Ax + Bu, \quad (12)$$

where  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ . Suppose that the system is controllable. By Heymann's Lemma (c.f., e.g., page 187 of [8]), one sees that for any  $v \in \mathbb{R}^m$  such that  $b := Bv \neq 0$ , there exists some  $K_0 \in \mathbb{R}^{m \times n}$  such that  $(A + BK_0, b)$  is itself controllable. Hence, the conclusion of single-input case that has just been proved above is applicable to the controllable pair  $(A + BK_0, b)$ , and one then sees that there exists some  $k \in \mathbb{R}^{1 \times n}$  such that  $\|e^{(A+BK_0+bk)t}\| \leq M \lambda^L e^{-\lambda t}$  for all  $t \geq 0$ . Hence, with  $K = K_0 + vk$ , it holds that

$$\|e^{(A+BK)t}\| \leq M \lambda^L e^{-\lambda t} \quad \forall t \geq 0. \quad (13)$$

This completes the proof.  $\square$

**Remark 2.2** In the above proof, we have used the fact that for a single input system  $(A, b)$  which is controllable, when it is not in the Brunovsky canonical form, one can find an invertible matrix  $T$  such that  $(T^{-1}AT, T^{-1}b)$  is in the canonical form. To be more precise, the matrix  $T$  can be chosen as (see e.g., [8]):

$$T = \begin{pmatrix} b & Ab & \dots & A^{n-1}b \end{pmatrix} \begin{pmatrix} a_{n-1} & \dots & a_1 & 1 \\ \vdots & \vdots & 1 & 0 \\ a_1 & 1 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix},$$

where  $a_1, \dots, a_{n-1}$  are as in the characteristic polynomial of  $A$  given by

$$\det(sI - A) = s^n + a_1 s^{n-1} + \dots + a_{n-2} s + a_{n-1}.$$

From this one can find an estimate of  $\|T\|$  and  $\|T^{-1}\|$ , which in turn will lead to an estimate of  $M_1$  in (11).

### 3 An Example

The design technique is demonstrated in the following example.

**Example 3.1** Consider the following controllable linear system:

$$A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 2 & 1 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

With the help of MATLAB, we first calculate the transfer matrix

$$T_1 = \begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

With the transfer matrix  $T_1$ , one has

$$T_1^{-1}A_1T_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 2 \end{pmatrix}, \quad T_1^{-1}B_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Calculation shows that  $\|T_1\| = 1.80193754431757$  and  $\|T_1^{-1}\| = 2.24697960199992$ . Taking  $\rho = n (= 3)$ , we have

$$L = \frac{(n-1)(n+2)}{2} = 5, \quad (14)$$

$$M = \|T_1\| \|T_1^{-1}\| n! n^{(n-1)(n-2)/2} \approx 218.642. \quad (15)$$

Suppose for some design purpose, a decay constant  $\lambda = 49.894$  is given. Choosing  $\lambda_1 = -\lambda$ ,  $\lambda_2 = \lambda_1 - 1$ ,  $\lambda_3 = \lambda_2 - 1$ , the feedback  $K_1$  can be easily calculated (under the normal form) as

$$\tilde{K}_1 \approx \begin{pmatrix} -151.681 & -7769.474 & -131773.562 \end{pmatrix}.$$

Back to the original coordinate frame, we have

$$K_1 = \tilde{K}_1 T_1^{-1} \approx \begin{pmatrix} -124155.769 & 7769.474 & -7617.793 \end{pmatrix}.$$

With such a choice of  $K_1$ , we get the desired decay estimate

$$\|e^{(A+BK)t}\| \leq M\lambda^L e^{-\lambda t} \quad \forall t \geq 0,$$

for the given decay constant  $\lambda = 49.894$  with  $L$  and  $M$  given as in (14)–(15).  $\square$

#### 4 Conclusion

In this note we show that if  $(A, B)$  is controllable, then for any  $\lambda > 0$ , a gain matrix  $K$  can be chosen such that the transition matrix  $e^{(A+BK)t}$  decays at the exponential rate  $e^{-\lambda t}$  and the overshoot of  $e^{(A+BK)t}$  can be bounded by  $M\lambda^L$  for some constants  $M$  and  $L$  that are independent of the decay constant  $\lambda$ . The result provides a convenient tool for control design, particularly for switched systems, see [1].

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#### Erratum

There is a mild flaw in the statement of Proposition 2.1 in the above paper (cf. [1]). We restate it as follows.

**Proposition** Let  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^{n \times m}$  be two matrices such that the pair  $(A, B)$  is controllable. Then for any  $\lambda \geq 1$ , there exists a matrix  $K \in \mathbb{R}^{m \times n}$  such that

$$\|e^{(A+BK)t}\| \leq M\lambda^L e^{-\lambda t}, \quad \forall t \geq 0, \quad (16)$$

where  $L = (n-1)(n+2)/2$  and  $M > 0$  is a constant, which is independent of  $\lambda$  and can be estimated precisely in terms of  $A, B$  and  $n$ .

The proof of Proposition 2.1 in [1] is only valid for the case when  $\lambda \geq 1$  (instead of the original version of  $\lambda > 0$ ), because the eigenvalues  $\lambda_1, \dots, \lambda_n$  were chosen to satisfy  $\lambda_1 \leq -1$ , and  $\lambda_k \leq \lambda_1$  for  $k \geq 1$ . For more details, we refer the reader to the discussions that followed formula (10) in [1].

A main motivation of the work in [1] was for us to develop the results in [2]. As in most applications of overshoot estimation for pole placements, the parameter  $\lambda$  in [2] was chosen as a number of large value. Hence, the correction does not affect our results in [2].

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#### References

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