On Controllability of Some Classes of Affine Nonlinear Systems*

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To Lennart for his 60's birthday.

Abstract: In this paper, we will present some recent progress in both global controllability and global asymptotical controllability of affine nonlinear systems. Our method is based on some basic facts in planar topology and in the geometric theory of ordinary differential equations. We will first present a necessary and sufficient condition for global controllability of general planar affine nonlinear systems, and then will give its generalizations to some high dimensional systems. Asymptotical controllability results will also be discussed and necessary and sufficient conditions will also be presented. Finally, we will show that the new controllability criterion can be easily applied to a number of practical examples.

1 Introduction

The controllability of nonlinear systems has been studied extensively over the past three decades, and considerable progress has been made in either analysis or synthesis by introducing some useful methods, including the well-known differential geometric method, see, for example, Isidori (1995), Jurdjevic (1997), Khalil (1996), Sontag (1998), Agrachev and Sachkov (2004), Sussmann (1978), Sussmann and Jurdjevic (1972), Hermann and Krener (1977). However, most of the existing results in the literature on controllability of general nonlinear systems are of local nature.

As for global controllability, there are two main approaches in the literature. The first one is to analyze the structure of reachable sets by using, for example, the Chow's Theorem Chow (1939) in differential manifold, see, Lobry (1970), Brockett (1972), Sussmann (1973), Hunt (1980) among others. The other approach is to study the relationship between local and global controllability, in which the local results are to be extended to global ones under certain conditions, where the global topological structure of the manifold in question plays a key role, see, for example, Hermes (1974), Hirschorn (1976), Aeyels (1985), Caines and Lemch (2003) among others. Of course, there are also several other results on global controllability, for example, Hirschorn (1990), Lukes (1972). Due to

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the difficulty of this problem, no complete characterization of global controllability has been established up to now, and most of the related results are rather complicated.

To the best of our knowledge, there is lack of a complete characterization of global controllability even for the following seemingly simple planar affine nonlinear system,

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})u.$$

For example, Kaya and Noakes (1997) gave some necessary conditions only, and Hunt (1980) gave both necessary and sufficient conditions which are distinct. Further, the paper Aeyels (1984) studied the systems with $\boldsymbol{g}$ being a constant vector field, where a sufficient condition is given. Some generalizations of these results may be found in Aeyels (1984) and Hunt (1982). The most general results up to now seem to have been obtained in the book Nikitin (1994), where under some hypotheses on $\boldsymbol{f}$ and/or $\boldsymbol{g}$, some necessary and sufficient conditions on global controllability of planar affine nonlinear system were obtained. Some ideas and results in Nikitin (1994) are similar to those to be presented in this paper, but the conditions used in Nikitin (1994) appear to be unnecessary and stringent since, for example, the main results in Nikitin (1994) (see, p. 44 and p. 109) cannot include the standard controllability criterion even for linear systems.

In this paper, we will investigate the global controllability by using a new method to analyze the reachable set. A necessary and sufficient condition together with a new criterion for global controllability of planar affine nonlinear systems with single input will be given, under some natural hypotheses on $\boldsymbol{f}$ and $\boldsymbol{g}$ Sun and Guo (2005b), Sun and Guo (2005a). We will also give some generalizations to the case where the vector field $\boldsymbol{g}$ has a singular point Sun et al. (2006) and the high dimensional systems with special structure Sun et al. (2006), Sun (2006). Our analysis is based on the use of Jordan curve-like Theorem, Poincare-Bendixson Theorem, Whitney's smooth extension theorem Whitney (1934), and some other basic facts in the geometric theory of ordinary differential equations in the plane.

In addition, we will also consider another basic concept of nonlinear control systems—the global asymptotical controllability, a concept closely related to but somewhat weaker than the global stability. The asymptotic controllability has also been studied previously in the literature, see e.g., Brockett (1983), Bacciotti (1992), Sontag (1982), but most of the results are of local nature. In this paper, we will present a necessary and sufficient condition for global asymptotic controllability of the planar affine nonlinear control systems Sun and Guo (2005c). Similar to the concept of global controllability, we will also give some generalizations to high dimensional systems.

The rest of this paper is organized as follows: Section 2 will introduce both concepts on global controllability and global asymptotical controllability. Section 3 will present the main results on planar systems and Section 4 will generalize the results to some high dimensional systems. Some illustrative examples will be given in Section 5. Finally, Section 6 will conclude the paper.

2 Basic Concepts

Consider the following affine nonlinear systems

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}) + \boldsymbol{G}(\boldsymbol{x})\boldsymbol{u}, \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state vector, and $\boldsymbol{u} \in \mathbb{R}^m$ is the input vector, $\boldsymbol{F}(\boldsymbol{x}) \in \mathbb{R}^{n\times 1}$, $\boldsymbol{G}(\boldsymbol{x}) \in \mathbb{R}^{n\times m}$ are $\mathrm{C}^1(\mathbb{R}^n)$ matrix functions. We need the following definition of global controllability of nonlinear systems Isidori (1995), Jurdjevic (1997), Sussmann and Jurdjevic (1972).

Definition 1. The control system (1) is said to be **globally controllable**, if for any two points $\boldsymbol{x}^0$ and $\boldsymbol{x}^1 \in \mathbb{R}^n$, there exists a right continuous control vector function $\boldsymbol{u}(\cdot)$ such that the trajectory of the system (1) under $\boldsymbol{u}(\cdot)$ satisfies $\boldsymbol{x}(0) = \boldsymbol{x}^0$ and $\boldsymbol{x}(T) = \boldsymbol{x}^1$ for some finite time $T \geqslant 0$.

Let $\boldsymbol{F}(\boldsymbol{0}) = \boldsymbol{0}$, i.e. the origin is a equilibrium point of the vector field $\boldsymbol{F}(\boldsymbol{x})$. Then we need the following definition of local and global asymptotical controllability of the system (1) Brockett (1983).

Definition 2. The system (1) is said to be **locally asymptotical controllable** at the origin, if there exist two neighborhood U_1 and U_2 of the origin, such that for any initial point $\boldsymbol{x}(0) = \boldsymbol{x}^0 \in U_1$, there exists a right continuous matrix control function $\boldsymbol{u}(\cdot)$ which keeps the trajectory $\boldsymbol{x}(t), t \geqslant 0$ in U_2 and drives the state converging to zero, i.e., $\boldsymbol{x}(t) \to 0$ as $t \to +\infty$. If $U_1 = U_2 = \mathbb{R}^n$, then the system (1) is said to be **globally asymptotical controllable.**[1]

In this paper, we will study both global controllability and global asymptotical controllability firstly for general planar systems, then for some high dimensional systems.

3 Planar Systems

We will consider singular and nonsingular cases separately.

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[1] In some literature the state trajectory does not need to be kept in U_2, e.g. Brockett (1983).

3.1 Planar Systems without Singularity

Consider the following planar affine nonlinear control systems

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2) + g_1(x_1, x_2)u \\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)u,\end{aligned} \tag{2}$$

where $f_i(x_1, x_2)$, $g_i(x_1, x_2)$ are **locally Lipschitz functions**, $i = 1, 2$, and u is the system control function taking values on $\mathbb{R}$. Denote $\boldsymbol{x} = (x_1, x_2)^T$, $\boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}), f_2(\boldsymbol{x}))^T$, $\boldsymbol{g}(\boldsymbol{x}) = (g_1(\boldsymbol{x}), g_2(\boldsymbol{x}))^T$ and assume that $\boldsymbol{g}(\cdot)$ is nonsingular, i.e., $\boldsymbol{g}(\boldsymbol{x}) \neq 0, \forall\, \boldsymbol{x} \in \mathbb{R}^2$.

First of all, it is easy to show that for any point $\boldsymbol{x}^0$ in $\mathbb{R}^2$ and for any function $u(t)$, if $\det(\boldsymbol{f}(\boldsymbol{x}^0), \boldsymbol{g}(\boldsymbol{x}^0)) \neq 0$, then the vector field of the control system (2) at $\boldsymbol{x}^0$ points to one side of the straight-line which passes through the point $\boldsymbol{x}^0$ with direction $\boldsymbol{g}(\boldsymbol{x}^0)$; and if $\det(\boldsymbol{f}(\boldsymbol{x}^0), \boldsymbol{g}(\boldsymbol{x}^0)) = 0$, then the vector field parallels to this straight-line.

The well known Jordan curve Theorem in topology (see, e.g., Armstrong (1983) pp. 112-115) says that a simple closed curve C in the plane separates the plane into two disjoint components, of which C is the common boundary. The proof of the Jordan curve Theorem in Armstrong (1983) actually gives the following assertion : the curve which is homeomorphic to the straight-line with its two ends extending to infinite separates the plane into two disjoint components[2]. We call this result as **Jordan curve-like theorem**.

We are now in a position to give the following definitions.

Definition 3. A **control curve** of the system (2) is defined to be a solution $(x_1(t), x_2(t))$ of the following differential equation on the plane:

$$\begin{aligned}\dot{x}_1 &= g_1(x_1, x_2) \\ \dot{x}_2 &= g_2(x_1, x_2),\end{aligned}$$

where $g_i(\boldsymbol{x}), i = 1, 2$ are the same as those in (2).

Lemma 1

Any control curve of the system (2) is homeomorphic to the straight-line with its two ends extending to infinite.

The following theorem gives the necessary and sufficient condition for global controllability of the system (2).

Theorem 1

The necessary and sufficient condition for global controllability of the control system (2) *is that* $g_1(\boldsymbol{x})f_2(\boldsymbol{x}) - g_2(\boldsymbol{x})f_1(\boldsymbol{x})$ *changes its sign over any control curve.*

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[2]The two ends of a curve $\Gamma(t), t \in \mathbb{R}$ extending to infinite means that: $\|\Gamma(t)\| \to +\infty$, when $t \to +\infty$ and $-\infty$.

We may call the function $g_1(\boldsymbol{x})f_2(\boldsymbol{x})-g_2(\boldsymbol{x})f_1(\boldsymbol{x})$ as the **criterion function** for global controllability, denoted as $C(\boldsymbol{x})$.

Next, we let $\boldsymbol{f}(\mathbf{0}) = \mathbf{0}$ and investigate the global asymptotical controllability of the system (2).

The local stabilization issue has been well studied in the literature (see, e.g., Isidori (1995), Khalil (1996) and Bacciotti (1992)), for example, if $(\frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial \boldsymbol{x}}|_{x=0}, \boldsymbol{g}(\mathbf{0}))$ is controllable, then a locally stabilizing controller can be easily constructed. Hence, the locally asymptotical controllability of many systems is easy to be verified.

Our purpose here, however, is to study the more difficult globally asymptotical controllability problems. To this end, we need to introduce several new concepts.

By the Jordan curve-like theorem, the curve which is homeomorphic to the straight-line with its two ends extending to infinite separates the plane into two disjoint components (see Fig 1).

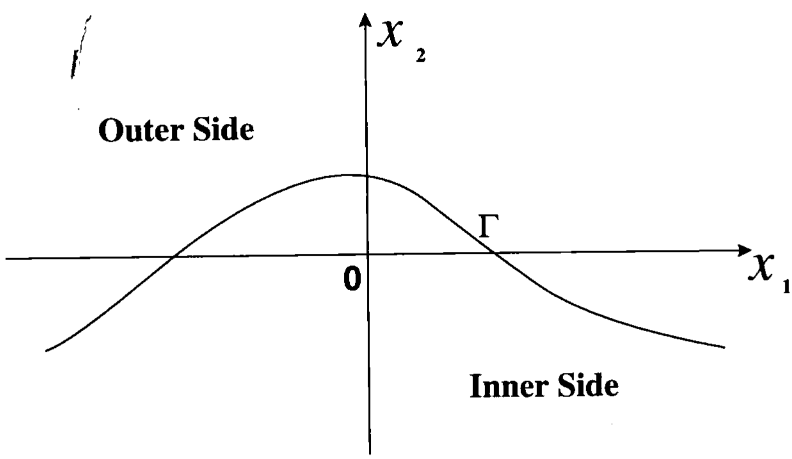

Figure 1

We are now in a position to give the following definitions.

Definition 4. The **inner side** of a curve which does not pass through the origin and is homeomorphic to the straight-line with its two ends extending to infinite, is defined as one of the above-mentioned components that contains the origin. The other component is accordingly called the **outer side** (see e.g., Fig 1).

Definition 5. A smooth curve $\Gamma : \gamma(s) \in \mathbb{R}^2, s \in \mathbb{R}$ which satisfies the conditions in Definition 4 is called a **P-curve** of system (2), if there exists $s_1 \in \mathbb{R}$ such that

$$L(s_1) < 0$$

holds for some function u, where $L(s) \triangleq \langle \boldsymbol{f}(\gamma(s)) + \boldsymbol{g}(\gamma(s))u, \boldsymbol{p}(s) \rangle$, and $\boldsymbol{p}(s)$ is a non-zero normal vector of $\gamma(s)$ which points to the outer side of Γ.

Proposition 1

A control curve $\Gamma : \gamma(s)$ of system (2) not passing through the origin is a P-curve if and only if there exists s_1, such that $\langle \boldsymbol{f}(\gamma(s_1)), \boldsymbol{p}(s_1) \rangle < 0$, where $\boldsymbol{p}(s)$ is non-zero normal vector of $\gamma(s)$, which points to the outer side of Γ.

This proposition is obvious, because $\boldsymbol{p}(s)$ is perpendicular to $\boldsymbol{g}(\gamma(s))$ for any s. Moreover, for any given control curve $\gamma(s)$ of system (2), $\boldsymbol{p}(s)$ can be taken as either $(-g_2, g_1)^T$ or $(g_2, -g_1)^T$. Consequently, $L(s)$ can be represented as

$$L(s) = \pm\{g_1(\gamma(s))f_2(\gamma(s)) - g_2(\gamma(s))f_1(\gamma(s))\}.$$

From this, we can immediately get the following easily verifiable condition for a P-curve.

Proposition 2

If the following function

$$g_1(\gamma(s))f_2(\gamma(s)) - g_2(\gamma(s))f_1(\gamma(s))$$

changes its sign over a control curve $\gamma(s)$ not passing through the origin, then $\gamma(s)$ is a P-curve.

The following theorem characterizes the additional condition needed for global asymptotical controllability, in addition to local asymptotical controllability.

Theorem 2

Let the system (2) be locally asymptotically controllable at the origin. Then the necessary and sufficient condition for global asymptotical controllability of the control system is that any control curve $\Gamma : \gamma(t)$ of the system not passing through the origin is a P-curve.

Remark 1. From Theorem 2, one can see that if there exists a control curve for the system (2) which is not a P-curve, then the system cannot be globally stabilized (see Khalil (1996)) by any control function u.

3.2 Planar Systems with Singularity

Now we consider the case where the control vector $g(x)$ is allowed to have a singular point, namely the following systems:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) + g_1(x_1, x_2)u \\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)u, \end{aligned} \tag{3}$$

where g or $-g$ is locally asymptotically stable at the origin and $g(\mathbf{0}) = \mathbf{0}$, $g(x) \neq \mathbf{0}, \forall\, x \in \mathbb{R}^2 \backslash \mathbf{0}$, f and g are the same as those in (2).[3] Without loss of generality, we suppose that $g(x)$ is locally asymptotically stable (see Khalil (1996)).

By Poincare-Bendixson Theorem and Jordan curve Theorem, every trajectory of the vector field $g(x)$ either tends to a singular point, or extends to infinite, or spirals around a limit cycle. We are now in position to give the following definition.

Definition 6. **A regular control curve** of the system (3) is defined to be a solution trajectory of the vector field $g(x)$ on the plane, which is either a nonzero closed curve or a curve whose two ends extend to infinite.

Theorem 3
For the control system (3), let $f(\mathbf{0}) \neq \mathbf{0}$ and $\mathcal{D}$ is the domain of attraction of $g(x)$. Then the system (3) is globally controllable, if and only if there are no points $x_+, x_- \in \mathcal{D}\backslash 0$ such that

$$C(\varphi(x_+, t)) \geqslant 0 \quad \forall\, t \in T_+,$$

$$C(\varphi(x_-, t)) \leqslant 0 \quad \forall\, t \in T_-,$$

and $C(x)$ changes its sign over every regular control curve, where $C(x)$ is the criterion function defined by $g_1(x)f_2(x) - g_2(x)f_1(x)$, $\varphi(x_*, t)$ denotes the trajectory of the vector field $g(x)$ passing through the point x_* and T_* is the existence interval of the trajectory $\varphi(x_*, t)$ and the star $*$ denotes the sign $+$ or $-$.

In particular, if the domain of attraction $\mathcal{D}$ is bounded, then the system (3) is globally controllable, if and only if the criterion function $C(x)$ changes its sign over every regular control curve.

Remark 2. Theorem 3 is a generalization of the Theorem 4.3 in Nikitin (1994), where the vector field g or $-g$ is assumed to satisfy a stronger condition, i.e. the global asymptotical stability condition. Here we only need local asymptotical stability, thanks to the Poincare-Bendixson Theorem.

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[3] A vector field $g(\cdot)$ is said to be locally asymptotical stable at the origin if its flow $\dot{x} = g(x)$ is so at the origin.

Now, we consider a kind of the degenerate case of the control system (3), where the domain of attraction $\mathcal{D}$ of $\boldsymbol{g}(\boldsymbol{x})$ degenerates into a point $\mathbf{0}$. If we view the point $\mathbf{0}$ as a degenerate control curve of the systems (3), we have the following corollary.

Corollary 1
Suppose $\boldsymbol{f}(\mathbf{0}) \neq \mathbf{0}$ and every control curve of the systems (3) is a closed curve. Then the control system (3) is globally controllable, if and only if the criterion function $C(\boldsymbol{x})$ changes its sign over every nonzero control curve.

In some cases, it is not difficult to verify whether every control curve of the systems (3) is a closed curve. For example, if there is a Lyapunov function $V(\boldsymbol{x})$, such that $V(\boldsymbol{x}) > 0$ for any $\boldsymbol{x} \in \mathbb{R}^2 \backslash \mathbf{0}$, $V(\mathbf{0}) = \mathbf{0}$, $V(\boldsymbol{x}) \to +\infty$ as $\|\boldsymbol{x}\| \to +\infty$, and $V(\boldsymbol{x})$ satisfies $\frac{\partial V}{\partial x_1} g_1 + \frac{\partial V}{\partial x_2} g_2 \equiv 0$, then every trajectory of the vector field $\boldsymbol{g}(\boldsymbol{x})$ is a closed curve, or every control curve of the systems (3) is a closed curve.

Next, we consider the global asymptotical controllability of the system (3). In this case, it is more complicated to define the P-curve, and we only consider a simple case where $\boldsymbol{f}(\mathbf{0}) = \mathbf{0}, \boldsymbol{f}(\boldsymbol{a}) \neq \mathbf{0}, \boldsymbol{g}(\boldsymbol{a}) = \mathbf{0}, \boldsymbol{g}(\boldsymbol{x}) \neq 0, \forall\, \boldsymbol{x} \in \mathbb{R}^2 \setminus \boldsymbol{a}$ and each solution trajectory of the vector field $\boldsymbol{g}(\boldsymbol{x})$ is a closed curve. Consequently, except $x = a$, every control curve of the system separates the plane into two disjoint components by the Jordan curve Theorem. Therefore, we may define the **inner side** of a control curve as the part that contains the origin, and the other part as the **outer side**. We are now in a position to give the following definition.

Definition 7. A smooth simple closed curve $\Gamma : \gamma(s), s \in \mathrm{I}$ on the plane, is called a **P-curve** of the system (3), if there exists $s_1 \in \mathrm{I}$ such that

$$L(s_1) < 0$$

holds for some function u, where $L(s) \triangleq \langle \boldsymbol{f}(\gamma(s)) + \boldsymbol{g}(\gamma(s))u, \boldsymbol{p}(s)\rangle$, and $\boldsymbol{p}(s)$ is a non-zero normal vector of $\gamma(s)$ which points to the outer side of Γ.

Proposition 3
A control curve $\Gamma : \gamma(s)$ of system (3) not passing through the origin is a P-curve if and only if there exists s_1, such that $\langle \boldsymbol{f}(\gamma(s_1)), \boldsymbol{p}(s_1)\rangle < 0$, where $\boldsymbol{p}(s)$ is non-zero normal vector of $\gamma(s)$, which points to the outer side of Γ.

Theorem 4
Let the system (3) be locally asymptotically controllable at the origin. Then the necessary and sufficient condition for global asymptotical controllability of the control system is that any control curve $\Gamma : \gamma(t)$ of the system not passing through the origin is a P-curve.

4 High Dimensional Systems

We will consider two kinds of generalizations in this section.

4.1 Systems with Triangular-like Structure

We will first generalize Theorem 1 to the following nonlinear affine control systems with a triangular-like structure :

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) + g_1(x_1, x_2)x_3 \\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)x_3 \\ \dot{x}_3 &= f_3(x_1, x_2, x_3) + g_3(x_1, x_2, x_3)x_4 \\ &\vdots \\ \dot{x}_n &= f_n(x_1, x_2, \ldots, x_n) + g_n(x_1, x_2, \ldots, x_n)u, \end{aligned} \tag{4}$$

where $f_i, g_i \in C^{n-2}$, $i = 1, 2, \ldots, n$, namely they are functions with $(n-2)$ times continuous partial derivatives, $g(x_1, x_2) = (g_1(x_1, x_2), g_2(x_1, x_2))^T \neq 0$ for any $(x_1, x_2)^T \in \mathbb{R}^2$, $g_i(x_1, x_2, \ldots, x_i) \neq 0$ for any $(x_1, x_2, \ldots, x_i)^T \in \mathbb{R}^i, i = 3, \ldots, n$ and u is a right-continuous control function taking values on $\mathbb{R}$.

The key idea here is to take the advantage of the triangular-like structure and to apply the results on planar affine nonlinear systems to the following x_1-x_2 subsystem :

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) + g_1(x_1, x_2)v \\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)v. \end{aligned} \tag{5}$$

Then we have the following theorem.

Theorem 5
The control system (4) is globally controllable if and only if its subsystem (5) is globally controllable, namely the criterion function $C(x)$ of the system (5) changes its sign over every control curve on the plane (x_1, x_2).

Remark 3. Similarly, if the subsystem (5) of the systems (4) has a singular point at the origin, but satisfies the conditions of Corollary 1, then Theorem 5 is also valid.

Similar to the previous section, we can also generalize Theorem 2 to the system (4) with a triangular-like structure.

The key idea here is also to take the advantage of the triangular-like structure and to apply the results on planar affine nonlinear systems to the x_1-x_2 subsystem (5) of the system (4). Let $f_i(\mathbf{0}) = 0, i = 1, 2, \ldots, n$, we then have the following theorem.

Theorem 6

Let the system (4) be locally asymptotically controllable at the origin. Then the necessary and sufficient condition for global asymptotical controllability of the control system is that its subsystem (5) is globally asymptotically controllable, namely any control curve of the subsystem (5) not passing through the origin is a P-curve on the plane (x_1, x_2).

Remark 4. Similarly, if the subsystem (5) of the systems (4) satisfies the conditions of Theorem 4, then Theorem 6 is also valid. Moreover if there exists a control curve for the subsystem (5) which is not a P-curve, then the system (4) cannot be globally stabilized by any control function u.

4.2 Systems with $n-1$ Controllers

Now, we generalize Theorem 1 to the n dimensional systems with $n-1$ controllers. Here we should note that a similar but more general model had been ever studied in Hunt (1982) and Nikitin (1994), however, the necessary and sufficient condition for global controllability is not given.

Consider the following system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \sum_{i=1}^{n-1} \boldsymbol{b}_i u_i, \tag{6}$$

where $\boldsymbol{f}(\boldsymbol{x})$ is **locally Lipschitz vector function**, $\boldsymbol{x} \in \mathbb{R}^n$, the vectors $\boldsymbol{b}_i (i = 1, 2, \ldots, n-1)$ are time-invariant and linearly independent, the control vector is denoted as $\boldsymbol{u} = (u_1, u_2, \ldots, u_{n-1})^T$.

Since $\boldsymbol{b}_i$ $(i = 1, 2, \ldots, n-1)$ are independent, there is a nonzero vector $\boldsymbol{c}$ such that

$$\langle \boldsymbol{c}, \boldsymbol{b}_i \rangle = 0, \qquad i = 1, 2, \ldots, n-1,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of two vectors. We need the following definition of control hyperplane.

Definition 8. The **control hyperplane** of the system (6) is a hyperplane which passes through any point $\boldsymbol{x}^0$ and takes the vector $\boldsymbol{c}$ as its normal vector, namely the hyperplane

$$\langle \boldsymbol{x} - \boldsymbol{x}^0, \boldsymbol{c} \rangle = 0, \qquad \boldsymbol{x} \in \mathbb{R}^n.$$

Theorem 7

The necessary and sufficient condition for global controllability of the system (6) is that the function $\det(\boldsymbol{f}(\boldsymbol{x}), \boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n)$ *changes its sign over every control hyperplane.*

Next, we generalize Theorem 2 to the n dimensional system (6) with $n-1$ controllers.

It is easy to know that every hyperplane in $\mathbb{R}^n$ separates $\mathbb{R}^n$ into two disjoint components. Now, we first give the following definition.

Definition 9. The **inner side** of a hyperplane which does not pass through the origin, is defined as one of the two above-mentioned components that contains the origin. The other component is accordingly called the **outer side**.

Definition 10. A hyperplane $\mathcal{H} : h(S)$, parameterized by S, not passing through the origin is called a P-**hyperplane** of system (6), if there exists S_1 such that

$$L(S_1) < 0$$

holds for some control vector $\boldsymbol{u}$, where

$$L(S) \triangleq \langle \boldsymbol{f}(h(S)) + \sum_{1}^{n-1} \boldsymbol{g}_i(h(S))u_i, \boldsymbol{p}(S)\rangle,$$

and $\boldsymbol{p}(S)$ is a non-zero normal vector of $h(S)$ which points to the outer side of the hyperplane $\mathcal{H}$.

Proposition 4
A control hyperplane of system (6) not passing through the origin is a P-hyperplane if and only if there exists S, such that $\langle \boldsymbol{f}(h(S)), \boldsymbol{p}(h(S))\rangle < 0$, where $\boldsymbol{p}(S)$ is the non-zero normal vector of $h(S)$, which points to the outer side of $\mathcal{H}$.

Let $\boldsymbol{f}(\mathbf{0}) = \mathbf{0}$. Then we have the following theorem.

Theorem 8
Let the system (6) be locally asymptotically controllable at the origin. Then the necessary and sufficient condition for global asymptotical controllability of the control system is that any control hyperplane of the system not passing through the origin is a P-hyperplane.

5 Some Examples

Example 1

Consider the following second-order linear systems

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + Bu, \tag{7}$$

where $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$

Let $\Delta = \det(B, AB)$, i.e. $\Delta = a_{21}b_1^2 + a_{22}b_1b_2 - a_{11}b_1b_2 - a_{12}b_2^2$. The control curves are

$$\begin{cases} x_1 = b_1 t + c_1 \\ x_2 = b_2 t + c_2 \end{cases}, t \in (-\infty, +\infty), \tag{8}$$

where c_1 and c_2 are arbitrary constants. Therefore, its criterion function $C(\boldsymbol{x})$ is

$$\begin{aligned} & b_2(a_{11}x_1 + a_{12}x_2) - b_1(a_{21}x_1 + a_{22}x_2) \\ = \; & b_2(a_{11}(b_1 t + c_1) + a_{12}(b_2 t + c_2)) \\ & -b_1(a_{21}(b_1 t + c_1) + a_{22}(b_2 t + c_2)) \\ = \; & (b_1 b_2 a_{11} + b_2^2 a_{12} - a_{21} b_1^2 - a_{22} b_1 b_2)t \\ & +(b_2 a_{11} c_1 + b_2 a_{12} c_2 - b_1 a_{21} c_1 - b_1 a_{22} c_2) \\ = \; & -\Delta\, t + (b_2 a_{11} c_1 + b_2 a_{12} c_2 - b_1 a_{21} c_1 \\ & -b_1 a_{22} c_2). \end{aligned}$$

Hence, $\Delta \neq 0$, i.e., (A, B) is controllable, is the necessary and sufficient condition for the criterion function $b_2(a_{11}x_1 + a_{12}x_2) - b_1(a_{21}x_1 + a_{22}x_2)$ to change its sign over any control curve defined by (8). This example shows how the standard result in linear systems can be deduced from Theorem 1. □

Next, we consider another example where the control curve cannot be solved explicitly.

Example 2

Consider the following planar affine nonlinear system

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) + \cos(x_1^2 + x_2^2)u \\ \dot{x}_2 &= f_2(x_1, x_2) + \sin(x_1^2 + x_2^2)u. \end{aligned} \tag{9}$$

Let us denote $\boldsymbol{x} = (x_1, x_2)^T$, $\boldsymbol{g}(\boldsymbol{x}) = (\cos(x_1^2 + x_2^2), \sin(x_1^2 + x_2^2))^T$, and let

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}, \boldsymbol{M}(\boldsymbol{x}) = \begin{pmatrix} 0 \\ x_1 \end{pmatrix}, \boldsymbol{N}(\boldsymbol{x}) = \begin{pmatrix} \sin(x_1^2 + x_2^2) \\ \cos(x_1^2 + x_2^2) \end{pmatrix}.$$

Furthermore, let D_1 and D_2 be two open discs centered at the origin with radius 1 and 2 respectively. By a standard result in differential manifold (see Berger and Gostiaus (1988) pp. 106–109), there exists a smooth function $\theta(\boldsymbol{x})$ on $\mathbb{R}^2$ which satisfies

$$0 \leqslant \theta(\boldsymbol{x}) \leqslant 1, \;\; \theta(\boldsymbol{x}) = \begin{cases} 1, & \boldsymbol{x} \in D_1 \\ 0, & \boldsymbol{x} \notin D_2. \end{cases}$$

Now, let $f(x) = \theta(x)M(x) + (1-\theta(x))N(x)$, then $f(x)$ is smooth function on $\mathbb{R}^2$ and $f(0) = 0$. It is easy to check that $(\frac{\partial f(x)}{\partial x}|_{x=0}, g(0))$ is controllable, and each solution of differential equations defining the control curve

$$\begin{aligned} \dot{x}_1 &= \cos(x_1^2 + x_2^2) \\ \dot{x}_2 &= \sin(x_1^2 + x_2^2) \end{aligned} \tag{10}$$

will extend to infinite. Therefore, for any control curve there must be some parts which lie outside of D_2.

It is easy to see that for x out of the disc D_2, the criterion function

$$\begin{aligned} g_1(x)f_2(x) - g_2(x)f_1(x) &= \cos^2(x_1^2 + x_2^2) - \sin^2(x_1^2 + x_2^2) \\ &= \cos(2(x_1^2 + x_2^2)) = \cos(2r^2), \end{aligned} \tag{11}$$

where $r = \sqrt{x_1^2 + x_2^2}$.

Since each control curve $\gamma(t), t \in \mathbb{R}$ defined by equations (10) will extend to infinite, we know that the function in (11) will change its sign on each control curve. Hence, by Theorem 1, the system is globally controllable. □

Example 3

Consider the following system

$$\begin{aligned} \dot{x}_1 &= -\sin x_2 \cos x_2 + \sin x_2 \exp(-x_1)u \\ \dot{x}_2 &= \sin^2 x_2 + \cos x_2 \exp(-x_1)u. \end{aligned} \tag{12}$$

It has been shown that this system is locally linearizable but not globally linearizable (see Cheng et al. (1985)). Here, we will show that this system is actually not globally controllable.

Note that one of the control curves of the system (12) is

$$\begin{cases} x_1 = \ln t, t > 0 \\ x_2 = \frac{\pi}{2}, \end{cases}$$

and that on this curve, we have its criterion function $C(x) = g_1(x)f_2(x) - g_2(x)f_1(x) = \frac{1}{t} > 0, \forall\, t > 0$ which does not change its sign. Hence, the system is not globally controllable by Theorem 1. Furthermore, it can be shown that the system is not globally stabilizable by Remark 1. □

Next, we consider a bacterial respiration model which was proposed by Degn and Harrison as a description of the existence of a maximal oxygen consumption rate at low oxygen concentration in Klebsiella Aerogenes cultures (see Colonius and Kliemann (2000) pp. 365-367).

Example 4

Consider the following bacterial respiration model

$$\begin{aligned}\dot{x}_1 &= b - x_1 - \frac{x_1 x_2}{1 + q x_1^2} + u \\ \dot{x}_2 &= a - \frac{x_1 x_2}{1 + q x_1^2},\end{aligned} \tag{13}$$

where a and q are positive constants, b is critical parameter, depending on the concentration rates in the underlying chemical reaction scheme.

Obviously, we have a control curve $\begin{cases} x_1 = t, \ t \in \mathbb{R} \\ x_2 = 0, \end{cases}$ and its criterion function $C(\boldsymbol{x}) = g_1(\boldsymbol{x}) f_2(\boldsymbol{x}) - g_2(\boldsymbol{x}) f_1(\boldsymbol{x}) = a > 0, \forall\, t \in \mathbb{R}$. Hence, the system (13) is not globally controllable. □

Now, we show that the new theorems can be easily applied to some engineering examples.

Example 5

A field-controlled DC motor can be described by

$$\begin{aligned}\dot{x}_1 &= -a x_1 + u \\ \dot{x}_2 &= -b x_2 + \rho - c x_1 x_3 \\ \dot{x}_3 &= \theta x_1 x_2 - d x_3,\end{aligned} \tag{14}$$

where x_1, x_2, x_3 and u represent the stator current, the rotor current, the angular velocity of the motor shaft and the stator voltage respectively, a, b, c, d, θ and ρ are positive constants (see Khalil (1996) pp. 51-52).

According to Theorem 5 and Remark 3, we need to investigate the global controllability of the following system first:

$$\begin{aligned}\dot{x}_2 &= -b x_2 + \rho - c x_3 v \\ \dot{x}_3 &= -d x_3 + \theta x_2 v.\end{aligned} \tag{15}$$

It is easy to know that the system (15) satisfies the condition of Corollary 1 and the trajectories of the vector field $(-c x_3, \theta x_2)^T$ are ellipses

$$(\lambda \sqrt{c} \cos(\sqrt{c\theta} t), \lambda \sqrt{\theta} \sin(\sqrt{c\theta} t)), \lambda > 0, t \in \mathbb{R}.$$

By Corollary 1, the control system (15) is globally controllable if and only if its criterion function change its sign for any control curve, i.e. the following function

$$(d - b) c \lambda s^2 + \sqrt{c} \rho s - c d \lambda, \qquad s \in [-1, 1] \tag{16}$$

change its sign for any $\lambda > 0$, where every λ corresponds a control curve.

If $d-b=0$, obviously, as long as λ is large enough, (16) will be negative for any $s \in [-1,1]$.

If $d-b<0$, we have

$$\Delta = c\rho^2 + 4(d-b)d(c\lambda)^2.$$

Obviously, as long as λ is large enough, Δ will be negative. Therefore (16) will be negative for any $s \in [-1,1]$.

If $d-b>0$, the equation $(d-b)c\lambda s^2 + \sqrt{c}\rho s - cd\lambda = 0$ has a positive and a negative roots. Because $(d-b)c\lambda(-1)^2 + \sqrt{c}\rho(-1) - cd\lambda = -bc\lambda - \sqrt{c}\rho < 0$, the negative root must be less than -1. Similarly, as long as λ is large enough, we have $(d-b)c\lambda + \sqrt{c}\rho - cd\lambda = -bc\lambda + \sqrt{c}\rho < 0$, therefore the positive root must be greater than 1. Hence, (16) will be negative for any $s \in [-1,1]$ as long as λ is large enough.

In summary, the system (15) is not globally controllable. Therefore the system (14) is not globally controllable by Theorem 5 and Remark 3.

Now, we investigate the global asymptotical controllability of the system (14). Obviously, the system has an equilibrium point $E = (0, \frac{\rho}{b}, 0)^T$. Since it can be locally exactly feedback linearized in some neighborhood of the point E (see Isidori (1995)), this system is locally stabilizable.

Let us make a transformation $y_1 = x_1, y_2 = x_2 - \frac{\rho}{b}, y_3 = x_3$. Then the system (14) changes to

$$\begin{aligned} \dot{y}_1 &= -ay_1 + u \\ \dot{y}_2 &= -by_2 - cy_1y_3 \\ \dot{y}_3 &= \theta y_1(y_2 + \frac{\rho}{b}) - dy_3. \end{aligned} \tag{17}$$

It can be shown that the corresponding control curves are ellipses

$$(\lambda\sqrt{c}\cos(\sqrt{c\theta}t) - \frac{\rho}{b}, \lambda\sqrt{\theta}\sin(\sqrt{c\theta}t)), \lambda > 0, t \in \mathbb{R}.$$

It is easy to know that $\lambda_0 = \frac{\rho}{b\sqrt{c}}$ corresponds to the control curve passing through the origin as in Figure 2, denoted by Γ_0. Therefore,

$$L(s) = (d-b)c\lambda s^2 + \sqrt{c}\rho s - cd\lambda, \qquad s \in [-1,1].$$

Next, we prove this system is globally asymptotically controllable by considering two cases.

Case 1. $d - b = 0$

In this case the function $L(s)$ reduces to a linear function. It can be seen that when $\lambda < \frac{\rho}{b\sqrt{c}}$, $L(s)$ changes its sign, but when $\lambda > \frac{\rho}{b\sqrt{c}}$, $L(s)$ is negative.

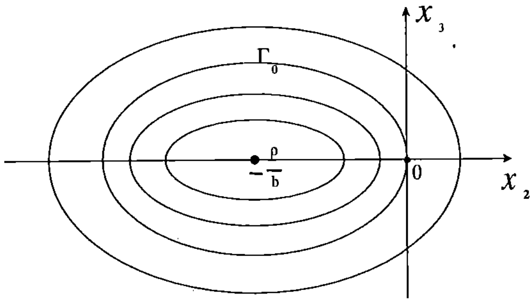

Figure 2

Case 2. $d - b \neq 0$

We have $L(-1) = -bc\lambda - \sqrt{c}\rho$ and $L(1) = -bc\lambda + \sqrt{c}\rho$. Therefore, by continuity, the range of the function $L(s)$ includes $[-bc\lambda - \sqrt{c}\rho, -bc\lambda + \sqrt{c}\rho]$.

Obviously, when $\lambda < \frac{\rho}{b\sqrt{c}}$, the function $L(s)$ changes its sign, and when $\lambda > \frac{\rho}{b\sqrt{c}}$, $L(s)$ remains to be negative.

In summery, any control curve of the subsystem (15) is a P-curve, therefore the system (14) is globally asymptotically controllable by Theorem 6 and Remark 4. □

Example 6

A field-controlled DC motor with negligible shaft damping term $-dx_3$ may be represented by a third-order model of the form

$$\begin{aligned} \dot{x}_1 &= -ax_1 + u \\ \dot{x}_2 &= -bx_2 + \rho - cx_1x_3 \\ \dot{x}_3 &= \theta x_1 x_2, \end{aligned} \tag{18}$$

where a, b, c, θ and ρ are positive constants (see Khalil (1996) p. 530).

In a similar method to the above example, the control system (18) is globally controllable if and only if the following function

$$-bc\lambda s^2 + \sqrt{c}\rho s = s(-bc\lambda s + \sqrt{c}\rho), \qquad s \in [-1, 1] \tag{19}$$

change its sign for any $\lambda > 0$, where each λ corresponds to a control curve.

It is easy to know that (19) change its sign for any $\lambda > 0$.

Hence, the control system (18) is globally controllable. □

Example 7

Consider the following model of a permanent magnet synchronous motor :

$$\begin{aligned} L_d \frac{\mathrm{d}i_d}{\mathrm{d}t} &= -R_s i_d + n_p \omega L_q i_q + u_d \\ L_q \frac{\mathrm{d}i_q}{\mathrm{d}t} &= -R_s i_q - n_p \omega L_d i_d - n_p \omega \Phi + u_q \\ J \frac{\mathrm{d}\omega}{\mathrm{d}t} &= \frac{3}{2} n_p [(L_d - L_q) i_d i_q + \Phi i_q] - \tau_L, \end{aligned} \tag{20}$$

where i_d and i_q are $d-q$ axis currents, ω is the motor speed, u_d and u_q are $d-q$ axis voltages, L_d, L_q and J are nonzero positive constants, R_s, n_p, Φ and τ_L are positive constants (see Guo et al. (2005)).

The control hyperplane of the system (20) is $\omega = c$, and it is easy to know that the function

$$\det(\boldsymbol{f}(\boldsymbol{x}), \boldsymbol{b}_1, \boldsymbol{b}_2) = \frac{\frac{3}{2} n_p [(L_d - L_q) i_d i_q + \Phi i_q] - \tau_L}{J}$$

changes its sign over every control hyperplane. Hence, by Theorem 7 we know the system (20) is globally controllable.

It may be noticeable that the global controllability of this system can also be guaranteed when either the control u_d or u_q fails.

First, let $(x_1, x_2, x_3)^T = (L_d i_d, L_q i_q, J\omega)^T$, $a = \frac{R_s}{L_d}$, $b = \frac{n_p}{J}$, $c = \frac{R_s}{L_q}$, $d = \Phi b$, $\beta = \frac{3(L_d - L_q)}{2 L_d L_q} n_p$, $h = \frac{3 n_p \Phi}{2 L_q}$. Then the system (20) can be re-represented as

$$\begin{aligned} \dot{x}_1 &= -a x_1 + b x_2 x_3 + u_d \\ \dot{x}_2 &= -c x_2 - b x_1 x_3 - d x_3 + u_q \\ \dot{x}_3 &= \beta x_1 x_2 + h x_2 - \tau_L. \end{aligned} \tag{21}$$

Here, we only investigate the case where the control u_q fails. Therefore the system (21) becomes to be

$$\begin{aligned} \dot{x}_1 &= -a x_1 + b x_2 x_3 + u_d \\ \dot{x}_2 &= -c x_2 - b x_1 x_3 - d x_3 \\ \dot{x}_3 &= \beta x_1 x_2 + h x_2 - \tau_L. \end{aligned} \tag{22}$$

By Remark 3, we need only to consider the following subsystem :

$$\begin{aligned} \dot{x}_2 &= -c x_2 - d x_3 - b x_3 v \\ \dot{x}_3 &= h x_2 - \tau_L + \beta x_2 v. \end{aligned} \tag{23}$$

Let us assume that $\beta > 0$ (or $L_d > L_q$). As in example 5, the control curves of the systems (23) are ellipses $(\lambda\sqrt{b}\cos(\sqrt{b\beta}t), \lambda\sqrt{\beta}\sin(\sqrt{b\beta}t))$, $\lambda > 0, t \in \mathbb{R}$, and the criterion function is

$$C(s) = \lambda(-c\beta b\lambda\cos^2 s - (d\beta - bh)\sqrt{b\beta}\lambda\sin s\cos s - \tau_L b\sqrt{\beta}\sin s),$$

where $s = \sqrt{b\beta}t$ and each $\lambda > 0$ corresponds to a control curve.

Note that $C(\frac{\pi}{2}) = -\lambda\tau_L b\sqrt{\beta} < 0$ and $C(-\frac{\pi}{2}) = \lambda\tau_L b\sqrt{\beta} > 0$. Therefore, by Theorem 5 and Remark 3, the system (20) is also globally controllable when the control u_q fails. □

Example 8

The Jet engine compression systems may be written as

$$\begin{aligned} \dot{R} &= -\sigma R^2 - \sigma R(2\phi + \phi^2) \\ \dot{\phi} &= -\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 - 3R\phi - 3R \\ \dot{\psi} &= -u, \end{aligned} \tag{24}$$

where R, ϕ, ψ and u are the normalized stall cell squared amplitude, the mass flow, the pressure rise and the mass flow through the throttle respectively, σ is constant positive parameter (see Krstic et al. (1995) pp. 67-68).

According to Theorem 5, we need only to investigate the global controllability of the following system:

$$\begin{aligned} \dot{R} &= -\sigma R^2 - \sigma R(2\phi + \phi^2) \\ \dot{\phi} &= -\frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 - 3R\phi - 3R - v. \end{aligned} \tag{25}$$

Since the control curve is $R = L$ with L being any constant. So the criterion function C is

$$\sigma R^2 + \sigma R(2\phi + \phi^2) = \sigma R(R + 2\phi + \phi^2) = \sigma R[(\phi + 1)^2 + R - 1].$$

By Theorem 1, the system (25) is not globally controllable, because the criterion function $C > 0$ over the control curve $R = L$ for any $\phi \in \mathbb{R}$ with $L > 1$. Hence, the system (24) is not globally controllable by Theorem 5. □

6 Concluding Remarks

In this paper, we summarized some recent results obtained on both global controllability and global asymptotical controllability of some classes of affine nonlinear systems. These results were obtained by introducing a new method based on some basic facts in planar topology and in the geometric theory of ordinary differential equations. Necessary and sufficient conditions (together with a new criterion) for controllability are established first for general planar affine nonlinear systems and then for two classes of high dimensional nonlinear systems. A number of examples, both mathematical and practical, are given to show that how our new criterion can be easily applied. For future investigation, it is desirable to extend the main results of this paper to more general high dimensional nonlinear control systems.

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