Jrl Syst Sci & Complexity (2008) 21: 406–415

ROBUST CONSENSUS AND SOFT CONTROL OF MULTI-AGENT SYSTEMS WITH NOISES*

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Received: 11 April 2008 / Revised: 18 April 2008

Abstract This paper considers the problem of robust consensus for a basic class of multi-agent systems with bounded disturbances and with directed information flow. A necessary and sufficient condition on the robust consensus is first presented, which is then applied to the analysis, control and decision making problems in the noise environments. In particular, the authors show how a soft control technique will synchronize a group of autonomous mobile agents without changing the existing local rule of interactions, and without assuming any kind of connectivity conditions on the system trajectories.

Key words Directed graph, robust consensus, soft control, time-varying systems.

1 Introduction

The interaction between information flow and system dynamics is an issue of wide interests. An important problem among these researches is consensus of multi-agent systems, which arise from many practical problems including unmanned air vehicles, micro-satellites, formation control of robots, flocks of birds, etc. To the best of our knowledge, almost all the existing theoretical investigations on the consensus problem have been carried out without considering the influence of the noise disturbance which is inevitable in the real world[1−5]. One exception is [6], where the robust consensus of a class of multi-agent systems with noises is studied under bidirectional information exchange.

However, there are a variety of practical applications where information only flows in one direction. For example, in soft control[5], the shill agent may be only equipped with a communication transmitter. For heterogeneous teams, different individuals may have different sensory abilities, thus individual i may sense the information of individual j while the opposite may not be valid. Hence, there is a need to extend the result of [6] to interaction topologies with directed information exchanges.

In this paper, by using the directed graph to represent information exchanges, we will study a more general class of multi-agent systems with noise. We will show that under certain assumptions, the robust consensus can be achieved under dynamically changing interaction topology if and only if the union of the collection of interaction graphs across some time intervals has a spanning tree frequently enough. Three typical examples, a class of multi-agent model, an application of soft control, and a decision-making process, are provided to illustrate the result.

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*This research is supported by the National Natural Science Foundation of China under Grant No. 60221301 and the Knowledge Innovation Project of Chinese Academy of Sciences under Grant No. KJCX3-SYW-S01.

The paper is organized as follows. In Section 2, we formally state the problem. The main result is established in Section 3, and its applications are given in Section 4. Section 5 gives some conclusions.

2 Problem Statement

Let $x_i(t) \in R,\ i = 1, 2, \cdots, n$, represent the ith agent's information state at time t. We consider the following discrete-time update law:

$$x_i(t+1) = \frac{1}{\sum\limits_{j=1}^{n} A_{ij}(t)} \sum_{j=1}^{n} A_{ij}(t)x_j(t) + w_i(t+1), \tag{1}$$

where $w_i(t+1) \in R$ is the noise influence on agent i, $A_{ij}(t) \geq 0$ is the weighting factor that agent j acts on agent i, and there will be no information flow from j to i if $A_{ij}(t) = 0$. In this paper, we assume every agent can sense its own information at any step, that is, $A_{ii}(t) > 0$. Thus, each agent updates its own state value by a weighted average of its own state value and the values received from other agents.

We can use directed graph $D_t = (V, E_t)$ to model the interaction topology among these agents at time t, where $V = \{1, 2, \cdots, n\}$ is the vertex set and $E_t \subset V \times V$ is the arc set. An arc $(i, j) \in E_t$ means that agent j can receive information from agent i at time t, that is, $A_{ji}(t) > 0$.

For a directed graph (digraph) D, a path from vertex i to j is a sequence of distinct vertexes $i_0, i_1, i_2, \cdots i_m$, where $i_0 = i$, $i_m = j$ and arc $(i_l, i_{l+1}) \in E$, $0 \leq l \leq m-1$. A digraph is called strongly connected if for any ordered pair of distinct vertices, i and j, there is a path in D connecting i to j. A digraph is said to have a spanning tree if and only if there exists a vertex $i \in V$, called root, such that there is a path from i to any other vertex. The union of a collection of digraphs $\{D_1, D_2, \cdots, D_k\}$ with the same vertex set V, is a digraph D with vertex set V and arc set equaling the union of the arc sets of all of the digraphs in the collection. The directed graph of matrix $B = [b_{ij}]_{n\times n}$, denoted by $\Gamma(B)$, is the directed graph on n nodes $1, 2, \cdots, n$ such that there is a directed arc in $\Gamma(B)$ from j to i if and only if $b_{ij} \neq 0$[7]. From the description of D_t, we know the digraph associated with the matrix $P(t) \doteq [P_{ij}(t)]$ is equivalent to the digraph D_t defined by the interaction relations of all agents at time t, where

$$P_{ij}(t) = \frac{A_{ij}(t)}{\sum\limits_{j=1}^{n} A_{ij}(t)}.$$

Furthermore, (1) can be written in matrix form as

$$x(t+1) = P(t)x(t) + w(t+1), \quad t = 1, 2, \cdots, \tag{2}$$

where $x(t) = [x_1(t), x_2(t), \cdots, x_n(t)]^{\mathrm{T}}$, $w(t) = [w_1(t), w_2(t), \cdots, w_n(t)]^{\mathrm{T}}$. Obviously, $\{P(t), t \geq 1\}$ is a sequence of stochastic matrix, where the notion of stochastic matrix is briefly described as follows. If all entries of a matrix $A = [a_{ij}]_{n\times n}$ satisfy $a_{ij} \geq 0$, then we say that A is nonnegative. Moreover, a nonnegative matrix A is called stochastic if the sum of each row satisfies $\sum_{j=1}^{n} a_{ij} = 1$, $i = 1, 2, \cdots, n$.

It is worth mentioning that almost all the existing analysis related with Vicsek model and consensus algorithm[1−4] would fall into the research of system (2) with $w(t+1) = 0$. The purpose of this paper is to study the noise effect as in [6]. To proceed, we need a few notions and concepts from [6]. The distance between a vector x and a subspace $X \subset R^n$ is defined by

$$d(x, X) = \inf_{y \in X} d(x, y) = \inf_{y \in X} \|x - y\|, \tag{3}$$

where $\|\cdot\|$ is the standard Euclidean norm. Moreover, we take X as the space spanned by the vector $[1, 1, \cdots, 1]^{\mathrm{T}}$, i.e., $X = \mathrm{span}\{[1, 1, \cdots, 1]^{\mathrm{T}}\}$, and denote the orthogonal subspace of X as M. Now, we define a function set and a noise set, respectively, as follows[6]:

$$K_0 = \big\{f(\cdot)|f : R^+ \to R^+,\ f(0) = 0,\ f(\delta) \text{ decreases to 0 as } \delta \to 0\big\};$$
$$\mathcal{B}(\delta) = \Big\{\{w(t)\}|\ \sup_{t \geq 0} d(w(t), X) \leq \delta\Big\}.$$

Definition 1 The system (2) is said to be robust consensus with respect to noise, if for any $\delta > 0,\ x(0) \in R^n$, and any sequence $\{w(t)\} \in \mathcal{B}(\delta)$, there always exist a function $f(\cdot) \in K_0$ and a constant $T > 0$ such that

$$d(x(t), X) \leq f(\delta), \quad \forall t \geq T. \tag{4}$$

In this paper, we shall generalize the results of [6] to the case of directed information flow.

3 Main Result

Suppose that the matrix sequence $\{P(t)\}_{t=1}^{\infty}$ satisfies the following assumption.

Assumption 1

1) For each t, $P(t)$ is a stochastic matrix with positive diagonal entries;

2) The non-zero entries of $P(t)$ have the following unanimous low bound:

$$\min_{(i,j):P_{ij}(t)>0} P_{ij}(t) \geq \alpha > 0, \quad \forall t \geq 1. \tag{5}$$

Intuitively, whenever $P_{ij}(t) > 0$, agent j communicates its current value $x_j(t)$ to agent i with a weighting factor larger than α. The following theorem is our main result.

Theorem 1 *Consider the system* (2) *under Assumption* 1. *Then, it is robust consensus if and only if there exists a constant $q > 0$ such that for any $t \geq 0$ the union of digraphs $\{D_{t+1}, D_{t+2}, \cdots, D_{t+q}\}$ associated with matrices $\{P(t+1), P(t+2), \cdots, P(t+q)\}$ has a spanning tree.*

To prove the main result, we will need some ideas. As in [6], we introduce a suitable projection operator, which can translate the distance between a vector and a subspace X to the norm of the projected vector. Thus, the problem of robust consensus can be transformed to a certain robust stability in the subspace. We decompose the space R^n into two orthogonal subspaces X and $M = X^{\perp}$. For any $x \in R^n$, there exist a unique pair of vectors $x_0 \in M$, $x_1 \in X$ such that $x = x_0 + x_1$. Furthermore, according to the property of projection, we have

$$\|x_0\| = \|x - x_1\| = \inf_{y \in X} \|x - y\| = d(x, X).$$

Denote P_M as the projector onto M, then $P_M x = x$ if and only if $x \in M$, $P_M x = 0$ if and only if $x \in X$, and

$$\|P_M x\| = d(x, X). \tag{6}$$

Take a standard orthogonal base $e_1, e_2, \cdots, e_n$ in the space R^n, where $e_n = (\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \cdots, \frac{1}{\sqrt{n}})^{\mathrm{T}}$, then $X = \operatorname{span}\{e_n\}$ and $M = \operatorname{span}\{e_1, e_2, \cdots, e_{n-1}\}$. We can get a detailed form of the projector P_M as follows:

$$\begin{aligned} P_M &= Q \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} Q^{\mathrm{T}} \\ &= I - \frac{1}{n} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} (1 \cdots 1) = I - \frac{1}{n} \mathbf{1} \cdot \mathbf{1}^{\mathrm{T}}, \end{aligned} \tag{7}$$

where $Q \doteq [e_1, e_2, \cdots, e_n]$, $\mathbf{1} \doteq [1, 1, \cdots, 1]^{\mathrm{T}}_{n \times 1}$. If P_1, P_2 are stochastic matrices, we have

$$\begin{aligned} P_M P_1 P_M P_2 &= P_M P_1 \Big(I - \frac{1}{n} \mathbf{1} \cdot \mathbf{1}^{\mathrm{T}}\Big) P_2 = P_M \Big(P_1 - \frac{1}{n} \mathbf{1} \cdot \mathbf{1}^{\mathrm{T}}\Big) P_2 \\ &= P_M P_1 P_2. \end{aligned}$$

Thus, for stochastic matrices $\{P_i,\ i = 1, 2, \cdots\}$, the projector P_M has the following property

$$\prod_{i=1}^{n} P_M P_i = P_M \prod_{i=1}^{n} P_i. \tag{8}$$

Let the projector P_M act on both sides of the system (2), we have

$$P_M x(t+1) = P_M P(t) x(t) + P_M w(t+1). \tag{9}$$

As in [6], set $\eta(t) = P_M x(t)$, $\nu(t) = P_M w(t)$, then $\eta(t)$, $\nu(t) \in M$, and

$$P_M P(t) \eta(t) = P_M P(t) x(t).$$

Thus, the system (9) is equivalent to

$$\eta(t+1) = P_M P(t) \eta(t) + \nu(t+1). \tag{10}$$

Set $\nu = \{\nu(t)\}_{t=1}^{\infty}$, $U_M(\delta) = \{\nu : \|\nu(t)\| \le \delta,\ \nu(t) \in M,\ \forall t \ge 1\}$. Here, the system (10) is said to be robust stable on the subspace M, if for any $\eta(0) \in M$ and any $\varepsilon > 0$, there exist constants $\delta \doteq \delta(\varepsilon, \eta(0)) > 0$, $T = T(\varepsilon, \eta(0)) > 0$, such that

$$\sup_{\nu \in U_M(\delta)} \sup_{t \ge T} \|\eta(t)\| \le \varepsilon.$$

Thus, the robust consensus of system (2) has been transformed to the robust stability of system (10) on the subspace M.

We define $\Phi(k, i)$ as the state transition matrix, that is,

$$\Phi(k+1, i) = P_M P(k) \Phi(k, i), \quad \Phi(i, i) = I, \quad \forall k \ge i \ge 0.$$

Then by (10), we have

$$\eta(t+1) = \Phi(t+1, 0) \eta(0) + \sum_{i=1}^{t+1} \Phi(t+1, i) \nu(i). \tag{11}$$

To motivate further study, we need the following exponential stability result whose proof is similar to Lemma 2.1.2 in [8], and is given in [9] for details.

Lemma 1 *Consider the system* (10) *under Assumption* 1*. If the system* (10) *is robust stable on the subspace M, then we have*

$$\| \Phi(t+h,t) \| \leq N\lambda^h, \qquad \forall t \geq 0, \quad \forall h \geq 1 \tag{12}$$

for some constants $N > 0$ and $\lambda \in (0,1)$.

We also need the following lemma which is about the algebraic multiplicity of eigenvalue 1 of stochastic matrix.

Lemma 2 *Let P be a stochastic matrix with P_M being its projector onto M. If $\rho(P_M P) < 1$, then* 1 *is a simple eigenvalue of P, where $\rho(A)$ is the spectral radius of a matrix A.*

Proof From the theorem of finite dimensional Markov chain[10], we know that there exists a matrix K such that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} P^i = K.$$

Thus,

$$\lim_{n\to\infty} P_M \frac{1}{n}\sum_{i=1}^{n} P^i = P_M K.$$

On account of $\rho(P_M P) < 1$, there exists a matrix norm $\|\cdot\|_l$ such that

$$\|P_M P\|_l \doteq \mu < 1.$$

By (8),

$$\begin{aligned}
\left\| P_M \frac{1}{n}\sum_{i=1}^{n} P^i \right\|_l &= \left\| \frac{1}{n}\sum_{i=1}^{n} (P_M P)^i \right\|_l \\
&\leq \frac{1}{n}\sum_{i=1}^{n} \|P_M P\|_l^i \\
&\leq \frac{1}{n}\sum_{i=1}^{n} \mu^i.
\end{aligned}$$

Thus,

$$\lim_{n\to\infty} \left\| \frac{1}{n}\sum_{i=1}^{n} (P_M P)^i \right\|_l = 0,$$

$$P_M K = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} (P_M P)^i = 0.$$

Because the rank of P_M is $n-1$, the rank of K is not greater than 1.

Obviously, 1 is an eigenvalue of P and $\rho(P) \leq 1$ for P is a stochastic matrix. Now, we prove 1 is simple. If not, from the Jordan canonical form, we have nonsingular matrix T such that

$$P = T \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ * & \lambda_2 & 0 & \cdots & 0 \\ 0 & * & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & * & \lambda_n \end{pmatrix} T^{-1},$$

where $|\lambda_i| \leq 1$ and $\lambda_1 = \lambda_2 = \cdots = \lambda_k = 1$ with $k \geq 2$. Thus,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} P^i = T \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ * & 1 & 0 & \cdots & 0 \\ * & * & * & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ * & * & \cdots & * & * \end{pmatrix} T^{-1},$$

which implies that the rank of K is larger than 1. Obviously it is a contradiction, so 1 is a simple eigenvalue of P. ∎

The following two lemmas will also be used in the proof of Theorem 1.

Lemma 3[3] *The eigenvalue* 1 *is simple for a stochastic matrix if and only if its associated digraph has a spanning tree.*

Lemma 4[3] *Let $\{P(t)\}$ be a sequence of stochastic matrix with positive diagonal entries, and with associated digraphs denoted by G_t. If for any sequence $\{i_1, i_2, \cdots, i_m\}$, the union of the directed graphs $\{G_{i_1}, G_{i_2}, \cdots, G_{i_m}\}$ has a spanning tree, then the matrix product $P(i_m)\cdots P(i_2)P(i_1)$ is SIA.*†

Proof of Theorem 1

By using Lemma 4, and following the same analysis in the proof of iii)⇒ i) of Theorem 3.1 in [6], one can get the desired conclusion directly.

As we have seen in the above, the robust consensus of the system (2) can be transformed to the stability of the system (10) on the subspace M. Define

$$P^{(t,r)} = P(t+r-1)\cdots P(t+1)P(t),$$

and from Lemma 1, we know that for any $t \geq 0$, there exists a constant integer q such that

$$\|P_M P^{(t,q)}\| < 1. \tag{13}$$

Let $P^{(t)}$ denote $P^{(t,q)}$, $D^{(t)}$ denote the associated directed graph of $P^{(t)}$, G_k denote the associated digraph of $P(k)$, and $G^{(t)}$ denote the union of digraphs $\{G_t, G_{t+1}, \cdots, G_{t+q}\}$.

From Lemma 2 and (13), we know that 1 is a simple eigenvalue of $P^{(t)}$. Furthermore, the digraph $D^{(t)}$ has a spanning tree according to Lemma 3. To complete the proof of the sufficiency part of Theorem 1, we need to show that $G^{(t)}$ has a spanning tree.

Now, let vertex r be the root of the graph $D^{(t)}$, then for any other vertex $j \in V$, there is a $r \to j$ path in digraph $D^{(t)}$, i.e., there is a sequence of arcs (r, i_1), (i_1, i_2), $(i_2, i_3), \cdots$, (i_{m-2}, i_{m-1}), (i_{m-1}, j) in $D^{(t)}$ connecting r to j. By the relations between $D^{(t)}$ and $P^{(t)}$, we know that the following elements of $P^{(t)}$: $P^{(t)}_{i_1,r}, P^{(t)}_{i_2,i_1}, \cdots, P^{(t)}_{i_{m-1},i_{m-2}}, P^{(t)}_{j,i_{m-1}}$ are all nonzero. Now, let us consider each nonzero entry $P^{(t)}_{i_{s+1},i_s}$, $s = 0, 1, \cdots, m-1$ (i_0 denotes r, i_m denotes j). By using the property of matrix product, we know that there exists a nonzero item $P_{i_{s+1},k_{q-1}}(t+q-1)P_{k_{q-1},k_{q-2}}(t+q-2)\cdots P_{k_1,i_s}(t)$ for some $k_1, k_2, \cdots, k_{q-1}$. Note that $P_{i,j}(k) \neq 0$ means there is an arc (j, i) in digraph G_k, thus, the item $P_{i_{s+1},k_{q-1}}(t+q-1)P_{k_{q-1},k_{q-2}}(t+q-2)\cdots P_{k_1,i_s}(t) \neq 0$ means there is a path in the union digraph $G^{(t)}$ connecting i_s with i_{s+1}. Hence, for each nonzero entry $P^{(t)}_{i_{s+1},i_s}$, $s = 0, 1, \cdots, m-1$, there is a path $i_s \to i_{s+1}$ in digraph $G^{(t)}$, and so it is obvious that there is a path in digraph $G^{(t)}$ connecting r with j. According to the arbitrariness of vertex j, we know that the digraph $G^{(t)}$ has a spanning tree rooted at r, i.e., the union of

†Stochastic matrix P is called indecomposable and aperiodic (SIA) if $B = \lim_{n\to\infty} P^n$ exists and all the rows of B are the same.

digraphs $\{G_t, G_{t+1}, \cdots, G_{t+q-1}\}$ associated with matrices $\{P(t), P(t+1), \cdots, P(t+q-1)\}$ has a spanning tree.

Hence, the proof of Theorem 1 is complete. ∎

4 Applications

4.1 Analysis of a Class of Multi-Agent Model with Noises

A typical multi-agent model that was studied by Vicsek et al.[11] contains the noise effects, and it was demonstrated by simulation that the system will synchronize if the population density is large and the noise is small. This paper intrigued a series of theoretical investigations initiated by Jadbabaie et al.[1]. However, to the best of our knowledge, all the existing theoretical researches do not consider the nonlinear Vicsek model with noises, mainly due to difficulties in the theoretical analysis.

Let us now consider a class of multi-agent systems with additive noise, which is a variation of the well-known Vicsek et al.[11] model: There are n autonomous agents moving in the plane with the same speed but with different headings. Each agent's heading is updated using a local rule based on the average of its own heading plus the heading of its neighbors with additive noise. Denote $x_i(t)$ as the position of agent i at time t, then the neighbor of agent i is defined as $N_i(t) = \{j| \ \|x_j(t) - x_i(t)\| \leq r_i,\}$, where $r_i \geq 0$ is the sensing radius of each agent. At each time step, the position $x_i(t)$ and heading $\theta_i(t)$ of the ith agent are updated according to

$$x_i(t+1) = x_i(t) + v_i(t), \tag{14}$$

where the velocity $v_i(t)$ is constructed to have an absolute value v and a direction given by the angle $\theta_i(t)$, which is updated according to the following equations:

$$\theta_i(t+1) = \pi_D\{\widehat{\theta}_i(t+1)\}, \tag{15}$$

$$\widehat{\theta}_i(t+1) = \arctan\left(\frac{\sum\limits_{j\in N_i(t)} \sin(\theta_j(t))}{\sum\limits_{j\in N_i(t)} \cos(\theta_j(t))}\right) + w_i(t+1), \quad t = 0, 1, \cdots, \tag{16}$$

where $\pi_D\{\cdot\}$ is the projection operator onto the bounded interval $[-\frac{\pi}{2} + b, \frac{\pi}{2} - b]$ with a given small number $b > 0$.

Remark 1 To analyze the synchronization property of the above model by using the stochastic matrix theory as in Theorem 1, we need the nonnegative property of the corresponding weighted average matrix, i.e., we need $\cos(\theta_i(t)) \geq 0$, or $\theta_i(t) \in (-\frac{\pi}{2}, \frac{\pi}{2})$. To guarantee this and to facilitate the analysis, a natural way is to introduce the above projection mechanism.

Now, to rewrite the above equation in the form of (2), we introduce a transformation: $\beta_i(t) = \tan\theta_i(t)$. Then, by using the following inequality

$$|\theta_i(t+1) - \widehat{\theta}_i(t+1)| \leq |w_i(t+1)|, \tag{17}$$

it is easy to obtain

$$\beta_i(t+1) = \sum_{j\in N_i(t)} \frac{\cos(\theta_j(t))}{\sum\limits_{m\in N_i(t)} \cos(\theta_m(t))}\beta_j(t) + \delta_i(t+1), \quad t = 0, 1, 2, \cdots, \tag{18}$$

where $\delta_i(t+1)$ is a "residual" process which is bounded by a constant times $\tan|w_i(t+1)|$ for small $|w_i(t+1)|$.

Next, we use digraph $G_t = (V, E_t)$ to express the neighbor relationships of agents at time t. In G_t, the ith node represents the ith agent, and a directed edge from i to j denoted as $(i, j) \in E_t$ represents $i \in N_j(t)$. Note that $(i, i) \in E_t$ for any t. Using digraph G_t, define $P(t) = [P_{ij}(t)]$ as follows:

$$P_{ij}(t) = \begin{cases} \dfrac{\cos(\theta_j(t))}{\sum\limits_{m \in N_i(t)} \cos(\theta_m(t))}, & (j, i) \in E_t, \\ 0, & (j, i) \notin E_t, \end{cases} \tag{19}$$

then (18) can be written in the form of (2):

$$\beta(t + 1) = P(t)\beta(t) + \delta(t), \qquad t = 0, 1, 2, \cdots, \tag{20}$$

where $\beta(t) = [\beta_1(t), \beta_2(t), \cdots, \beta_n(t)]^{\mathrm{T}}$, $\delta(t) = [\delta_1(t), \delta_2(t), \cdots, \delta_n(t)]^{\mathrm{T}}$. It is easy to see that $P(t)$ satisfies Assumption 1. Thus, we can use Theorem 1 to get the following result.

Theorem 2 *For the multi-agent model* (15) *and* (16)*, if there exists an integer $q > 0$ such that for any $t \geq 0$, the union of the neighbor digraphs $\{G_{t+1}, G_{t+2}, \cdots, G_{t+q}\}$ has a spanning tree, then the headings of the group have robust consensus property.*

4.2 Application of Soft Control

In this subsection, we consider the intervention in the collective behavior of multi-agent systems. In many real applications, it is rather difficult or even impossible to change the existing local rules of agents, such as the flying strategies of birds. If we need to have such system to avoid danger or improve efficiency, we should resort to a new mechanism-soft control, which was introduced by J. Han et al. in [5]. The key idea of soft control is to intervene the global behavior of the system while keeping the basic local rule of the existing agents in the systems. This can be realized, for example, by adding one (or a few) special agent called shill[5] which can be controlled, with the expectation that the collective behavior will emerge from the system. Here, we try to design a soft control law which will synchronize the class of multi-agent systems as discussed in the last subsection. The key point is to use soft control to guarantee the required joint connectivity of the underlying multi-agent systems.

To be precise, we consider the flocking for the same group of n agents as in the last subsection, with an added member (say agent 0) acting as the group's shill. The shill will move with a constant heading $\theta_0(0) \in [-\frac{\pi}{2} + b, \frac{\pi}{2} - b]$, and will not be influenced by other agents. Thus,

$$\theta_0(t + 1) = \theta_0(t), \qquad t = 0, 1, 2, \cdots. \tag{21}$$

On the other hand, the other agents labeled as 1 to n, will treat this shill as an ordinary agent. In other words, they will continue to use the same heading update rule (15), (16) as before.

Let

$$P^*(t) \doteq \begin{bmatrix} 1 & 0 \\ \gamma(t) & P(t) \end{bmatrix}, \tag{22}$$

where $\gamma(t) = [\gamma_1(t), \gamma_2(t), \cdots, \gamma_n(t)]^{\mathrm{T}}$ with

$$\gamma_i(t) = \begin{cases} \dfrac{\cos(\theta_0(t))}{\sum\limits_{m \in N_i(t)} \cos(\theta_m(t))}, & (0, i) \in E_t, \\ 0, & (0, i) \notin E_t, \end{cases}$$

and the (i, j)-th element of the matrix $P(t)$ is defined the same as (19).

Thus, the update law about the heading of all the agents can be written in the following form:

$$\beta^*(t+1) = P^*(t)\beta^*(t) + \delta^*(t), \quad t = 0, 1, 2, \cdots, \tag{23}$$

where $\beta^*(t) = [\beta_0(t), \beta^{\mathrm{T}}(t)]^{\mathrm{T}}$ with $\beta_0(t) = \tan\theta_0(t)$, $\delta^*(t) = [0, \delta^{\mathrm{T}}(t)]^{\mathrm{T}}$. Here $\beta(t)$, $\delta(t)$ are the same as in (20).

Next, we proceed to design a simple control law to guarantee the topological condition required for the dynamical graphs associated with the multi-agent systems. Suppose that the position $x_0(t)$ of the shill can be controlled at any time step t, in the following way:

$$\|x_0(t) - x_i(t)\| \le r_i, \quad t = kn + i, \quad i = 1, 2, \cdots, n, \quad k = 0, 1, 2, \cdots. \tag{24}$$

It is easy to check that for any $t \ge 0$, the union of the neighbor graphs $\{G_{t+1}, G_{t+2}, \cdots, G_{t+n}\}$ has a spanning tree which is rooted at the shill vertex 0. Thus, according to Theorem 1, we know that the headings of all agents are robust consensus around the desired heading $\theta_0(0)$. To conclude this subsection, we remark that a more "intelligent" shill can be designed based on the neighbor information of the ordinary agents[12].

4.3 The Decision-Making Process

When a committee make a decision, an assembly of persons is usually called together for discussion. Each individual is aware of other's opinions, and may modify its own attitude by taking into account of other opinions. There have been a range of articles[13–14] analyzing the conditions under which a consensus can be obtained for such a decision-making process.

Following [14], we consider a collective decision-making model as follows. Let the initial opinion of n individuals be denoted as $x_0 = [x_1(0), x_2(0), \cdots, x_n(0)]^{\mathrm{T}}$, and let $P_{ij}(0) \ge 0$ be the initial weight of the influence of the jth individual to the ith individual. After the first round of information interchanges, the ith individual's opinion becomes

$$x_i(1) = \sum_j P_{ij}(0)x_j(0),$$

where the $P_{ij}(0)'s$ can be normalized so that

$$\sum_j P_{ij}(0) = 1, \quad i = 1, 2, \cdots, n.$$

Similarly, we may denote $P(t) = [P_{ij}(t)]$, $i, j = 1, 2, \cdots, n$, as the weight matrix at time $t \ge 1$, which is a stochastic matrix after proper normalization.

Let $x(t) = [x_1(t), x_2(t), \cdots, x_n(t)]^{\mathrm{T}}$ be the opinion vector after t interchanges, it follows that

$$x(t) = P(t-1)x(t-1), \qquad t = 1, 2, \cdots. \tag{25}$$

This model is closely related to inhomogeneous Markov chain, see [13–15], and most analytical results provide only sufficient conditions for the consensus by conditions on the properties of $\{P(t)\}$.

In the present paper, we consider the noise effects on the consensus property of this group of individuals, whose opinions are updated by

$$x(t+1) = P(t)x(t) + w(t), \quad t = 0, 1, 2, \cdots, \tag{26}$$

where $w(t)$ is the noise process.

The weight may reflect the individual's trust when the opinions are diverse, and we can use digraph to represent the trust relationships among all individuals. For convenience, we say i has trust in j at time t if and only if $P_{ij}(t) > 0$, and in this case we may draw an arc in digraph $G_t = (V, E_t)$ describing the trust relationships, where $V = \{1, 2, \cdots, n\}$.

By using Theorem 1, we have the following theorem.

Theorem 3 *Consider the decision-making model* (26)*, suppose that at each time step, every individual trusts himself and gives a trust weight no less than* $\alpha > 0$ *to anyone he trusts. Then the opinions of all individuals will be robust consensus if and only if there exists a constant integer* $q > 0$ *such that for any* $t \geq 0$ *the union of trust relationship digraphs* $\{G_{t+1}, G_{t+2}, \cdots, G_{t+q}\}$ *has a spanning tree.*

This theorem shows that if all individuals have a positive trust level α in at least one common individual at any round of discussion, then their opinions will converge to a range bounded by the level of the disturbance.

5 Conclusions

This paper studies the robust consensus problem for a class of multi-agent systems with noises. We have used directed graphs to represent information exchanges among the agents, and generalized the results of [6] to the case of directed graphs, with applications to a class of soft control systems and a decision-making processes. An interesting point is that both the troublesome connectivity condition and the noise influence can be treated easily by resorting to a soft control. Of course, many interesting problems still remain open, which belong to further investigation.

References

[1] A. Jadbabaie, J. Lin, and A. S. Morse, Coordination of groups of mobile autonomous agents using nearest neighbor rules, *IEEE Trans. on Automatic Control*, 2003, **48**(6): 988–1001.

[2] F. Cucker and S. Smale, Emergent behavior in flocks, *IEEE Trans. on Automatic Control*, 2007, **52**(5): 852–862.

[3] W. Ren and R. W. Beard, Consensus seeking in multi-agent systems under dynamically changing interaction topologies, *IEEE Trans. on Automatic Control*, 2005, **50**(5): 655–661.

[4] L. Moreau, Stability of multi-agent systems with time-dependent communication links, *IEEE Trans. on Automatic Control*, 2005, **50**(2): 169–182.

[5] J. Han, M. Li, and L. Guo, Soft control on collective behavior of a group of autonomous agents by a shill agent, *Journal of System Science and Complexity*, 2006, **19**(1): 54–62.

[6] L. Wang, Z. X. Liu, and L. Guo, Robust consensus of multi-agent systems with noise, in *Proceedings of the* 26*th Chinese Control Conference*, Zhangjiajie, Hunan, 2007, 737–741.

[7] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University, New York, 1985.

[8] L. Guo, *Time-Varying Stochastic Systems*, Jilin Science and Technology Press, Jilin, 1993.

[9] L. Wang and Z. X. Liu, Robust consensus of multi-agent systems with noise (Tech. Report), Academy of Mathematics and Systems Science, Chinese Academy of Sciences, 2007.

[10] J. L. Doob, *Stochastic Processes*, John Wiley, New York, 1953.

[11] T. Vicsek, A. Czirok, I. Ben Jacob, I. Cohen, and O. Schochet, Novel type of phase transitions in a system of self-driven particles, *Phys. Rev. Lett.*, 1995, **75**(6): 1226–1229.

[12] Z. X. Liu, J. Han, and L. Guo, Analysis and regulation of dynamical networks based on Vicsek's model, *Complex Networks* (ed. by L. Guo and X. X. Xu), Shanghai Scientific & Technological Education press, Shanghai, 2006, 49–66.

[13] M. H. DeGroo, Reaching a consensus, *J. Amer. Statist. Assoc.*, 1974, **69**(345): 118–121.

[14] S. Chatterjee and E. Seneta, Towards consensus: some convergence theorems on repeated averaging, *J. Appl. Prob*, 1977, **14**(1): 89–97.

[15] E. Seneta, *Non-Negative Matrices and Markov Chains* (2nd ed.), Springer, New York, 1984.