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On Feedback Capability in a Class of Nonlinearly Parameterized Uncertain Systems

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***Abstract*—In this technical note, we will investigate the maximum capability and limitations of the feedback mechanism in globally stabilizing a basic class of discrete-time nonlinearly parameterized dynamical systems with multiple unknown parameters. Both "possibility" and "impossibility" theorems together with a fairly complete characterization on the capability of feedback will be presented. It will be seen that to characterize the feedback capability, the growth rates of the sensitivity functions of the nonlinear dynamics with respect to the uncertain parameters play a crucial role, and a suitable decomposition of the family of the nonlinearly parameterized functions in question turns out to be necessary.**

***Index Terms*—Feedback capability, global stabilization, nonlinear systems, parameterized uncertainty.**

I. Introduction

It is well known that feedback is a key concept in control systems, which is mainly used to deal with uncertainties in dynamical systems to be controlled. Robust control and adaptive control are two typical techniques for feedback design in the presence of structural uncertainties, and comparisons between the two from various aspects are also available (cf, e.g., [27]). It is conceivable that adaptive control has the ability to deal with larger class of uncertainties since an on-line estimation loop is usually imbedded in the feedback control design.

There has been much progress in adaptive control of linear systems(cf., e.g., [2], [4], [6], [7]), or nonlinear systems with nonlinearity having linear growth rate(cf. e.g. [20], [22] and [25]). Furthermore, it is also possible to design globally stabilizing adaptive controls for a wide class of nonlinear continuous-time systems (see, e.g., [9] and [11]) and for infinite-dimensional systems. However, while many results in continuous-time case can be extended to the discrete-time one (see, e.g., [8]), fundamental difficulties arise for adaptive control of discrete-time nonlinear systems with either nonparametric uncertainty ([26]) or parametricuncertainty but the nonlinearity having a growth rate faster thanlinearity (cf. [5], [10], [24]), partly because the high gain or nonlinear damping methods that are so useful in the control design of continuous-time systems are no longer effective in the discrete-time case. Similarly, forsampled-data control of nonlinear uncertain systems, the design of stabilizing sampled-data feedback is shown to be possible only forthe case where the sampling rate is high enough (cf.e.g., [18]). In fact, difficulties will again emergewhen the sampling rate is a prescribed value (may not be smallenough). This is so even for nonlinear systems with nonlinearityhaving a linear growth rate (cf. [23]).

Given the above difficulties that we encountered in the adaptive control of discrete-time (or sampled-data) nonlinear systems, one may be curious to know whether or not such difficulties are caused by the inherent limitations of the feedback principle. To investigate this fundamental problem, we have to place ourselves into a framework that is somewhat beyond those of the traditional robust control and adaptive control, because one needs to study the fundamental limitations of the full feedback mechanism which includes all (nonlinear and time-varying) feedback laws and which is not restricted to a specific class of feedback laws.

An initial step in this direction was made in [5] for a basic class of nonlinear stochastic systems with a scalar unknown parameter. It was found that the system is globally stabilizable by feedback if and only if the nonlinear function has a growth rate not faster than x^4 when $x \to \infty$ (see, [5], [16]). This critical rate appears to be somewhat surprising! This result was subsequently extended to systems with multiple unknown parameters by introducing a polynomial criterion (see, [24], [25]). Not long ago, [17] proved that the polynomial rule of [24] does indeed provide a necessary and sufficient condition for global feedback stabilization of a wide class of nonlinear systems with bounded multiple unknown parameters and with bounded noises, by using a somewhat complicated purely deterministic method. Recently, by introducing a simple stochastic imbedding approach, a new critical theorem on the feedback capability was obtained for the case where the input channel contains an uncertain parameter [14]. It should be pointed out that the related existing results (e.g. [3], [5], [14]–[17]) on feedback capability concern only with linearly parameterized discrete-time nonlinear models, i.e., the unknown parameter enters into the systems in a linear way. This has obvious shortcomings since, as is well-known, nonlinearly parameterized models are usually encountered in practical systems, as well as used in system approximation and identification (see, e.g., [1], [19] and [21]).

In this technical note, we will investigate the feedback capability in stabilizing a basic class of discrete-time nonlinearly parameterized dynamical systems with nonlinear dynamics having nonlinear growth rates, and with multiple unknown parameters and bounded noises. We will not only present a theorem on the maximum capability of feedback, but also give a theorem on the fundamental limitations of the feedback mechanism. It will be seen that the growth rates of the sensitivity functions with respect to the uncertain parameters play an important role in characterizing the feedback capability, but it turns out that such a characterization in the present nonlinearly parameterized case is more complicated than the linearly parameterized case previously studied. Some preliminary results were presented in [12].

The rest of the technical note is organized as follows. In the next section, we will present the main theorems of the technical note, with their

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proofs given in Section III. Some concluding remarks will be given in Section IV.

II. Main Results

To explore the fundamental limits of the feedback capability in nonlinearly parameterized control systems, we consider the following model:

$$y_{t+1} = f(\theta, y_t) + u_t + w_{t+1} \tag{1}$$

where $\theta \in \mathbb{R}^p$, $p \geq 1$, is an unknown parameter vector in the p-dimensional Euclidean space, y_t, u_t and w_t are the system output, input and noise signals, respectively, $f(\cdot) : \mathbb{R}^{p+1} \to \mathbb{R}$ is a known nonlinear measurable function. We assume that the parameter vector and the noise sequence satisfy the following conditions:

A1) The unknown parameter vector θ belongs to a certain ball in $\mathbb{R}^p : \Theta_0 = \{\theta : \|\theta\| \leq R\}$, where $R > 0$ may not be known *a priori*.

A2) The noise sequence is an arbitrarily bounded sequence with an upper bound $w > 0$, i.e.

$$\sup_{t\geq 1} |w_t| \leq w. \tag{2}$$

We are interested to know whether or not the above uncertain systems can be stabilized by feedback. The precise definitions of "feedback" and "global stabilization" by feedback are referred to e.g. [14] and [26]. To establish our main results of the technical note, we need some further structural conditions on the nonlinear function $f(\cdot,\cdot)$ as follows.

A3) The sensitivity function of θ defined by $f'(\theta, x) \triangleq (\partial f(\theta, x)/\partial \theta) \triangleq (f_1'(\theta, x), \cdots, f_p'(\theta, x))^\tau$ exists and is continuous, which has the following growth rates as $x \to \infty$

$$\left|f_i'(\theta, x)\right| = \Theta\left(|x|^{b_i}\right), \quad 1 \leq i \leq p \tag{3}$$

uniformly in θ, where $b \triangleq (b_1, \cdots, b_p) \in \Omega$ with[1] $\Omega \triangleq \{b : b_1 > \cdots > b_p > 0 \text{ with } b_1 > 1\}$. In the following, all functions $f(\cdot,\cdot)$ satisfying Assumption A3) will be denoted as a class $\mathcal{F}(b)$.

For any given $b = (b_1, \cdots, b_p)$, let us introduce a polynomial $P(x)$ defined by

$$P(x) = x^{p+1} - b_1 x^p + \sum_{i=1}^{p-1} (b_i - b_{i+1}) x + b_p. \tag{4}$$

Let us define $\Omega_s \triangleq \{b \in \Omega : P(x) > 0 \text{ for } \forall x \in (1, b_1)\}$ as the stabilizability set and $\Omega_u \triangleq \{b \in \Omega : b_1 \geq 4\}$ as the unstabilizability set.

Our main results of this technical note will justify the above nomenclature by showing that, for any function $f(\cdot,\cdot) \in \mathcal{F}(b)$, the system (1) is globally stabilizable for $b \in \Omega_s$ and is not stabilizable for $b \in \Omega_u$. We first give a result on stabilizability.

Theorem 2.1: Under the Assumptions A1)-A2), for any $f(\cdot,\cdot) \in \mathcal{F}(b)$, the corresponding uncertain system (1) is globally stabilizable by feedback provided that $b \in \Omega_s$.

Moreover, we have the following "Impossibility Theorem":

Theorem 2.2: Under the Assumptions A1)-A2), for any $f(\cdot,\cdot) \in \mathcal{F}(b)$, the corresponding uncertain system (1) is **not** globally stabilizable by feedback if $b \in \Omega_u$.

[1]In the case where the growth rates are bounded by linear, i.e., $0 \leq b_1 \leq 1$, the system (1) is already known to be globally stabilizable by feedback (see, [13]).

Remark 2.1: By Theorems 2.1 and 2.2, we obviously have $\Omega_u \subset \Omega_s^c$. But by [24, Corollary 3], it is not hard to see that $\Omega_u \neq \Omega_s^c$ for $p \geq 2$. This demonstrates that there is a gap between the stabilizable set and unstabilizable set. However, we remark that both Theorems 2.1 and 2.2 are the best possible results that one could have under the given conditions, since the function class $\mathcal{F}(b)$ is rather general and hence "too rich". This will be explained in detail in the following.

First, we point out that in Theorem 2.1, the condition $b \in \Omega_s$ cannot be relaxed without further specifications on $f(\cdot,\cdot)$, in addition to the condition $f(\cdot,\cdot) \in \mathcal{F}(b)$ as given in A3). To show this, we consider the following polynomial regression model:

$$y_{t+1} = \theta_1 y_t^{b_1} + \theta_2 y_t^{b_2} + \cdots + \theta_p y_t^{b_p} + u_t + w_{t+1} \tag{5}$$

which obviously belongs to $\mathcal{F}(b)$. By [17], we know that for any $b \in \Omega_s^c$, the system (5) is not globally stabilizable. This shows that the stabilizability set Ω_s in Theorem 2.1 cannot be expanded anymore under the general condition $f(\cdot,\cdot) \in \mathcal{F}(b)$. It is worthy noting that although for some $f(\cdot) \in \mathcal{F}(b), b \in \Omega_s$ is a necessary and sufficient condition for stabilizability, this fact may not be true for other functions $f(\cdot) \in \mathcal{F}(b)$. The example (6) below is such a counterexample.

Similarly, the condition $b \in \Omega_u$ in Theorem 2.2 also cannot be relaxed under the general condition $f(\cdot,\cdot) \in \mathcal{F}(b)$. The following example may explain this point:

$$y_{t+1} = \theta_1 y_t^{b_1} \exp\left\{\sum_{i=2}^{p} \frac{\theta_i}{|y_t|^{b_1 - b_i} + 1}\right\} + u_t + w_{t+1} \tag{6}$$

which can be easily verified to belong to $\mathcal{F}(b)$. However, as will be shown later in the proof of Theorem 2.3 (see (25) in Section III-C), the system (6) is stabilizable whenever $b \in \Omega_u^c$. Hence, the unstabilizability set Ω_u cannot be enlarged in general. For the same reason, $b \in \Omega_u^c$ cannot be a necessary and sufficient condition for the stabilizability for all $f(\cdot) \in \mathcal{F}(b)$, as has been shown by the example (5).

We remark that even though $\Omega_s \subset \Omega_u^c$ in general, in the special case where $p = 1$, we do have $\Omega_s = \Omega_u^c = \{b_1 : 1 < b_1 < 4\}$. As a corollary, we obtain a critical value for feedback capability of nonlinearly parameterized uncertain systems, which is similar, but in fact, an extension of that established in [5] and [16] for the linearly parameterized case.

Corollary 2.1: Let $p = 1$, and the Assumptions A1)-A2) hold. Then, the uncertain system (1) with $f(\cdot,\cdot) \in \mathcal{F}(b)$ is globally stabilizable by feedback **if and only if** $b \in \Omega_s$.

For the case where $p > 1$, there will indeed be a gap between the sets Ω_s and Ω_u^c. Now, let us further consider the connections between Theorems 2.1 and 2.2. As we already know, Ω_s and Ω_u are the stabilizable set and unstabilizable set respectively, and $\Omega_s \subset \Omega_u^c$. One may be curious to know the following question: Whether or not there exists a parameter set Ω_α with $\Omega_s \subset \Omega_\alpha \subset \Omega_u^c$, such that Ω_α can serve as a universal criterion of the feedback capability for each $f(\cdot,\cdot) \in \mathcal{F}(b)$ with $b \in \Omega$? The answer is no, because the family of functions $\mathcal{F}(b)$ is too rich, as will be demonstrated in the following lemma.

Lemma 2.1: For any given parameter set Ω_α with $\Omega_s \subset \Omega_\alpha \subset \Omega_u^c$, there always exists a function $f_\alpha(\theta, x, b) \in \mathcal{F}(b)$ such that under Assumptions A1)-A2), the uncertain system (1) with $f(\theta, x) = f_\alpha(\theta, x, b)$ is globally stabilizable by feedback **if and only if** $b \in \Omega_\alpha$.

Conversely, we can also find a mapping from any function $f(\cdot,\cdot) \in \mathcal{F}(b), b \in \Omega$ to a parameter set of b, which characterizes the feedback capability.

Lemma 2.2: For any function $f_\alpha(\cdot,\cdot) \in \mathcal{F}(b), b \in \Omega$, let it be explicitly parameterized as $f_\alpha(\cdot,\cdot,b)$. Then, there is a corresponding parameter set Ω_α with $\Omega_s \subset \Omega_\alpha \subset \Omega_u^c$, such that the system (1) with $f(\theta, x) = f_\alpha(\theta, x, b)$ is stabilizable by feedback **if and only if** $b \in \Omega_\alpha$.

Now, we are in a position to answer the following question: Can the complicated class of functions $\mathcal{F}(b)$ be decomposed as the union of finite or countable number of subclasses, with each subclass corresponds to a criterion for feedback capability? The answer is again no, because the union of such a decomposition has the cardinality of the continuum. A complete characterization is given in the following theorem.

Theorem 2.3: Consider the function class $\mathcal{F}(b)$ as defined in Assumption A3). Let Ω_α be any set of parameter b such that $\Omega_s \subset \Omega_\alpha \subset \Omega_u^c$. Then $\mathcal{F}(b)$ can be decomposed as follows:

$$\mathcal{F}(b) = \bigcup_{\alpha} \mathcal{F}_\alpha(b)$$

where $\mathcal{F}_\alpha(b) \subset \mathcal{F}(b)$ are disjoint and nonempty families of functions for different α, such that for each $f(\cdot,\cdot) \in \mathcal{F}_\alpha(b)$, the corresponding system (1) is stabilizable by feedback **if and only if** $b \in \Omega_\alpha$.

III. Proofs of Theorems 2.1–2.3

We will first prove Theorem 2.1.

A. Proof of Theorem 2.1

First, by (3) we know that there exist some $x' > 1$ and $c_2 \geq c_1 > 0$ such that

$$c_1 \leq \frac{|f_i'(\theta, x)|}{|x|^{b_i}} \leq c_2, \quad \forall |x| \geq x',\ i = 1, 2, \cdots, p. \tag{7}$$

For any $t \geq 1$, let us define $\{i_j(t), 1 \leq j \leq p\}$ sequentially by

$$i_1(t) := \arg\max_{0 \leq i \leq t-1} |y_i|; \tag{8}$$

$$i_j(t) := \arg\max_{\substack{0 \leq i \leq t-1 \\ c|y_i| < \left|y_{i_{j-1}(t)}\right|}} |y_i|, \quad 2 \leq j \leq p, \tag{9}$$

where $c = \sqrt[\min_{1 \leq k \leq p-1}(b_k - b_{k+1})]{(c_1^p(p-1))^{-1}(c_2^p p! p)}$.

Without loss of generality (see [17, pp. 281]), we may suppose $|y_{t_0}| \geq x'$, where x' is given in (7). The key lemma to estimate the parameter vector is as follows.

Lemma 3.1: Consider the equations for any $t \geq 1$ and $j = 1, 2 \cdots, p$:

$$F_{i_j(t)} = f\left(\theta, y_{i_j(t)}\right) + u_{i_j(t)} + w_{i_j(t)+1} - y_{i_j(t)+1}. \tag{10}$$

Then there exists a map $g_t \in C^1$ such that $\theta = g_t(v_t)$, where $v_t = (w_{i_1(t)+1}, \cdots, w_{i_p(t)+1})^\tau$. Furthermore, let $D_t^{-1}(\theta) \triangleq (f'(\theta, y_{i_1(t)}), \cdots, f'(\theta, y_{i_p(t)}))^\tau$. Then $g_t(v_t)$ is differentiable with respect to v_t, and satisfies $g_t'(v_t) = -D_t(\theta)$, where $g_t'(v_t) = \partial g_t(v_t)/\partial v_t$.

Proof: By the system (1), the vector defined by $F_t(\theta, v_t) \triangleq (F_{i_1(t)}, \cdots, F_{i_p(t)})^\tau = 0$, and for any $t \geq 1$, it is differentiable on the domain $\Theta_0 \times [-w, w]^p$. Moreover, by (8), we have $(\partial F/\partial\theta)(\theta, v_t) = D_t^{-1}(\theta) \neq 0$, then the first assertion is obviously true by the implicit function theorem. Next, let $F_{v_t}'(\theta, v_t) = (\partial F/\partial v_t)(\theta, v_t)$ and $F_\theta'(\theta, v_t) = (\partial F/\partial\theta)(\theta, v_t)$, the implicit function theorem also gives

$$g_t'(v_t) = -\left[F_\theta'\left(g_t(v_t), v_t\right)\right]^{-1}\left[F_{v_t}'\left(g_t(v_t), v_t\right)\right].$$

Note that $F_{v_t}'(g_t(v_t), v_t) = I$ and $F_\theta'(g_t(v_t), v_t) = D_t^{-1}(\theta)$, we obtain the second assertion. ■

Now, we proceed to design the controller. Let the parameter estimator be defined as

$$\begin{cases} \hat{\theta}_t = 0 & \text{for } 0 \leq t \leq t_0 - 1; \\ \hat{\theta}_t = g_t(0) & \text{for } t \geq t_0. \end{cases} \tag{11}$$

Then, we may let $u_t = -f(\hat{\theta}_t, y_t)$, where $\hat{\theta}_t$ is defined by (11). Thus, the system (1) at time $t \geq t_0$ can be written as

$$\begin{aligned} y_{t+1} &= f\left(g_t(v_t), y_t\right) - f\left(g_t(0), y_t\right) + w_{t+1} \\ &= -f'^\tau(\theta^*, y_t) D_t(\theta^*) v_t + w_{t+1}, \end{aligned} \tag{12}$$

where $\theta^* = g_t(v_t^*)$, and v_t^* is some vector. Note that $A^{-1} = \mathrm{Adj}\, A \cdot \det^{-1} A$ for any $\det A \neq 0$, where $\mathrm{Adj}\, A$ is the adjoint matrix of A. By some simple manipulations we have

$$\begin{aligned} |y_{t+1}| &\leq \left|f'^\tau(\theta^*, y_t) D_t(\theta^*) v_t\right| + w \\ &\leq c_2 w \sum_{i=1}^p |y_t|^{b_i} \frac{\sum_{k=1}^p |d_{k,i}(\theta^*, t)|}{\left|\det D_t^{-1}(\theta^*)\right|} + w, \end{aligned}$$

where $d_{k,i}(\theta^*, t)$ is the (k, i)-th cofactor of $\det D_t^{-1}(\theta^*)$.

The rest of the proof is similar to that in [17, Section IV, pp. 283] by using a contradiction argument, which gives $\sup_{t \geq 0} |y_t| < \infty$, and completes the proof of Theorem 2.1.

B. Proof of Theorem 2.2

The proof is based on a proposition, which considers the following more general nonlinearly parameterized model:

$$y_{t+1} = f(\theta, \phi_t) + w_{t+1} \tag{13}$$

where $\theta \in \mathbb{R}^p$, $p \geq 1$ is an unknown parameter vector, $\phi_t = (y_t, u_t; \cdots; y_{t-d}, u_{t-d})$ with $d \geq 0$, u_t and w_t are the system regression vector, feedback law and noise signal respectively, and where $f(\cdot,\cdot) : \mathbb{R}^{2(d+1)+p} \to \mathbb{R}$ is a nonlinear measurable function. Denote the partial derivative of $f(\theta, \phi)$ with respect to θ by $f'(\theta, \phi)$, which is assumed to exist and be continuous.

Before the proposition is presented, we first give a regularity assumption on the nonlinear function $f(\cdot,\cdot)$.

A4) For any $\varepsilon > 0$, there exists a non-increasing and nonnegative function $h(\varepsilon)$ such that for any $\phi \in R^{2d+2}$ with $\|\phi\| > h(\varepsilon)$, the set $\Delta_{\varepsilon,\phi}$ defined by

$$\Delta_{\varepsilon,\phi} \triangleq \left\{\theta \in \Theta_0 : |f(\theta, \phi)| < \varepsilon \max_\theta \left\|f'(\theta, \phi)\right\|\right\} \tag{14}$$

satisfies $L(\Delta_{\varepsilon,\phi}) \leq M\varepsilon$, where $L(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}^p$ and $M > 0$ is some constant. Moreover, assume that for $\|\phi\| > h(\delta)$, $f_i'(\theta, \phi) \neq 0$ for $\theta \in \Theta_0$, $i = 1, 2, \cdots, p$, where δ is any real number in $(0, (16M)^{-1}\pi R^2]$.

Proposition 3.1: Let Assumption A4) be satisfied. Then, there exist a parameter $\theta \in \Theta_0$ and a noise sequence $\{w_i\}$ satisfying A2) such that the corresponding outputs of system (13) satisfy:

$$y_{t+1}^2 \geq \frac{1}{K_1(t+1)^4}\left(\frac{d_t}{r_{t-1}} - 1\right) - K_2 \tag{15}$$

provided that the regressors satisfy $\|\phi_i\| > h((i+1)^{-2}\delta)$ for all $i \leq t$, where

$$d_t = \sum_{j=1}^p \min_{\theta \in \Theta_0} \left|f_j'(\theta, \phi_t)\right|^2$$

$$r_{t-1} \triangleq 1 + t^4 \max_{\theta \in \Theta_0}\left[\sum_{i=0}^{t-1} f'^\tau(\theta, \phi_i) f'(\theta, \phi_i)\right] \tag{16}$$

with $r_{-1} \triangleq K$ and $K_1, K_2 > 0$ are some constants.

Proposition 3.1 can be proven by a stochastic imbedding method (cf. [14]), and the detailed proof will be omitted here due to space limitation. The interested readers are referred to [15]. To prove Theorem 2.2 by applying Proposition 3.1, we need to prove that Assumption A4) holds automatically for sufficiently large $\|\phi\|$ by Assumption A3). This is the content of the following lemma.

Lemma 3.2: Let $f(\theta,\phi):\mathbb{R}^{p+1}\to\mathbb{R}$ be a nonlinear function with $\phi\in\mathbb{R}$. Then, Assumption A4) holds under A3).

Proof: Let $\theta,\theta'\in\Delta_{\varepsilon,\phi}$, where $\Delta_{\varepsilon,\phi}$ is defied in Assumption A4). Since (7) is satisfied with $x=\phi$, by the definition of $\Delta_{\varepsilon,\phi}$, it is easy to see that if $|\phi|\geq x'$,

$$\begin{aligned}\left|f(\theta,\phi)-f(\theta',\phi)\right| &\leq 2\varepsilon\max_{\theta}\left\|f'(\theta,\phi)\right\|\\ &\leq 2\varepsilon\sqrt{p}c_2|\phi|^{b_1}. \end{aligned}\tag{17}$$

Note that $\|\theta\|,\|\theta'\|\leq R$ and $|\phi|^{b_1}\geq|\phi|^{b_i}$ for $i=2,3,\cdots,p$, by (7),

$$\begin{aligned}\left|f(\theta,\phi)-f(\theta',\phi)\right| &=\left|\sum_{i=1}^{p}f_i(\theta^*,\phi)\left(\theta_i-\theta_i'\right)\right|\\ &\geq c_1|\phi|^{b_1}\left|\theta_1-\theta_1'\right|-2Rc_2(p-1)|\phi|^{b_2},\end{aligned}$$

where $\theta^*\in\Theta$. So, by (17), we immediately obtain

$$\left|\theta_1-\theta_1'\right|\leq\frac{2c_2\varepsilon\sqrt{p}+2Rc_2(p-1)|\phi|^{b_2}}{c_1|\phi|^{b_1}}$$

As a result, for any $|\phi|>\sqrt[b_1-b_2]{R(p-1)(\sqrt{p}\varepsilon)^{-1}}$, we have $|\theta_1-\theta_1'|\leq 4c_2\sqrt{p}\varepsilon c_1^{-1}$.

Since θ and θ' are arbitrary points in Δ_ε and $|\theta_i|\leq R, i=2,\cdots,p$ for all $\theta\in\Theta$, we have $L(\Delta_\varepsilon)\leq 2^{p+1}R^{p-1}c_2\sqrt{p}\varepsilon c_1^{-1}$. Let $M=2^{p+1}R^{p-1}c_2\sqrt{p}c_1^{-1}$ and

$$h(\varepsilon)=\max\left\{\left(\frac{R(p-1)}{\sqrt{p}\varepsilon}\right)^{\frac{1}{b_1-b_2}},x'\right\}\tag{18}$$

where x' is defined by (7). According to A3), we immediately conclude Assumption A4). ■

Proof of Theorem 2.2: Let $h(\cdot)$ be defined by (18). By Proposition 3.1, if

$$|y_t|>h\left(\frac{\delta}{(t+1)^2}\right)$$

there exist a θ and a $\{w_t\}$ such that the corresponding output sequence satisfies (15). Since $f_i'(\theta,y_t)=\Theta(y_t^{b_i})$ uniformly in θ whenever $|y_t|\geq x'$, we have for any $i=1,\cdots,p$ and $1\leq l\leq t$, there is a $M_2>0$ such that $r_{t-1}\leq M_2t^4\sum_{i=1}^{p}\sum_{l=0}^{t-1}y_l^{2b_i}+1$, where r_{t-1} is defined by (16). Consequently, for some $M_3>0$,

$$y_{t+1}^2\geq\frac{1}{M_3(t+1)^8}\left(\frac{y_t^{2b_1}}{\sum_{i=1}^{p}\sum_{l=0}^{t-1}y_l^{2b_i}+1}-1\right)-K_2,\ t\geq 0.\tag{19}$$

Furthermore, since $b_1>4$, there exists a root $\lambda\in(1,b_1)$ of the polynomial $x^2-b_1x+b_1$. We now prove by induction that for $t\geq 1$, there are two constants $W',W>0$ such that

$$|y_{t+1}|\geq\frac{1}{W'(t+1)^4}\left|\frac{y_t}{y_{t-1}}\right|^{b_1},\tag{20}$$

$$\begin{aligned}|y_{t+1}| &\geq\frac{W^{\lambda-1}t^{4\lambda}}{(t+1)^4}|y_t|^\lambda\\ &>h\left(\frac{\delta}{(t+2)^2}\right),\end{aligned}\tag{21}$$

whenever $|y_0|\geq 1$ is large enough, where δ is defined in Proposition 3.1. In fact, by (19),

$$y_1^2\geq\frac{1}{M_3}\left(y_0^{2b_1}-1\right)-K_2.\tag{22}$$

Then, it is easy to see that there is a $W'>0$ such that (20) holds for $t=1$ by (19) when $|y_0|$ is sufficiently large. Let $W>0$ be a constant such that

$$W\geq\sup_t\left(\frac{t+1}{t}\right)^{4b_1}W'.\tag{23}$$

Similar to (22), by (20) for $t=1$, the inequality (21) holds when $|y_0|$ is sufficiently large.

Now, suppose (20) and (21) hold for $t\leq k$, where $k\geq 1$ is some integer. For $t=k+1$, by the induction assumption it is easy to obtain (20) from (19). As a result, by (20) and (23),

$$W(k+2)^4|y_{k+2}|\geq\left|\frac{W(k+1)^4y_{k+1}}{Wk^4y_k}\right|^{b_1}.$$

Next, let $a_k=Wk^4|y_k|$. The above inequality can be rewritten by $a_{k+2}\geq|a_{k+1}/a_k|^{b_1}$. Since (21) holds for $t\leq k$, which means $a_{t+1}\geq a_t^\lambda$, $t=1,\cdots,k$. By [17, Lemma 3.3], we immediately get $a_{k+2}\geq a_{k+1}^\lambda$. That is

$$|y_{k+2}|\geq\frac{W^{\lambda-1}(k+1)^{4\lambda}}{(k+2)^4}|y_{k+1}|^\lambda$$

which also implies the second inequality in (21). By induction, we have (20) and (21) hold for all $t\geq 1$. Hence, $\{y_t\}$ diverges to infinity exponentially. ■

C. Proof of Theorem 2.3

We first prove Lemmas 2.1 and 2.2.

Proof of Lemma 2.1: We adopt a constructive method to prove this theorem. First, for any given Ω_α with $\Omega_s\subset\Omega_\alpha\subset\Omega_u^c$, we construct a function as follows:

$$f_\alpha(\theta,x,b)\triangleq\begin{cases}\theta_1\exp\left\{\sum_{i=2}^{p}\frac{\theta_i}{|x|^{b_1-b_i}+1}\right\}x^{b_1}, & b\in\Omega_\alpha\\ \theta_1x^{b_1}+\theta_2x^{b_2}+\cdots+\theta_px^p, & b\notin\Omega_\alpha.\end{cases}\tag{24}$$

We now first prove that if $b\in\Omega_\alpha$, the system (1) with $f\in\mathcal{F}_\alpha(b)$ is stabilizable. Note that for $b\in\Omega_\alpha$, $b_1<4$ and

$$f_\alpha(\theta,x,b)=\theta_1\exp\left\{\sum_{i=2}^{p}\frac{\theta_i}{|x|^{b_1-b_i}+1}\right\}x^{b_1}.\tag{25}$$

We only need to prove that the system (1) with $f(\theta,x)=f_\alpha(\theta,x,b)$ is stabilizable for any $b_1<4$.

The idea of the proof is similar to that of [16], so we only write down the differences in the proofs. Without loss of generality, suppose $y_0\neq 0$. For any $t\geq 1$, let

$$i_t:=\arg\max_{0\leq i\leq t-1}|y_i|.\tag{26}$$

Let $\theta_t=\theta_1\exp\{\sum_{i=2}^{p}(\theta_i/(1+|y_t|^{b_1-b_i})\}$ and I_t be the domain of all θ_t's possible values. By (25) and the system (1), θ_{i_t} is actually

$$\theta_{i_t}=\frac{y_{i_t+1}-u_{i_t}-w_{i_t+1}}{y_{i_t}^b}.$$

Since w_{i_t+1} is unknown and $|w_{i_t+1}|\leq w$, then $|I_{i_t}|\leq(w/|y_{i_t}|^b)$, where $|\cdot|$ is the length of an interval. Note that

$$\frac{\theta_t}{\theta_{i_t}}=\exp\left\{\sum_{i=2}^{p}\left(\frac{\theta_i}{1+|y_t|^{b_1-b_i}}-\frac{\theta_i}{1+|y_{i_t}|^{b_1-b_i}}\right)\right\}$$

we have

$$\frac{\theta_t}{\theta_{i_t}} \le \max\left\{\exp\left\{\sum_{i=2}^{p}\frac{\theta_i}{1+|y_t|^{b_1-b_i}}\right\},\ \exp\left\{\sum_{i=2}^{p}\frac{\theta_i}{1+|y_{i_t}|^{b_1-b_i}}\right\}\right\} \le \exp\left\{\sum_{i=2}^{p}\bar{\theta}_i\right\}.$$

So, the domain I_t of all θ_t's possible values can be shown to be an interval similarly and satisfies

$$|I_t| \le \exp\left\{\sum_{i=2}^{p}\bar{\theta}_i\right\}|I_{i_t}| \le \frac{\exp\left\{\sum_{i=2}^{p}\bar{\theta}_i\right\}w}{|y_{i_t}|^b}. \tag{27}$$

Let $\hat{\theta}_t$ be the center point of I_t, we define the control sequence as the following:

$$u_0 = 0; \quad u_t = -\hat{\theta}_t y_t^b, \quad \text{for } t \ge 1. \tag{28}$$

Then for $t \ge 1$, the closed-loop dynamics is

$$y_{t+1} = (\theta_t - \hat{\theta}_t)y_t^b + w_{t+1}. \tag{29}$$

Therefore, noting that the noise are bounded, by (27)

$$|y_{t+1}| \le \frac{\exp\left\{\sum_{i=2}^{p}\bar{\theta}_i\right\}w}{|y_{i_t}|^b}|y_t|^b + w, \quad \text{for } \forall t \ge 1. \tag{30}$$

Then by (26), we have

$$|y_{t+1}| \le \frac{\exp\left\{\sum_{i=2}^{p}\bar{\theta}_i\right\}w}{\max\limits_{0\le i\le t-1}|y_i|^b}|y_t|^b + w, \quad \text{for } \forall t \ge 1. \tag{31}$$

The rest arguments are the same as that in [16, Section 3, pp. 454], and will not be repeated. Thus, the system (1) with $f(\theta,x) = f_\alpha(\theta,x,b)$ is stabilizable for any $b \in \Omega_\alpha$.

On the other hand, if $b \notin \Omega_\alpha$, we have by (24)

$$f_\alpha(\theta,x,b) = \theta_1 x^{b_1} + \theta_2 x^{b_2} + \cdots + \theta_p x^p.$$

By [17], we immediately know that if there is a $x \in (1, b_1)$ such that $P(x) \le 0$, where $P(x)$ is defined by (4), then, the system (1) with $f(\theta,x) = f_\alpha(\theta,x,b)$ is not globally stabilizable by any feedback. Since $b \notin \Omega_\alpha$ implies $b \notin \Omega_s$, the above condition is obviously satisfied, and the necessary part is also proved. This completes the proof of Lemma 2.1. ■

Proof of Lemma 2.2: Given any $f_\alpha(\cdot,\cdot,b) \in \mathcal{F}(b)$, the system (1) can be written as

$$y_{t+1} = f_\alpha(\theta,x,b) + u_t + w_{t+1}, \quad b \in \Omega \tag{32}$$

where θ and $\{w_t\}$ satisfy Assumptions A1) and A2), respectively. Let us define $\Omega_\alpha \subset \Omega$ by

$$\Omega_\alpha = \{b \in \Omega : \text{the system (32) is stabilizable by feedback}\}.$$

Obviously, Ω_α must satisfy $\Omega_s \subset \Omega_\alpha \subset \Omega_u^c$, because by Theorems 2.1 and 2.2 and the definition of Ω_α, we know that $\Omega_s \cap \Omega_\alpha^c$ and $\Omega_\alpha \cap \Omega_u$ are both empty sets. ■

Proof of Theorem 2.3: For any Ω_α with $\Omega_s \subset \Omega_\alpha \subset \Omega_u^c$, let us define

$$\mathcal{F}_\alpha(b) \triangleq \{f(\cdot,\cdot) \in \mathcal{F}(b) : \text{the system (1) is stabilizable by feedback if and only if } b \in \Omega_\alpha\}.$$

Then, by Lemmas 2.1 and 2.2, it is easy to see that the theorem is true. ■

IV. Conclusion

The primary motivation and contribution of this technical note is to explore the maximum capability and fundamental limitations of the feedback principle in stabilizing uncertain nonlinear dynamical systems. To the best of our knowledge, this technical note seems to be the first to address these issues for discrete-time nonlinearly parameterized uncertain systems with sensitivity functions having growth rates faster than linear. Such growth rates, parameterized by $(b_1, \cdots, b_p)$, are used in characterizing both "possibility" and "impossibility" theorems on the capability of feedback as established in the technical note. Finally, we would like to point out that, while various extensions or improvements of the technical note may be expected, the impossibility results established in Theorems 2.2 and 2.3 are actually applicable to any larger class of uncertain systems that includes the basic model class (1) as a subclass.

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Theories and Ultra Efficient Computation of Joint Spectral Radius for Estimating First Passage Time Distribution of Markov Set-Chain

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***Abstract*—This technical note is concerned with the tail distribution of the first passage time of Markov set chains (MSC). An original two-part idea—a *more progressive* relation and a *sortedness* test—is conceived to characterize such chains. The theoretical construction based on this idea further results in an algorithm that can compute the tightest exponent bound of the tail distribution for high-dimensional problem instances with surprising ease. To understand the computational implication of this algorithm, note that the problem is equivalent to computing the joint spectral radius (JSR) of a special independent column polytope (one that defines Markov set chains) of nonnegative matrices. In this context, the reported algorithm can compute the exact JSR value for cases of 100 × 100 matrices in less than a second in Matlab. Problems of this size is far beyond the scope of known JSR techniques. It is worth noting that the fields of MSC and JSR have not had significant overlap as one may expect, despite their conceptual akiness. Meanwhile, the present technical note is a contribution that belongs to both fields.**

***Index Terms*—Joint spectral radius (JSR), Markov set chains (MSC).**

I. Introduction

This technical note is concerned with the first passage time of finite Markov chains with uncertain and possibly time-varying transition probabilities. More specifically, we are interested in estimating the exponent bounds of the tail distribution of the first passage time as it approaches infinity, with the only knowledge that the transition probability matrix is contained in some set at all times. Note also that we do not assume slowly varying transition matrices. In this pursuit, some interesting structures are discovered, which lead to an algorithm that computes the exponents with surprising efficiency.

Two bodies of literature are particularly relevant: Markov set-chains (MSC), such as [11], [20], [21], and *Joint Spectral Radius (JSR)*, such as [2], [4], [9], [18]. The former is a standard model of Markov chains with uncertain and time-varying parameters. The MSC literature contains more theoretical results such as conditions of ergodic properties, while the JSR research concentrates more on computational issues. It is well known that the spectral structure of nonnegative matrices plays a key role in Markov chains. However, the actual impact from JSR research to MSC is limited (see for example [12], which is useful for slowly varying inhomogeneous chains). One observation is that the uncertainty model in MSC is a polytope of transition matrices, while the JSR problem is most often considered for a finite set of matrices. Although the JSR of a polytope of matrices can be reduced to that of the extreme points [3], that would result in a huge number of points to consider. To give an idea, note that the set of all $n \times n$ substochastic matrices has $(n+1)^n$ extreme points. On the other hand, the JSR problem is very hard for even a pair of matrices [22]. A set of $(n+1)^n$ matrices (or something of similar order of magnitude) would be out of the question for $n > 10$. More detailed discussion of known results in MSC and JSR will be given in a separate section.

In the present technical note, the Markov set-chains under consideration are analysed in terms of a *more/less progressive* relation to *sorted* homogeneous chains. Both ideas of progressivity and sortedness are original. This construction further results in an algorithm (Algorithm 1.) with surprising efficiency for computing the exact sought-after exponents. In related JSR terms, what we compute is the JSR of some set of nonnegative matrices that represents independent column uncertainties. It has been proved in [5] that the JSR in this case is less elusive than the general case, as it coincides with the spectral radius of some extreme point of that set. But the algorithm that we provide has much less computation cost than computing the spectral radius of all extreme points, or the method in [5]. Numerically, cases of $n = 100$ are solved by our algorithm in less than a second in Matlab. This is not possible with a straightforward adoption of any known technique.

The fact that our algorithm computes the exact JSR value is formally proved, while the claim that the algorithm always finishes very quickly is in an empirical sense based on a large number of numerical experiments. The formal proof of the latter is most likely non-trivial and most likely probabilistic. It is also true that the present technical note specializes in nonnegative matrices, while many works on JSR apply to general matrices. In short, this technical note enhances the state of the art of JSR research in a particular direction.

Applicationwise, finite-state model is an effective approach for many problems, but it is only an approximation of the much richer reality. In many potential application areas of the probability theory, there is a difficulty of assigning exact probabilities to real-world events. This motivates us to consider what would happen if the transition probabilities in the finite-state model are uncertain within some range. It is worth noting that the modeling of hybrid systems using MSC has been suggested in [1].

In what follows, Section II formulates the problem. A detailed discussion of the MSC and JSR literature is then given in Section III. Section IV proves a rather general theorem for comparing uncertain, inhomogeneous processes against time-homogeneous Markov chains (Theorem 1). Section V then discusses a particularly useful construction (Theorem 2) involving two new concepts that we call *progressivity* and *sortedness*. The former is a new relation between substochastic matrices and the latter is a property that certifies a substochastic matrix

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