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关于自适应非线性镇定的某些结果

骆 曼†, 郭 雷

(中国科学院数学与系统科学研究院 系统科学研究所 系统控制重点实验室, 北京 100190)

摘要: 不确定非线性系统的反馈控制一直是控制科学的中心问题之一, 迄今已经取得很大进展. 然而, 目前现有大部分工作所研究的反馈控制规律, 或是连续时间形式的, 或是采样反馈形式但需要采样频率充分快, 或是离散时间反馈形式, 但需要被控离散时间系统的非线性函数增长速度不超过线性. 要消除或减弱这些约束条件, 一般来讲是相当困难的. 这就促使我们探究反馈机制的最大能力和根本局限. 尽管近年来在这个方向有许多重要进展, 但仍有许多非平凡的重要问题有待研究. 例如, 在反馈通道中有时滞情形, 或者系统状态是高维的情形. 在本文中, 我们将探索两类比较特殊的离散时间不确定非线性动力系统的控制问题, 给出关于全局自适应反馈镇定的某些初步结果.

关键词: 反馈; 自适应控制; 不确定性; 非线性系统; 全局稳定性

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Some results on adaptive nonlinear stabilization

LUO Man†, GUO Lei

(Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science,
Chinese Academy of Sciences, Beijing 100190, China)

Abstract: Feedback control of uncertain nonlinear dynamical systems has been a central issue in control theory, and considerable progress has been made up to now. However, most of the existing works concern with either continuous-time feedback laws, or sampled-data feedback laws with sufficiently fast sampling, or with discrete-time feedback laws for parametric nonlinear systems with nonlinearities having a linear growth rate. Removing these constraints turns out to be quite difficult in general, which motivates the study of the maximum capability and fundamental limitations of the feedback mechanism. Although much effort has been made in this direction in recent years, many problems still remain open. For example, the case where there is a pure time-delay in the feedback channel or the case where the system state is of high dimension remains to be unexplored, which appears to be highly nontrivial. In this paper, we shall present some preliminary results on global adaptive nonlinear stabilization, by investigating two special classes of discrete-time uncertain nonlinear dynamical systems with delayed feedback and with two dimensional state signal, respectively.

Key words: feedback; adaptive control; uncertainty; nonlinear systems; global stabilization

1 Introduction

It is well known that feedback is a most important concept of control systems, which distinguishes the area of control science with any other branches of science and technology. The main purpose of using feedback in control systems is to deal with the influences of various internal and external uncertainties on the performance of the systems to be controlled. Over the past 80 years, significant progress has been made in control theory[1], and various control methods have been investigated and proposed to deal with uncertainties in control systems, which include, for example, PID control, adaptive control, robust control, and fuzzy control, etc. Most of the existing works concern with either continuous-time feedback laws, or sampled-data feedback laws with sufficiently fast sampling, or with discrete-time feedback laws for parametric nonlinear systems with nonlinearities having a linear growth rate. However, in practice, most of the control laws are implemented with computers and the processing or feedback rates are usually not allowed to be arbitrarily high due to various constrains in e.g., communication, computation, and actuation, etc. Removing or relaxing the above-mentioned theoretical constraints turns out to be quite difficult in

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†Corresponding author. E-mail: luoman@amss.ac.cn; Tel.: +86 18811782479.

general, which motivates the study of the maximum capability and fundamental limitations of the feedback mechanism. Initiated by the work of Guo[2] in understanding the maximum capability of feedback, substantial progress has been made in recent years in various aspects which includes, for example, parametric nonlinear systems[3–6], nonparametric systems[7–8], sampled-data control systems[9–10]. These results have provided a series of ‘critical values’ and/or ‘impossibility theorems’ on the maximum capability of feedback, and reveal that the feedback capability depends on the nature of uncertainty, the structure of nonlinearity, and the timely use of feedback information.

Notwithstanding, there are still many important situations which have not been investigated up to now. Here, we just mention two cases as follows. One is the case where there is a time-delay in the feedback channel. As is well known, the presence of time-delay in feedback is very common in practical systems, for example, hierarchy control systems, communication systems, network systems and mechanical systems, etc[11–13]. It turns out that the investigation of feedback control for time-delay uncertain systems is not only important, but also highly nontrivial theoretically. Another is the case where the state variable is of high dimension with nonlinearities having a nonlinear growth rate. It appears that this case is more difficult to investigate in theory, since the previously used method for the scalar variable case cannot be applied directly in general.

In this paper, we shall investigate global adaptive stabilization problems for two special classes of discrete-time uncertain nonlinear dynamical systems with time-delayed feedback and high-dimensional state, respectively. In the former case, we will provide a sufficient condition for global feedback stabilization in terms of the Lipschitz constant for the uncertain class of Lipschitz functions, and reveal how the time-delay in the feedback channel influences the feedback capability. In the later case, we will establish global feedback stabilizability of a class of two dimensional parametric uncertain nonlinear dynamical systems, with nonlinearities having nonlinear growth rate. Part of the results of the paper has been reported in [14].

2 Time-delay systems

In this section, we consider the following first-order discrete time nonlinear dynamical systems with time-delay in the input channel:

$$y_{t+1} = f(y_t) + u_{t-d} + w_{t+1},\ t > 0,\ y_0 \in \mathbb{R}^1, \quad (1)$$

where $\{y_t\}$, $\{u_t\}$ and $\{w_t\}$ are output, input and noise signals of the system, respectively, and $d \geqslant 0$ is the time-delay. The nonlinear function $f(\cdot): \mathbb{R}^1 \to \mathbb{R}^1$ is unknown a priori, but belongs to a class of generalized Lipschitz functions denoted by

$$\mathcal{F}(L) \triangleq \{f(\cdot): |f(x) - f(y)| \leqslant L|x-y| + c,\\ L > 0,\ c > 0,\ \forall x, y \in \mathbb{R}^1\}, \quad (2)$$

Assume further that $\{w_t\}$ is a sequence of ‘unknown but bounded noises’ with unknown bound $w > 0$, i.e.,

$$|w_t| \leqslant w,\ \forall t \geqslant 0. \quad (3)$$

To investigate the global adaptive stabilization problem, we need the following definitions first.

Definition 1[6] A sequence $\{u_t\}$ is called a feedback control law if at each step $t \geqslant 0$, u_t is a causal function of the observation $\{y_t\}$, i.e.,

$$u_t = h_t(y_0, \cdots, y_t), \quad (4)$$

where $h_t(\cdot): \mathbb{R}^t \to \mathbb{R}^1$ can be an arbitrary (nonlinear and time-varying) mapping at each step t.

With the feedback mechanism defined as above, our objective is to investigate how much uncertainty in $f(\cdot)$ can be dealt with by the delayed feedback control u_{t-d} in Eq.(1). The general $d \geqslant 1$ case appears to be quite complicated to investigate, which remains to be an open problem. Though out the paper we only consider the case where $d = 1$.

Let us denote

$$\begin{cases} \bar{b}_t \triangleq \max\limits_{0 \leqslant i \leqslant t} y_i, \\ \underline{b}_t \triangleq \min\limits_{0 \leqslant i \leqslant t} y_i,\ t \geqslant 0, \end{cases} \quad (5)$$

and

$$i_{t-1} \triangleq \arg \min_{0 \leqslant i \leqslant t-2} |y_{t-1} - y_i|.$$

i.e.,

$$|y_{t-1} - y_{i_{t-1}}| = \min_{0 \leqslant i \leqslant t-2} |y_{t-1} - y_i|,\ t \geqslant 2. \quad (6)$$

At any time instant $t \geqslant 2$, the estimate of $f(y_{t-1})$ is defined as

$$\hat{f}_t(y_{t-1}) = y_{i_{t-1}+1} - u_{i_{t-1}-1}, \quad (7)$$

which can be written as

$$\hat{f}_t(y_{t-1}) = f(y_{i_{t-1}}) + w_{i_{t-1}+1},\ t \geqslant 2. \quad (8)$$

We denote

$$j_t \triangleq \arg \min_{0 \leqslant j \leqslant t-2} |y_j - (\hat{f}_t(y_{t-1}) + u_{t-2})|,$$

i.e.,

$$|y_{j_t} - (\hat{f}_t(y_{t-1}) + u_{t-2})| =\\ \min_{0 \leqslant j \leqslant t-2} |y_i - (\hat{f}_t(y_{t-1}) + u_{t-2})|,\ t \geqslant 2. \quad (9)$$

Thus, y_{j_t} can be regarded as an estimate of $y_t = f(y_{t-1}) + u_{t-2} + w_t$. Consequently, respectively, an estimate of $f(y_t)$ can be defined as

$$\hat{f}_t(y_t) \triangleq y_{j_t+1} - u_{j_t-1}, \quad (10)$$

which can be rewritten as

$$\hat{f}_t(y_t) = f(y_{j_t}) + w_{j_t+1}. \tag{11}$$

Now, we define the feedback control law as

$$\begin{cases} u_{-1} \triangleq 0, \\ u_0 \triangleq 0, \\ u_{t-1} \triangleq -\hat{f}_t(y_t) + \dfrac{1}{2}(\underline{b}_{t-1} + \bar{b}_{t-1}),\ t \geqslant 2. \end{cases} \tag{12}$$

Theorem 1 For any $f \in \mathcal{F}(L)$ with $L < \dfrac{\sqrt{5+4\sqrt{2}}-1}{2}$, the feedback control (12) globally stabilizes the corresponding system (1) with $d = 1$, i.e.,

$$\limsup_{t\to\infty} |y_t| < \infty,\ \forall y_0 \in \mathbb{R}^1. \tag{13}$$

Remark 1 As is known, for the case where $d = 0$ in Eq.(1), Xie and Guo[8] has shown that the critical value for global feedback stabilization is $L = \frac{3}{2} + \sqrt{2}$. The current Theorem 1 shows how the unit time-delay will affect the capability of feedback. Moreover, observe that $L < \dfrac{\sqrt{5+4\sqrt{2}}-1}{2} \approx 1.284$, which shows that the class of uncertain systems $\mathcal{F}(L)$ is not open-loop stable and Theorem 1 is indeed nontrivial. Of course, whether or not the above value is critical remains unknown, but we conjecture that there is not much room for nontrivial generalization to the case where $d \geqslant 2$.

3 Proof of the Theorem 1

First we note that in the case where $L < 1$, the result is trivial, so now we just consider the case $L \geqslant 1$.

First, we need some notations [8]. Denote

$$B_t \triangleq [\underline{b}_t, \bar{b}_t],\ \Delta B_t \triangleq B_t - B_{t-1}, \tag{14}$$

and

$$|B_t| \triangleq \bar{b}_t - \underline{b}_t,\ |\Delta B_t| \triangleq |B_t| - |B_{t-1}|, \tag{15}$$

where $\Delta B_0 \triangleq B_0$, $\underline{b}_t$ and $\bar{b}_t$ are defined in (5). By the definition (5) we have

$$\underline{b}_t \leqslant \underline{b}_{t-1},\ \bar{b}_t \geqslant \bar{b}_{t-1},$$

and

$$\begin{aligned} &(\underline{b}_t - \underline{b}_{t-1})(\bar{b}_t - \bar{b}_{t-1}) = 0, \\ &|\underline{b}_t - \underline{b}_{t-1}| + |\bar{b}_t - \bar{b}_{t-1}| = |\Delta B_t|. \end{aligned} \tag{16}$$

Clearly, the interval sequence $\{B_t, t > 0\}$ is nondecreasing and that ΔB_t is also an interval (can be a null set $\varnothing$) and

$$B_t = \bigcup_{i=0}^{t} \Delta B_i,\ \Delta B_i \cap \Delta B_j = \varnothing,\ i \neq j. \tag{17}$$

For any point $a \in \mathbb{R}^1$ and any set $B \subset \mathbb{R}^1$, define a distance function $\mathrm{dis}(\cdot,\cdot)$ as

$$\mathrm{dis}(a, B) \triangleq \inf_{b\in B} |a - b|, \tag{18}$$

and if $B = \{b\}$, we rewrite $\mathrm{dis}(a, B)$ as $\mathrm{dis}(a, b) \triangleq |a - b|$. Then, it is easy to see that $|\Delta B_t| = \mathrm{dis}(y_t, B_{t-1})$, $t \geqslant 1$.

Now, we divide the proof into five steps.

Step 1 We analyze some properties of the notations (14)–(15).

First, it is obvious that

$$|B_{t+1}| = \begin{cases} |B_t|, & y_{t+1} \in B_t, \\ |y_{t+1} - \frac{1}{2}(\underline{b}_t + \bar{b}_t)| + \frac{1}{2}|B_t|, & y_{t+1} \notin B_t. \end{cases} \tag{19}$$

Since

$$|y_{t+1} - \frac{1}{2}(\underline{b}_t + \bar{b}_t)| > \frac{1}{2}|B_t| \Longleftrightarrow y_{t+1} \notin B_t,$$

we have

$$|B_{t+1}| = \max\{|y_{t+1} - \frac{1}{2}(\underline{b}_t + \bar{b}_t)| + \frac{1}{2}|B_t|, |B_t|\}. \tag{20}$$

Now, we proceed to prove that

$$|y_{t-1} - y_{i_{t-1}}| \leqslant \max_{0\leqslant i\leqslant t} |\Delta B_i|,\ \forall t \geqslant 2, \tag{21}$$

where i_t is defined in Eq.(6). We consider two cases separately.

Case 1 If $y_{t-1} \notin B_{t-2}$, then by definitions (5)–(6)(14)−(15), we have

$$|y_{t-1} - y_{i_{t-1}}| = |B_{t-1}| - |B_{t-2}| = |\Delta B_{t-1}|.$$

Case 2 If $y_t \in B_{t-2}$, then by (17), we know $y_t \in \Delta B_i$ for some $0 \leqslant i \leqslant t-2$. Then by Eq.(6) we have

$$|y_{t-1} - y_{i_{t-1}}| \leqslant |\Delta B_i|, \text{ for the same } i.$$

Combining the two cases above, we can see that inequality (21) is true.

Step 2 We get an error bound concerning y_t.

For simplicity of expression, we denote

$$\varGamma \triangleq f(y_{i_{t-1}}) + u_{t-2} + \omega_{i_{t-1}+1}, \tag{22}$$

and note that

$$y_t = f(y_{t-1}) + u_{t-2} + \omega_t.$$

Thus, by Eq.(2) and inequality (21), we have the following inequality about $|y_t - \varGamma|$:

$$\begin{aligned} |y_t - \varGamma| = &|[f(y_{t-1}) + u_{t-2} + \omega_t] - \\ &[f(y_{i_{t-1}}) + u_{t-2} + \omega_{i_{t-1}+1}]| \leqslant \\ &|f(y_{t-1}) - f(y_{i_{t-1}})| + |\omega_t - \omega_{i_{t-1}+1}| \leqslant \\ &L|y_{t-1} - y_{i_{t-1}}| + c + 2\omega \leqslant \\ &L \max_{0\leqslant i\leqslant t} |\Delta B_i| + c + 2\omega. \end{aligned} \tag{23}$$

Step 3 We consider the distance between $\varGamma$ and y_{j_t}. We will discuss it based on three different cases of $\varGamma$:

Case 1 If $\varGamma \in B_{t-2}$, by the definition of y_{j_t} in (9) and the same reason as Step 1, we have

$$|\varGamma - y_{j_t}| \leqslant \max_{0\leqslant i\leqslant t} |\Delta B_i|. \tag{24}$$

Case 2 If $\varGamma \in B_t - B_{t-2}$, then by the definition of y_{j_t} in (9), we have

$$|\Gamma - y_{j_t}| = \mathrm{dis}(\Gamma, B_{t-2}) \leqslant |B_t| - |B_{t-2}| \leqslant 2 \max_{0\leqslant i\leqslant t} |\Delta B_i|. \tag{25}$$

Case 3 If $\Gamma \notin B_t$, we consider two cases separately as below:

i) If $y_t \in B_{t-2}$, by Eq.(23), then we have

$$|\Gamma - y_{j_t}| = \mathrm{dis}(\Gamma, B_{t-2}) \leqslant |\Gamma - y_t| \leqslant L \max_{0\leqslant i\leqslant t} |\Delta B_i| + c + 2\omega. \tag{26}$$

ii) If $y_t \notin B_{t-2}$, by the triangle inequality of distance and Eq.(23), we have

$$\begin{aligned}&|\Gamma - y_{j_t}| = \mathrm{dis}(\Gamma, B_{t-2}) \leqslant \\ &\mathrm{dis}(\Gamma, y_t) + \mathrm{dis}(y_t, B_{t-2}) \leqslant \\ &|\Gamma - y_t| + |B_t| - |B_{t-2}| = \\ &|\Gamma - y_t| + |\Delta B_t| + |\Delta B_{t-1}| \leqslant \\ &L \max_{0\leqslant i\leqslant t} |\Delta B_i| + c + 2\omega + 2 \max_{0\leqslant i\leqslant t} |\Delta B_i| = \\ &(L+2) \max_{0\leqslant i\leqslant t} |\Delta B_i| + c + 2\omega.\end{aligned} \tag{27}$$

Combining the three cases above, we have

$$|\Gamma - y_{j_t}| \leqslant (L+2) \max_{0\leqslant i\leqslant t} |\Delta B_i| + c + 2\omega. \tag{28}$$

Step 4 We get a final bound between y_t and y_{j_t}. We consider two cases separately.

Case 1 $y_t \in B_{t-2}$.

i) If $\Gamma \in B_{t-2}$, by the triangle inequality, (23) and (24), we have

$$\begin{aligned}&|y_t - y_{j_t}| \leqslant |y_t - \Gamma| + |\Gamma - y_{j_t}| \leqslant \\ &L \max_{0\leqslant i\leqslant t} |\Delta B_i| + c + 2w + \max_{0\leqslant i\leqslant t} |\Delta B_i| = \\ &(L+1) \max_{0\leqslant i\leqslant t} |\Delta B_i| + c + 2w.\end{aligned}$$

ii) If $\Gamma \notin B_{t-2}$, by the definitions of y_{j_t} in (9), we have

$$|\Gamma - y_{j_t}| = dis(\Gamma, B_{t-2}).$$

So y_{j_t} must be just between Γ and y_t, then by Eq.(23) we have

$$|y_t - y_{j_t}| \leqslant |y_t - \Gamma| \leqslant L \max_{0\leqslant i\leqslant t} |\Delta B_i| + c + 2w.$$

Case 2 $y_t \notin B_{t-2}$.

i) If $\Gamma \in B_{t-2}$, by the triangle inequality and (23), (24), we have

$$\begin{aligned}&|y_t - y_{j_t}| \leqslant |y_t - \Gamma| + |\Gamma - y_{j_t}| \leqslant \\ &L \max_{0\leqslant i\leqslant t} |\Delta B_i| + c + 2w + \max_{0\leqslant i\leqslant t} |\Delta B_i| = \\ &(L+1) \max_{0\leqslant i\leqslant t} |\Delta B_i| + c + 2w.\end{aligned}$$

ii) If $\Gamma \notin B_{t-2}$, meanwhile, Γ and y_t are at the same side of B_{t-2}, then by the definition of y_{j_t} in (9), we can know that y_{j_t} is the bound of B_{t-2} which is nearer to Γ, therefore it is nearer to y_t too. Then we have

$$|y_t - y_{j_t}| = dis(y_t, B_{t-2}) \leqslant |B_t| - |B_{t-2}| \leqslant 2 \max_{0\leqslant i\leqslant t} |\Delta B_i|.$$

iii) If $\Gamma \notin B_{t-2}$ and Γ and y_t are at different sides of B_{t-2}, then y_{j_t} will be somewhere just between y_t and Γ, so by (23) we have

$$|y_t - y_{j_t}| \leqslant |y_t - \Gamma| \leqslant L \max_{0\leqslant i\leqslant t} |\Delta B_i| + c + 2w.$$

Combining the two cases above, we have

$$|y_t - y_{j_t}| \leqslant (L+1) \max_{0\leqslant i\leqslant t} |\Delta B_i| + c + 2w. \tag{29}$$

Step 5 Now, we proceed to find a recursive inequality on $\{|\Delta B_t|, t \geqslant 0\}$.

By Eqs.(10)–(12), we have

$$\begin{aligned}u_{t-1} = &-y_{j_t+1} + u_{j_t-1} + \frac{1}{2}(\underline{b}_{t-1} + \bar{b}_{t-1}) = \\ &-f(y_{j_t}) + \frac{1}{2}(\underline{b}_{t-1} + \bar{b}_{t-1}) - \omega_{j_t+1}, \\ y_{t+1} = &\ f(y_t) - f(y_{j_t}) + \frac{1}{2}(\underline{b}_{t-1} + \\ &\bar{b}_{t-1}) - \omega_{j_t+1} + \omega_{t+1}.\end{aligned}$$

Then by inequalities (2)(16) and (29), we have

$$\begin{aligned}&|y_{t+1} - \frac{1}{2}(\underline{b}_t + \bar{b}_t)| = \\ &|f(y_t) - f(y_{j_t}) - \frac{1}{2}((\underline{b}_t + \bar{b}_t) - \\ &(\underline{b}_{t-1} + \bar{b}_{t-1})) + w_{j_t+1} - w_{t+1}| \leqslant \\ &|f(y_t) - f(y_{j_t})| + 2\omega + \frac{1}{2}(|\underline{b}_t - \underline{b}_{t-1}| + \\ &|\bar{b}_t - \bar{b}_{t-1}|) \leqslant \\ &L|y_t - y_{j_t}| + c + 2\omega + \frac{1}{2} \max_{0\leqslant i\leqslant t} |\Delta B_i| \leqslant \\ &L((L+1) \max_{0\leqslant i\leqslant t} |\Delta B_i| + c + 2\omega) \\ &+c + 2\omega + \frac{1}{2} \max_{0\leqslant i\leqslant t} |\Delta B_i| = \\ &(L^2 + L + \frac{1}{2}) \max_{0\leqslant i\leqslant t} |\Delta B_i| + (L+1)(c + 2w).\end{aligned} \tag{30}$$

By the property of $f(\cdot) \in \mathcal{F}(L)$, (2) and (20), we have

$$\begin{aligned}&|B_t| \leqslant |B_{t+1}| = \\ &\max\{|y_{t+1} - \frac{1}{2}(\underline{b}_t + \bar{b}_t)| + \frac{1}{2}|B_t|, |B_t|\} \leqslant \\ &\max\{(L^2 + L + \frac{1}{2}) \max_{0\leqslant i\leqslant t} |\Delta B_i| + \\ &(L+1)(c + 2w) + \frac{1}{2}|B_t|, |B_t|\}.\end{aligned}$$

Hence, by (15) and the definition of $|B_t|$, we have

$$
\begin{aligned}
&0 \leqslant |\Delta B_{t+1}| = \\
&\max\{|y_{t+1} - \frac{1}{2}(\underline{b}_t + \bar{b}_t)| - \frac{1}{2}|B_t|, 0\} \leqslant \\
&\max\{(L^2 + L + \frac{1}{2}) \max_{0\leqslant i\leqslant t} |\Delta B_i| + \\
&\qquad (L+1)(c+2w) - \frac{1}{2}|B_t|, 0\} = \\
&((L^2 + L + \frac{1}{2}) \max_{0\leqslant i\leqslant t} |\Delta B_i| + \\
&(L+1)(c+2w) - \frac{1}{2}|B_t|)^+ = \\
&((L^2 + L + \frac{1}{2}) \max_{0\leqslant i\leqslant t} |\Delta B_i| + \\
&(L+1)(c+2w) - \frac{1}{2}\sum_{i=0}^{t} |\Delta B_i|)^+.
\end{aligned}
$$

If $L < \dfrac{\sqrt{5+4\sqrt{2}}-1}{2}$, we have

$$L^2 + L + \frac{1}{2} < \frac{3}{2} + \sqrt{2},$$

and thus it follows by Lemma 3.3 of [13] that

$$\sum_{i=0}^{\infty} |\Delta B_i| < \infty, \tag{31}$$

$$\text{i.e.,}\ \lim_{t\to\infty} |B_t| < \infty. \tag{32}$$

Thus, we have

$$\limsup_{t\to\infty} |y_t| < \infty.$$

This complete the proof of Theorem 1.

4 High-dimensional systems

We now consider the following high-dimensional discrete time nonlinear dynamical systems

$$X_{t+1} = F(\Theta, X_t) + U_t + W_{t+1}, \tag{33}$$

where $U_t \in \mathbb{R}^n, X_t \in \mathbb{R}^n$ and $W_t \in \mathbb{R}^n$ are the system input, output and noise, respectively, $\Theta \in \mathbb{R}^m$ is an unknown parameter, and $F(\cdot) : \mathbb{R}^{m+n} \to \mathbb{R}^n$ is a known nonlinear function. The dimension n is assumed to be $n \geqslant 2$, since the scalar case $n = 1$ has been investigated previously in [15].

To understand the feedback capability of the system (33) in the currently fairly general form is quite complicated even in the case of $n = 2$. Here, we only consider the following two dimensional discrete time nonlinear dynamical systems:

$$
\begin{cases}
x_{t+1} = \theta^1(x_t^a + y_t^b) + u_t^1 + \omega_{t+1}^1, \\
y_{t+1} = \theta^2(x_t^a + y_t^b) + u_t^2 + \omega_{t+1}^2,
\end{cases} \tag{34}
$$

where x_t, y_t are the outputs of the system, u_t^1, u_t^2 are the inputs of the system, θ^1, θ^2 are two unknown parameters, a, b are integers, ω_t^1 and ω_t^2 are unknown but bounded noise signals with unknown bound $\omega > 0$, i.e.

$$|w_t^1| \leqslant w,\ |w_t^2| \leqslant w,\ \forall t \geqslant 0. \tag{35}$$

Assume further that at the time $t = 0$, we have the following a priori knowledge about the unknown parameters θ^1, θ^2:

$$\theta^1 \in [\underline{\theta}^1, \bar{\theta}^1] \subset \mathbb{R}^1,\ \theta^2 \in [\underline{\theta}^2, \bar{\theta}^2] \subset \mathbb{R}^1. \tag{36}$$

We are interested in designing a feedback control law which robustly stabilizes the system (34) with respect to all possible θ^1, θ^2 and ω_t^1, ω_t^2.

To establish some concrete theoretical results, we need the following definitions first.

Definition 2 A sequence $\{u_t^1, u_t^2\}$ is called a feedback control law if at any time $t \geqslant 0$, u_t^1, u_t^2 are (causal) functions of all the observations $\{x_i, y_i, i \leqslant t\}$ up to the time t, i.e.:

$$
\begin{cases}
u_t^1 = h_t^1(x_0, y_0, x_1, y_1, \cdots, x_t, y_t), \\
u_t^2 = h_t^2(x_0, y_0, x_1, y_1, \cdots, x_t, y_t),
\end{cases} \tag{37}
$$

where $h_t^1 : \mathbb{R}^{2(t+1)} \to \mathbb{R}^1, h_t^2 : \mathbb{R}^{2(t+1)} \to \mathbb{R}^1$ can be any Lebesgue measurable (nonlinear) mappings.

Definition 3 The system (34) is said to be robust feedback stabilizable, if there exists a feedback control law $\{u_t^1, u_t^2\}$ such that for any $\{x_0, y_0\} \in \mathbb{R}^2$ and any $\{\theta^1, \theta^2\}$, $\{\omega_t^1, \omega_t^2\}$, the outputs of the closed-hoop system are bounded as follows:

$$\sup_{t\geqslant 0}[|x_t| + |y_t|] < \infty. \tag{38}$$

Theorem 2 The uncertain system (34) is robust feedback stabilizable if $a < 4,\ b < 4$.

Remark 2 We remark that in the scalar state case where, for example, we only have the first state x_t, we already know that the condition $a < 4$ is also necessary for global feedback stabilization (see [2] and [15]).

5 Proof of the Theorem 2

We need to design an adaptive control law, which robustly stabilizes the system (34) for any $a < 4, b < 4$. We remark that to implement this control law, the bounds $[\underline{\theta}^1, \bar{\theta}^1]$, $[\underline{\theta}^2, \bar{\theta}^2]$ and ω need not to be known.

Without loss of generality, suppose that there is $t_0 \geqslant 0$ such that $x_{t_0}^a + y_{t_0}^b \neq 0$. In fact, if $x_t^a + y_t^b \equiv 0, \forall t \geqslant 0$.we can define the control sequence as the following:

$$u_t^1 \equiv 0,\ u_t^2 \equiv 0,\ \forall t \geqslant 0. \tag{39}$$

Then according to (34) and (35), we have

$$|x_t| \leqslant \omega,\ |y_t| \leqslant \omega,\ \forall t \geqslant 0. \tag{40}$$

Thus the proof is finished. For simplicity, we take $t_0 = 0$.

For any $t \geqslant 1$, let

$$i_t \triangleq \arg \max_{0 \leqslant i \leqslant t-1} |x_i^a + y_i^b|, \tag{41}$$

that is

$$|x_{i_t}^a + y_{i_t}^b| = \max_{0 \leqslant i \leqslant t-1} |x_i^a + y_i^b|. \tag{42}$$

The parameters estimates at time $t \geqslant 1$ are chosen to be

$$\begin{cases} \hat{\theta}_t^1 \triangleq \dfrac{x_{i_t+1} - u_{i_t}^1}{x_{i_t}^a + y_{i_t}^b}, \\ \hat{\theta}_t^2 \triangleq \dfrac{y_{i_t+1} - u_{i_t}^2}{x_{i_t}^a + y_{i_t}^b}. \end{cases} \tag{43}$$

It is easy to check by Eq.(34) that the estimation errors are

$$\begin{cases} \tilde{\theta}_t^1 \triangleq \theta_t^1 - \hat{\theta}_t^1 = \dfrac{-\omega_{i_t+1}^1}{x_{i_t}^a + y_{i_t}^b}, \\ \tilde{\theta}_t^2 \triangleq \theta_t^2 - \hat{\theta}_t^2 = \dfrac{-\omega_{i_t+1}^2}{x_{i_t}^a + y_{i_t}^b}. \end{cases} \tag{44}$$

Now, we define the control sequence as the following:

$$\begin{cases} u_0^1 \triangleq 0,\ u_0^2 \triangleq 0, \\ u_t^1 \triangleq -\hat{\theta}_t^1 (x_t^a + y_t^b),\ \text{for } t \geqslant 1, \\ u_t^2 \triangleq -\hat{\theta}_t^2 (x_t^a + y_t^b),\ \text{for } t \geqslant 1. \end{cases} \tag{45}$$

Then for $t \geqslant 1$, the closed-loop dynamics is

$$\begin{cases} x_{t+1} = \tilde{\theta}_t^1 (x_t^a + y_t^b) + \omega_{t+1}^1 = \\ \qquad \dfrac{-\omega_{i_t+1}^1}{x_{i_t}^a + y_{i_t}^b}(x_t^a + y_t^b) + \omega_{t+1}^1, \\ y_{t+1} = \tilde{\theta}_t^2 (x_t^a + y_t^b) + \omega_{t+1}^2 = \\ \qquad \dfrac{-\omega_{i_t+1}^2}{x_{i_t}^a + y_{i_t}^b}(x_t^a + y_t^b) + \omega_{t+1}^2. \end{cases} \tag{46}$$

Therefore, noting that the noise are uniformly bounded as in (35), we have

$$\begin{cases} |x_{t+1}| \leqslant \dfrac{\omega}{|x_{i_t}^a + y_{i_t}^b|} |x_t^a + y_t^b| + \omega, \\ |y_{t+1}| \leqslant \dfrac{\omega}{|x_{i_t}^a + y_{i_t}^b|} |x_t^a + y_t^b| + \omega, \end{cases} \text{for any } t \geqslant 1. \tag{47}$$

Then by inequality (42), we have

$$\begin{cases} |x_{t+1}| \leqslant \dfrac{\omega}{\max\limits_{0 \leqslant i \leqslant t-1} |x_i^a + y_i^b|} |x_t^a + y_t^b| + \omega, \\ |y_{t+1}| \leqslant \dfrac{\omega}{\max\limits_{0 \leqslant i \leqslant t-1} |x_i^a + y_i^b|} |x_t^a + y_t^b| + \omega, \\ \text{for any } t \geqslant 1. \end{cases} \tag{48}$$

Now, we use a contradiction argument to prove that the outputs are uniformly bounded as inequality (38). Suppose on the contrary, there exist some $x_0, y_0 \in \mathbb{R}^1$, some θ^1, θ^2 and a sequence of $\{\omega_t^1, \omega_t^2\}$ such that for the feedback control law proposed above,

$$\sup_{t \geqslant 0} |x_t| + |y_t| = \infty. \tag{49}$$

First, we prove that if

$$\sup_{t \geqslant 0} [|x_t| + |y_t|] = \infty, \tag{50}$$

then we will have

$$\sup_{t \geqslant 0} |x_t^a + y_t^b| = \infty. \tag{51}$$

Suppose on the contrary, we have

$$\sup_{t \geqslant 0} |x_t^a + y_t^b| \leqslant M, \tag{52}$$

for some $0 < M < \infty$. Now we take:

$$u_t^1 \equiv 0,\ u_t^2 \equiv 0,\ \forall t \geqslant 0. \tag{53}$$

Then by inequality (35) we have

$$\begin{cases} |x_{t+1}| \leqslant |\theta^1||x_t^a + y_t^b| + |u_t^1| + |\omega_{t+1}^1| \leqslant \\ \qquad M|\theta^1| + \omega < \infty, \\ |y_{t+1}| \leqslant |\theta^2||x_t^a + y_t^b| + |u_t^2| + |\omega_{t+1}^2| \leqslant \\ \qquad M|\theta^2| + \omega < \infty. \end{cases} \tag{54}$$

i.e.,

$$\sup_{t \geqslant 0} [|x_t| + |y_t|] < \infty. \tag{55}$$

This contradicts with inequality (38), so we have

$$\sup_{t \geqslant 0} |x_t^a + y_t^b| = \infty \tag{56}$$

Next, from the sequence $\{|x_t^a + y_t^b|, t \geqslant 0\}$, we can pick out a subsequence $\{|x_{t_k}^a + y_{t_k}^b|, k \geqslant 1\}$ which monotonously increasing to infinity, and satisfies for any $k = 1, 2, \cdots$,

$$|x_t^a + y_t^b| \leqslant |x_{t_k}^a + y_{t_k}^b| < |x_{t_{k+1}}^a + y_{t_{k+1}}^b|, \\ \text{for any } t_k < t < t_{k+1}. \tag{57}$$

For any $k = 2, 3, \cdots$, by inequality (48), we have

$$\begin{cases} |x_{t_{k+1}}| \leqslant \xi |x_{t_{k+1}-1}^a + y_{t_{k+1}-1}^b| + \omega, \\ |y_{t_{k+1}}| \leqslant \xi |x_{t_{k+1}-1}^a + y_{t_{k+1}-1}^b| + \omega, \\ \text{for any } t \geqslant 1, \end{cases} \tag{58}$$

where $\xi = \dfrac{\omega}{\max\limits_{0 \leqslant i \leqslant t_{k+1}-2} |x_i^a + y_i^b|}$. By inequality (57), it is easy to check that

$$|x_{t_{k+1}-1}^a + y_{t_{k+1}-1}^b| \leqslant |x_{t_k}^a + y_{t_k}^b|, \tag{59}$$

and

$$\max_{0 \leqslant i \leqslant t_{k+1}-2} |x_i^a + y_i^b| \geqslant |x_{t_{k-1}}^a + y_{t_{k-1}}^b|. \tag{60}$$

Hence, by inequality (58), we have

$$\begin{cases} |x_{t_{k+1}}| \leqslant \dfrac{\omega}{|x_{t_{k-1}}^a + y_{t_{k-1}}^b|} |x_{t_k}^a + y_{t_k}^b| + \omega, \\ |y_{t_{k+1}}| \leqslant \dfrac{\omega}{|x_{t_{k-1}}^a + y_{t_{k-1}}^b|} |x_{t_k}^a + y_{t_k}^b| + \omega. \end{cases} \tag{61}$$

Now, take logarithm on both sides of inequality (61), we have

$$\begin{cases}\log|x_{t_{k+1}}| \leqslant \log(\dfrac{\omega}{|x^a_{t_{k-1}}+y^b_{t_{k-1}}|}|x^a_{t_k}+y^b_{t_k}|+\omega),\\ \log|y_{t_{k+1}}| \leqslant \log(\dfrac{\omega}{|x^a_{t_{k-1}}+y^b_{t_{k-1}}|}|x^a_{t_k}+y^b_{t_k}|+\omega).\end{cases} \tag{62}$$

Noticing that

$$|x^a_{t_k}+y^b_{t_k}| < |x^a_{t_{k+1}}+y^b_{t_{k+1}}|, \tag{63}$$

so we have

$$\begin{aligned}&\log|x^a_{t_{k+1}}+y^b_{t_{k+1}}| \leqslant\\ &\log(|x^a_{t_{k+1}}|+|y^b_{t_{k+1}}|) \leqslant\\ &\log(2\max\{|x^a_{t_{k+1}}|,|y^b_{t_{k+1}}|\}) \leqslant\\ &\max\{\log(2|x^a_{t_{k+1}}|),\log(2|y^b_{t_{k+1}}|)\} \leqslant\\ &\max\{a\log|x_{t_{k+1}}|, b\log|y_{t_{k+1}}|\}+\log 2 \leqslant\\ &\max\{a,b\}\log(\frac{\omega}{|x^a_{t_{k-1}}+y^b_{t_{k-1}}|}|x^a_{t_k}+y^b_{t_k}|+\\ &\omega)+\log 2 <\\ &\max\{a,b\}\log(2\omega\frac{|x^a_{t_k}+y^b_{t_k}|}{|x^a_{t_{k-1}}+y^b_{t_{k-1}}|})+\log 2 <\\ &\max\{a,b\}(\log|x^a_{t_k}+y^b_{t_k}|-\\ &\log|x^a_{t_{k-1}}+y^b_{t_{k-1}}|)+(a+b)\log 4\omega.\end{aligned} \tag{64}$$

Then it follows by Lemma 3.5[8] that for $\max\{a,b\} < 4$, $\{\log|x^a_{t_k}+y^b_{t_k}|\}$ cannot be monotonously increasing to infinity, which contradicts to the definition of $\{|x^a_{t_k}+y^b_{t_k}|\}$. This concludes both the contradiction argument and the proof of the theorem.

6 Concluding remarks

In this paper, we have provided some further results on global adaptive stabilization for two special classes of uncertain nonlinear dynamical systems. One is concerned with the case of time-delay, and another is concerned with states in the plane. We remark that the results are just preliminary ones, and much effort need to be made towards a comprehensive theory on feedback capability. This paper also rises some concrete open problems for further investigation.

References:

[1] BASAR T. *Control Theory: Twenty-Five Seminal Papers (1932–1981)* [M]. New Jersey: Wiley-IEEE Press, 2000.

[2] GUO L. On critical stability of discrete-time adaptive nonliear control [J]. *IEEE Transations on Automatic Control*, 1997, 42(11): 1488 – 1499.

[3] LI C Y, XIE L L, GUO L. A polynomial criterion for adaptive stabilizability of discrete-time nonlinear systems [J]. *Communications in Information and Systems*, 2006, 6(4): 273 – 298.

[4] LI C Y, GUO L. A new critical theorem for adaptive nonlinear stabilization [J]. *Automatica*, 2010, 46(6): 999 – 1007.

[5] LI C Y, GUO L. On feedback capability in nonlinearly parameterized uncertain dynamical systems [J]. *IEEE Transactions on Automatic Control*, 2011, 56(12): 2946 – 2951.

[6] XIE L L, GUO L. Fundamental limitations of discrete-time adaptive nonlinear control [J]. *IEEE Transations on Automatic Control*, 1999, 44(9): 1777 – 1782.

[7] HUANG C D, GUO L. On feedback capability for a class of semiparametric uncertain systems [J]. *Automatica*, 2012, 48(5): 873 – 878.

[8] XIE L L, GUO L. How much uncertainty can be dealt with by feedback [J]. *IEEE Transations on Automatic Control*, 2000, 45(12): 2203 – 2217.

[9] XUE F, GUO L. On limitations of the sampled-data feedback for nonparametric dynamical systems [J]. *Journal of Systems Science & Complexity*, 2002, 15(3): 225 – 250.

[10] REN J L, CHENG Z B, GUO L. Further results on limitations of sampled-data feedback [J]. *Journal of Systems Science & Complexity*, 2004, 27(5): 817 – 835.

[11] DUGARD L, VERRIEST I. *Stability and Control of Time-Delay Systems, Lecture Notes in Control and Information Sciences* [M]. Heidelberg, Berlin: Springer, 1998, Vol.228.

[12] GU K, KHARITONOV V L, CHEN J. *Stability of Time-Delay Systems* [M]. New York: Birkhäuser Basel, 2003.

[13] RICHARD J . P. Time-delay systems: an overview of some recent advances and open problems [J]. *Automatica*, 2003, 39(10): 1667 – 1694.

[14] LUO M,GUO L. On the capability of feedback for unknown nonparametric systems with input delay [C] //*Chinese Control Conference*, Nanjing: IEEE, 2014, 2604 – 2607.

[15] LI C Y, XIE L L. On robust stability of discrete-time adaptive nonlinear control [J]. *Systems & Control Letters*, 2006, 55(6): 452 – 458.

Authors' Introduction:

LUO Man (1988–), Female, She received the Bachelor's degree from the School of Mathematics and Statistics of Wuhan University, China, in 2012 and is currently a doctor degree candidate of the Academy of Mathematics and Systems Science, Chinese Academy of Sciences. Her current research interests are adaptive control and feedback capability. E-mail: luoman@amss.ac.cn;

GUO Lei (1961–), Lei GUO received his B.S. degree in mathematics from Shandong University in 1982, and Ph.D. degree in control theory from the Chinese Academy of Sciences in 1987. He is currently a professor at the Academy of Mathematics and Systems Science, Chinese Academy of Sciences. He has worked on problems in adaptive control, system identification, adaptive signal processing, and stochastic systems. His current research interests include the capability of feedback, multi-agent systems, complex adaptive systems, and quantum control systems, among others. E-mail: lguo@amss.ac.cn.