Consensus of Flocks under M-Nearest-Neighbor Rules*

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Abstract This paper investigates a class of flocks with an M-nearest-neighbor rule, where each agent's neighbors are determined according to M nearest agents with M being a given integer, rather than all the agents within a fixed metric distance as in the well-known Vicsek's model. Such a neighbor rule has been validated by biologists through experiments and the authors will prove that, similar to the Vicsek's model, such a new neighbor rule can also achieve consensus under some conditions imposed only on the system's speed and the number M, n, without resorting to any priori connectivity assumptions on the trajectory of the system. In particular, the authors will prove that if the number M is proportional to the population size n, then for any speed v, the system will achieve consensus with large probability if the population size is large enough.

Keywords Consensus, multi-agent systems, M-nearest neighbor, random geometric graph, topological distance.

1 Introduction

Recently, much research attention from the fields of biology, physics, computer science, mathematics and systems science has been paid to the collective behavior of flocks or multi-agent systems , as a starting point for the investigation of complex systems, and a vast literature has appeared including the practical observations, simulation experiments and theoretical analysis. A central issue is to understand how local interactions between the agents lead to global behavior of the system. For some classes of multi-agent systems, it has been found that without centralized control, the system can spontaneously produce some kinds of interesting "macro" phenomena, such as synchronization, whirlpool, etc., mainly based on local information exchange among agents. Among the mathematic models to simulate these collective behaviors[1–3], the Vicsek's model[1] owning to a simple and intuitive local rule, can exhibit some kind of phase transitions from random motion to synchronization.

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In Vicsek's model, each agent's neighbors are the ones within a prescribed geometric distance from it. Actually, most of the existing flocking models assume that the agents interact with each other according to a geometric distance, which is used to decide the interaction strengths among agents[1, 2]. However, for a given practical flocks, whether the interactions are indeed determined by the geometric distance is still a question. Not long ago, a group of European scientists had given an alternative distance in [4], that is the "topological distance", intuitively, the neighbors are defined as the M nearest individuals away from a given agent. They reconstructed the three-dimensional positions of individual birds in airborne flocks of a few thousand members, and showed that the interaction does not depend on the geometric distance, but rather on the topological distance. Simply speaking, each agent interacts with a fixed number of agents nearest to it rather than the ones within a fixed distance from it. In fact, it was discovered that each bird interacts on average with six to seven neighbors in the experiments of [4].

The same group of scientists[4] also tried to give some explanation to such an interaction rule. Through experiments, they argue that the rule seems to be more suitable to keep cohesion when the group encounters strong density fluctuations. And it has been observed that stragglers and small groups are significantly more to be preyed compared with the animals belonging to large and highly cohesive aggregations[4]. So, topological interaction is the only mechanism granting such robust cohesion with higher biological fitness.

The point we are trying to make is, the M-nearest-neighbor rule has existed in the research of wireless networks, in which links are formed by nodes choosing the power levels at which they transmit. This raises the question: How many neighbors should each node be connected to, in order that the overall network then becomes connected? To answer this question, the wireless network can be modeled as nodes located randomly on the plane according to a Poisson point process and each node is connected to a fixed number of nearest neighbors[5]. And it has been proved that each node should be connected to $\Theta(\log n)$ nearest neighbors in order to achieve asymptotically connectivity[5, 6].

Following [7], this paper will consider the model which consists of n agents on the plane and each agent moves with the same constant speed. At each time step, every agent's neighbors are the nearest M ones from it (including itself). We call this neighbor rule as the "M-nearest-neighbor rule". Each agent updates its heading by making an average of its neighbors' current headings. This model is different from the well-known Vicsek's model with the geometric distance based neighbor rule there changed to the M-nearest-neighbor rule to be studied in the present paper.

To the best of the authors' knowledge, there is few complete theoretical research on flocks with M-nearest-neighbor rule, but there is indeed a vast literature on the theoretical investigation of the Vicsek's model, see, e.g., [8–16]. These have not only promoted the development of the research on flocking model, but also offered much inspiration to our study in the current paper. Here, we only mention two representative works. Firstly, Jadbabaie, et al.[8] initiated a theoretical study for the consensus of a partially linearized Vicsek's model. What Jadbabaie, et al. showed was that the system will achieve consensus if the associated dynamical neighbor graphs are jointly connected within some contiguous and bounded time intervals. However, how to sidestep or verify the troublesome connectivity condition imposed on the dynamic graphs turns out to be a difficult and challenging issue in theory, because the underlying dynamical equation is strongly coupled and highly nonlinear. Subsequently, a major advance towards resolving this bottleneck issue was made by Tang and Guo[7], where a random framework as originally considered by Vicsek, et al.[1] is introduced in their analysis. By carrying out a detailed analysis of both the system's nonlinear dynamical properties and the spectral gap of random geometric graphs, Tang and Guo[7] proved that the overall multi-agent system will achieve consensus with large probability as long as the size of the population is large enough.

Concerning the flocking model with the M-nearest-neighbor rule, we should mention that Wang[17] appears to be the first one to study it, where a sufficient condition on consensus is introduced, without resorting to any dynamical connectivity assumptions on the system trajectories. However, the condition imposed on the parameters of the initial graph there are hard to verify, especially for a random geometric graph. We should also remark that, the method used in [7] is no longer valid under the new M-nearest-neighbor rule, because the neighbor graphs with topological distance are directed , and it appears to be an essential difficulty when estimating the spectrum gap of a graph.

In this paper, we will work with a random framework, and give an easily verifiable condition on the consensus of the model with the M-nearest-neighbor rule. Some new techniques will be introduced to sidestep the trouble of estimating the spectrum of a directed graph.

The rest of this paper is organized as follows: In Section 2, we will present the formulation of the problem and the main results. The proof of the results will be put in Section 3. Then, some simulations of the results will be presented in Section 4. In Section 5, we give some concluding remarks.

2 Main Results

Let us assume that n autonomous agents move in the plane with the same speed $v_n (v_n > 0)$ but with different headings. At any time t, the position and heading of the agent i are denoted by $X_i(t)(\in \mathbb{R}^2)$ and $\theta_i(t)(\in (-\pi, \pi))$ respectively. The distance between agents i and j is denoted by $d_{ij}(t)$ and $d_{ij}(t) = \|X_i(t) - X_j(t)\|_2$, where $\|\cdot\|_2$ denotes the Euclidean norm. These are the same as the models in [7], [15], and [16]. The difference lies in: For any agent $i(1 \le i \le n)$, the neighbors of i means the nearest M_n individuals from its recent position, where M_n is a pre-defined value depending on n, so j is i's neighbor if j is one of the nearest M_n agents from i. If at time t there is more than one agent which can be treated as the M_n-th nearest neighbor of agent i, then agent i chooses the one who is the nearest at time $t-1$. The neighbor set of i at time t is denoted by $\mathcal{N}_i(t)$. Particularly, we define that each agent is a neighbor of itself, i.e., $i \in \mathcal{N}_i(t), \forall t > 0, 1 \le i \le n$. For $t = 1, 2, \cdots$, the updating rules of the agents' positions are as follows:

$$X_i(t) = X_i(t-1) + v_n(\cos\theta_i(t), \sin\theta_i(t)), \tag{1}$$

the updating rules of the agents' headings are as follows:

$$\theta_i(t) = \frac{1}{|\mathcal{N}_i(t-1)|} \sum_{j \in \mathcal{N}_i(t-1)} \theta_j(t-1), \tag{2}$$

where $|S|$ denotes the cardinality of the set S. Because the number of any agent's neighbors is the fixed number M_n, we can rewrite (2) as follows:

$$\theta_i(t) = \frac{1}{M_n} \sum_{j \in \mathcal{N}_i(t-1)} \theta_j(t-1). \tag{3}$$

This paper will mainly investigate the consensus property of the models (1)–(3). Following Tang and Guo[7], we give the definition of "consensus":

Definition 2.1 If there exists a constant $\overline{\theta} \in (-\pi, \pi]$ such that $\lim_{t\to\infty} \theta_i(t) = \overline{\theta}$, $\forall 1 \leq i \leq n$, then we say the models (1)–(3) can achieve consensus.

In this paper, we will consider the models (1)–(3) in the following probabilistic framework: Let $(\Omega, \mathcal{F}, \mathcal{P})$ be the underlying probability space, and assume that the initial positions $\{X_i(0), 1 \leq i \leq n\}$ are i.i.d. random vectors uniformly distributed in $[0,1]^2$, that the initial headings $\{\theta_i(0), 1 \leq i \leq n\}$ are i.i.d. random variables uniformly distributed on the interval $(-\pi, \pi]$, and that the initial positions and initial headings are mutually independent.

Before listing the main results, some new variables will be introduced:

For $M_n \gg \log n^\dagger$, denote $\{g_n, n \in \mathbb{N}\}$ as positive sequences satisfying

$$1 \ll g_n \ll \frac{M_n}{\log n}. \tag{4}$$

Set $\{\varepsilon_n, n \in \mathbb{N}\}$ satisfying

$$\varepsilon_n = \sqrt{\frac{3 \log n (g_n + 1)}{M_n}}. \tag{5}$$

Obviously, when $M_n \gg \log n, \varepsilon_n = o(1)$.

The main results of this paper are listed as follows:

Theorem 2.2 *Suppose that the* n *agents are independently and uniformly distributed in* $[0,1]^2$ *at the initial time* $t = 0$*, and* $M_n \gg \log n$*. Then under any initial headings' configuration, for any* $\eta_n \in (0,1)$*, the flocks models* (1)–(3) *will achieve consensus with probability* $1 - O(n^{-g_n})$ *if the speed satisfies*

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†Throughout this paper, we will use some standard mathematical notations as follows:

- For positive sequences $\{g_1(n)\}$ and $\{g_2(n)\}$, $g_1(n) = O(g_2(n))$ if there exists a constant $c > 0$ and a value $n_0 > 0$ such that $g_1(n) \leq c(g_2(n))$ for any $n \geq n_0$.
- For positive sequences $\{g_1(n)\}$ and $\{g_2(n)\}$, $g_1(n) = \Theta(g_2(n))$ if there exists constants $c_1 > 0$ and $c_2 > 0$ and a value $n_0 > 0$ such that $c_1 g_2(n) \leq g_1(n) \leq c_2 g_2(n)$ for any $n \geq n_0$.
- For positive sequences $\{g_1(n)\}$ and $\{g_2(n)\}$, $g_1(n) = o(g_2(n))$ or $g_1(n) \ll (g_2(n))$ or $g_2(n) \gg (g_1(n))$ if $\lim_{n\to\infty} \frac{g_1(n)}{g_2(n)} = 0$.
- For matrixes $A = [a_{ij}]$, $B = [b_{ij}]$, $A \geq B$ if $a_{ij} \geq b_{ij}$.
- $\lceil x \rceil$ denotes the smallest integer not less than x.

$$v_n \le \frac{(\frac{n}{M_n})^{K_n} a_n^{2(K_n-1)} \eta_n r_n}{\Delta_1 K_n}, \tag{6}$$

where r_n *satisfies*

$$n\pi[(1+\eta_n)r_n]^2(1+\varepsilon_n) = M_n, \tag{7}$$

and $a_n = \frac{1}{\sqrt{5}}(1-\eta_n)r_n$, $K_n = 2\big(\left\lceil \frac{1}{a_n} \right\rceil + 1\big)$, $\Delta_1 = \max_{i,j}\{\theta_i(1) - \theta_j(1)\}$.

Remark 2.3 From [5], $\Theta(\log n)$ is the least neighbors number to guarantee the connectivity of the initial graph, so the condition $M_n \ge \Theta(\log n)$ is natural. Here we only provide the consensus results under $M_n \gg \log n$ because the analysis under $M_n = \Theta(\log n)$ is much more complicated. We should search for other methods to deal with the latter case.

Remark 2.4 From the proof below, (6) still holds if we substitute Δ_0 for Δ_1, where $\Delta_0 = \max_{i,j}\{\theta_i(0) - \theta_j(0)\}$. For convenience we denote the new condition as (6*). So we obtain the speed condition for consensus only using the initial headings and system parameters n, M_n. It means that for arbitrary initial headings configuration, the system will achieve consensus provided that the speed is small enough.

Corollary 2.5 *Assume that* $M_n \gg \log n$*, and suppose that at the initial time* $t = 0$*, the* n *agents are independently and uniformly distributed in* $[0,1]^2$*, with headings independently and uniformly distributed in* $(-\pi, \pi]$*. Then the flocks models* (1)–(3) *will achieve consensus almost surely, provided that*

$$v_n = o\left((5\pi)^{-2\sqrt{5\pi n/M_n}} M_n/\log n\right). \tag{8}$$

With the preparations of the above results, we obtain an interesting corollary:

Corollary 2.6 *Consider the probabilistic framework and suppose that* $M_n = \alpha n(1+o(1))$*, where* α *is a fixed constant. Then for any* $v > 0$*, the flock model* (1)–(3) *will achieve consensus almost surely, as the population size* n *is large enough.*

3 Proofs of the Main Result

To analyze the consensus behavior, some concepts need to be introduced first. Similar to [12], we will use a graph sequence $\{G(t), t = 0, 1, \cdots\}$ to describe the relationship among neighbors. For $t \ge 0$, define

$$G(t) = G(\{X_1(t), X_2(t), \cdots, X_n(t)\}, E(t))$$

to be the position graph of the model at time t, where $E(t) = \{(i,j) : j \in \mathcal{N}_i(t)\}, 1 \le i \le n$, notice that for all $1 \le i \le n$ and $t \ge 0$, $(i,i) \in E(t)$, and the graphs formed in this way are directed. Denote $P(t)$ as the average matrix of the graph $G(t)$, i.e.,

$$(P(t))_{ij} = \begin{cases} \dfrac{1}{M_n}, & \text{if } (i,j) \in E(t), \\ 0, & \text{else}, \end{cases} \quad \forall i,j = 1, 2, \cdots, n.$$

Notice that $P(t)$ is a stochastic matrix. Set $\theta(t) := (\theta_1(t), \theta_2(t), \cdots, \theta_n(t))^{\mathrm{T}}$, then the iteration rule of the headings and positions of the model based on (1) and (3) can be rewritten as

$$\begin{cases} \theta(t) = P(t-1)\theta(t-1), \\ X_i(t) = X_i(t-1) + v_n(\cos\theta_i(t), \sin\theta_i(t)), \end{cases} \quad \forall t \geq 1, 1 \leq i \leq n. \tag{9}$$

Let us use the measure $\tau(P)$ to describe the convergence speed of the infinite products of a stochastic matrix P:

$$\tau(P) = \frac{1}{2}\max_{i,j}\sum_{s=1}^{n}|P_{is} - P_{js}| = 1 - \min_{i,j}\sum_{s=1}^{n}\min(P_{is}, P_{js}). \tag{10}$$

A lemma related to $\tau(P)$ will be presented first.

Lemma 3.1 (see [17]) *Let $y = [y_1, y_2, \cdots, y_n]^{\mathrm{T}} \in \mathbb{R}^n$ be an arbitrary vector, and $P = [P_{ij}] \in \mathbb{R}^{n\times n}$ be a stochastic matrix. If $z = [z_1, z_2, \cdots, z_n]^{\mathrm{T}} = Py$, then*

$$\max_{s,s'}|z_s - z_{s'}| \leq \tau(P)\max_{j,j'}|y_j - y_{j'}|. \tag{11}$$

Let $\mathcal{X}_n(t) = \{X_1(t), X_2(t), \cdots, X_n(t)\}$ be the set of the positions of the n agents at time t, hence $\mathcal{X}_n(0)$ denotes the set including n points which are independently and uniformly distributed in $[0,1]^2$. Also, for convenience, we use $B(X_i(t), r)$ to denote the circle centered at $X_i(t)$ with radius r, so $\mathcal{X}_n(t)\bigcap B(X_i(t), r)$ is the set which consists of the positions of the agents lying in $B(X_i(t), r)$ at time t.

Lemma 3.2 *Suppose that $M_n \gg \log n$, then for any $\eta_n \in (0,1)$,*

$$\begin{aligned} &P\Big\{\Big||\mathcal{X}_n(0)\cap B(X_i(0),(1+\eta_n)r_n))| \\ &\qquad - n\pi[(1+\eta_n)r_n]^2\Big| \leq \varepsilon_n n\pi[(1+\eta_n)r_n]^2\Big\} \\ &= 1 - O(n^{-g_n}), \end{aligned} \tag{12}$$

where r_n satisfies (7).

Proof First consider the small disk $B(X_i(0),(1+\eta_n)r_n)$. Denote Y_j as the indicator function of the event where agent j falls into $B(X_1(0),(1+\eta_n)r_n)$. Then $\{Y_j, 1 \leq j \leq n\}$ are i.i.d. Bernoulli random variables with success probability $p = \pi[(1+\eta_n)r_n]^2$ and $|\mathcal{X}_n(0)\cap B(X_1(0),(1+\eta_n)r_n)| = \sum_{j=1}^{n}Y_j$. According to Chernoff Bound, for given ε_n, it is true that

$$P\{||\mathcal{X}_n(0)\cap B(X_1(0),(1+\eta_n)r_n)| - np| > \varepsilon_n np\} \leq 2\exp\left(-\frac{\varepsilon_n^2 np}{3}\right). \tag{13}$$

Obviously, $\{|\mathcal{X}_n(0)\cap B(X_i(0),(1+\eta_n)r_n))|, 1 \leq i \leq n\}$ are identically distributed random variables, hence using the Boole's inequality

$$\begin{aligned}
&P\left\{\max_{1\le j\le n}||\mathcal{X}_n(0)\cap B(X_j(0),(1+\eta_n)r_n)|-np|\le\varepsilon_n np\right\}\\
\ge&1-\sum_{j=1}^{n}P\left\{||\mathcal{X}_n(0)\cap B(X_j(0),(1+\eta_n)r_n)|-np|>\varepsilon_n np\right\}\\
\ge&1-2n\exp\left(-\frac{\varepsilon_n^2 np}{3}\right)\\
=&1-2\exp\left(\log n-np\cdot\frac{1}{3}\cdot\frac{3(g_n+1)\log n}{M_n}\right)\\
=&1-O(n^{-g_n}).
\end{aligned}$$

The last equality holds because the condition that r_n satisfies (7).

For fixed $\eta_n\in(0,1)$, denote the set

$$\begin{aligned}
B_1^{\eta_n}=&\Big\{\omega\in\Omega:\Big|\,|\mathcal{X}_n(0)\cap B(X_i(0),(1+\eta_n)r_n)|\\
&-n\pi[(1+\eta_n)r_n]^2\Big|\le\varepsilon_n n\pi[(1+\eta_n)r_n]^2,1\le i\le n\Big\},
\end{aligned}$$

hence from Lemma 3.2, under the condition $M_n\gg\log n$,

$$P(B_1^{\eta_n})=1-O(n^{-g_n}). \tag{14}$$

The proof is finished. ∎

For the same η_n, let us partition the unit square $[0,1]^2$ into $\lceil 1/a_n\rceil^2$ small squares with the length of each side equal to $\frac{1}{\lceil 1/a_n\rceil}$, where $a_n=\frac{1}{\sqrt{5}}(1-\eta_n)r_n$. If $M_n\gg\log n$, we can deduce that $a_n\gg\sqrt{\log n/n}$. Furthermore, we label these small squares as S_j, $j=1,2,\cdots,\lceil 1/a_n\rceil^2$, from left to right and from bottom to top. Denote the set

$$\begin{aligned}
B_2^{\eta_n}=\Big\{\omega\in\Omega:na_n^2(1-o(1))\le|\mathcal{X}_n(0)\cap S_j|\\
\le na_n^2(1+o(1)),1\le j\le\lceil 1/a_n\rceil^2\Big\}.
\end{aligned}$$

The following lemma can be deduced directly from Lemma 4 in [7].

Lemma 3.3 *Assume $a_n\gg\sqrt{\log n/n}$, then $P(B_2^{\eta_n})=1-O(n^{-g_n})$.*

Proof of Theorem 2.2 Since the distribution of initial positions are independent and uniform, then for any $\eta_n\in(0,1)$, by Lemmas 3.2 and 3.3, $B_1^{\eta_n}\bigcap B_2^{\eta_n}$ happens with the probability $1-O(n^{-g_n})$. The following discussion is restricted on the set $B_1^{\eta_n}\cap B_2^{\eta_n}$. Then the number of agents lying in $B(X_i(0),(1+\eta_n)r_n)$ is not more than $n\pi(1+\eta_n){r_n}^2(1+\varepsilon_n)$, from (7), that is M_n, so for arbitrary $r<(1+\eta_n)r_n$, we have $\mathcal{X}_n(0)\bigcap B(X_i(0),r)\subset\mathcal{N}_i(0)$. For the grids we just delineated, notice that any two agents in two adjacent squares have a distance less than $(1-\eta_n)r_n$, so they are neighbors of each other at the initial time. See Figure 1.

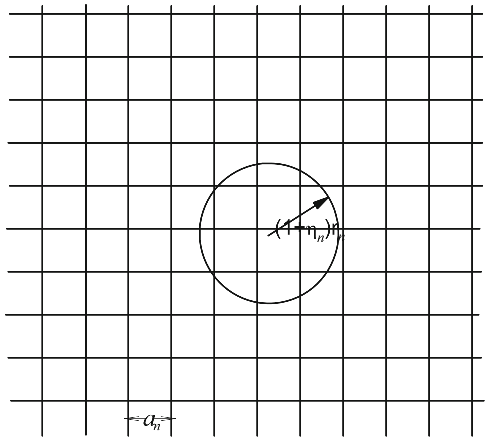

Figure 1 For this grid, the side length $a_n = \frac{1}{\sqrt{5}}(1-\eta_n)r_n$, for some i, the vertexes in the disk with i as the center and $(1+\eta_n)r_n$ as the radius are all neighbors of i, so the vertexes in the adjacent squares are neighbors of each others

When $t \geq 0$, we have

$$\begin{aligned} d_{ij}(t+1) &= \|X_i(t+1) - X_j(t+1)\|_2 \\ &\leq d_{ij}(t) + v_n \left| 2\sin\frac{\theta_i(t+1)-\theta_j(t+1)}{2} \right| \\ &\leq d_{ij}(t) + v_n \left|\theta_i(t+1) - \theta_j(t+1)\right|, \end{aligned}$$

set $\Delta_t = \max_{i,j}\{\theta_i(t) - \theta_j(t)\}$, we obtain

$$d_{ij}(t+1) \leq d_{ij}(t) + v_n\Delta_{t+1}. \tag{15}$$

From (3) we know that Δ_t is monotonously decreasing. Actually, there exists $L_n = 1 - (\frac{n}{M_n})^{K_n} a_n^{2(K_n-1)}(1+o(1))$ such that

$$\Delta_{k\cdot K_n} \leq L_n^k \Delta_1, \quad \forall k \geq 0, \tag{16}$$

where $K_n = 2(\left\lceil \frac{1}{a_n} \right\rceil + 1)$ is defined in Lemma 2.2. Actually, (15) still holds if we substitute Δ_t for Δ_{t+1}, so does (16) if Δ_1 is substituted by Δ_0. Then taking the same analysis just as that being showed later we can obtain (6*). Next, we use induction to prove (16).

Obviously, the conclusion holds for the case $k = 0$.

Assume $\Delta_{k\cdot K_n} \leq L_n^k\Delta_1$ holds for all $k \leq T$. From the monotonicity of Δ_t, for $\forall t \in [lK_n, (l+1)K_n), l \leq T$, $\Delta_t \leq L_n^l\Delta_1$ holds. So for arbitrary i, j and $\forall t \in [TK_n, (T+1)K_n)$, from (15) we have

$$\begin{aligned} d_{ij}(t+1) &\leq d_{ij}(0) + v_n \sum_{l=1}^{t} \Delta_l \\ &\leq d_{ij}(0) + K_n v_n(1 + L_n + L_n^2 + \cdots + L_n^{\mathrm{T}})\Delta_1 \\ &< d_{ij}(0) + K_n v_n \frac{1}{1-L_n}\Delta_1, \end{aligned}$$

since v_n satisfies (6), we have

$$d_{ij}(t+1) < d_{ij}(0) + \eta_n r_n. \tag{17}$$

Similarly, we can get

$$d_{ij}(0) \leq d_{ij}(t+1) + \eta_n r_n. \tag{18}$$

So

$$d_{ij}(t+1) \begin{cases} < r_n, & \text{if} \quad d_{ij}(0) < (1-\eta_n)r_n; \\ > r_n, & \text{if} \quad d_{ij}(0) > (1+\eta_n)r_n. \end{cases} \tag{19}$$

From (19), we obtain that when $TK_n \leq t \leq T(K_n+1)-1$,

$$\mathcal{X}_n(0) \cap B(X_i(0), (1-\eta_n)r_n) \subset \mathcal{X}_n(t) \cap B(X_i(t), r_n), \tag{20}$$

and at the time t, the agents outside the circle $B(X_i(0), (1+\eta_n)r_n)$ cannot move into the circle $B(X_i(t), r_n)$, the agents in $B(X_i(0), (1-\eta_n)r_n)$ cannot run off from $B(X_i(t), r_n)$, then

$$\mathcal{X}_n(t) \cap B(X_i(t), r_n) \subset B(X_i(0), (1+\eta_n)r_n), \tag{21}$$

so

$$|\mathcal{X}_n(t) \cap B(X_i(t), r_n)| < M_n,$$

that means $\mathcal{X}_n(t) \cap B(X_i(t), r_n) \subset \mathcal{N}_i(t)$, hence from (20) we get

$$\mathcal{X}_n(0) \cap B(X_i(0), (1-\eta_n)r_n)(0) \subset \mathcal{N}_i(t), \tag{22}$$

that means the neighbor relationship at time $t=0$ will not change during the time interval $TK_n \leq t \leq T(K_n+1)-1$.

Next, we prove that

$$P((T+1)K_n-1)P((T+1)K_n-2)\cdots P(TK_n) \geq \frac{c_n}{M_n} I \cdot I^{\mathrm{T}}, \tag{23}$$

where I is the vector with all the entries equal to 1 and $c_n = (\frac{na_n^2}{M_n})^{K_n-1}(1+o(1))$. Define e_i to be a vector whose i's entry is 1 and other entries are 0 , $1 \leq i \leq n$, then (23) is equal to

$$P((T+1)K_n-1)P((T+1)K_n-2)\cdots P(TK_n)e_i \geq \frac{c_n}{M_n} I. \tag{24}$$

We define the value of the j'th entry of the vector e_i as agent j's potential at time TK_n-1, and $P(t)\cdots P(TK_n)e_i$ as the potential of agent j at time t. Intuitively, (24) can be interpreted as follows: when $t = TK_n - 1$, the agent i has potential 1 and other agents have potential 0, when the system evolves to $t = (T+1)K_n - 1$, all the agents get potential not less than $\frac{c_n}{M_n}$, so the potential of the system is dispersed from one agent to all other agents. As a result, we only need to prove that for any $j \neq i$, at the time $t = (T+1)K_n - 1$, j has the potential not less than $\frac{c_n}{M_n}$.

Without loss of generality, assume that at the time TK_n, agent i lies in S_{m1} and agent j lies in S_{m2}. Then there exists a path from left to right and bottom to top joining S_{m1} and S_{m2}, and such a path is denoted as $S_{m1} \to S_{m1+1} \cdots S_{m2-1} \to S_{m2}$. See Figure 2.

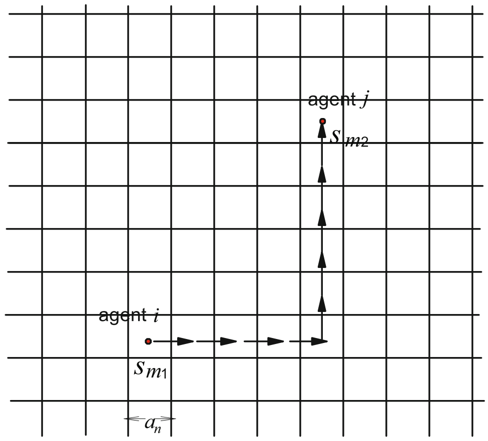

Figure 2 The only path connecting S_{m1} and S_{m2} is displayed by solid arrows

From (22), we know that agent i is a neighbor of all the agents in squares S_{m1}, S_{m1+1}, so the entries in the i' th column of $P(TK_n)$ whose responding agents are also in the set S_{m1}, S_{m1+1} have the values $\frac{1}{M_n}$, that means the corresponding entries of the vector $P(TK_n)e_i$ have values $\frac{1}{M_n}$. As a result, the agents which lie in the sets S_{m1}, S_{m1+1} have the potential of $\frac{1}{M_n}$.

When $t = TK_n + 1$, the topology of the graph $G(TK_n + 1)$ has changed compared with $G(TK_n)$, because every agent has moved during the time interval. For example, some agents in S_k may escape from S_k but some others not belonging to S_k at the beginning may run into S_k, for some k. Despite of this, from (22), we know that all the neighbor relationships among the agents in the sets $S_{m1}, S_{m1+1}, \cdots, S_{m2-1}, S_{m2}$ at time 0 do not change during the time interval $0 \le t \le (T+1)K_n - 1$. Hence, under $P(TK_n + 1)$, the agents in S_{m1+2} have potential increasing from 0 to not less than

$$\frac{1}{M_n}\left(|S_{m1+1}| \cdot \frac{1}{M_n}\right) = \frac{na_n^2}{M_n^2}(1 + o(1)),$$

and at this time, the potential of agents belonging to S_{m1}, S_{m1+1} does not decrease.

Similarly, there exists $t_0 \le (T+1)K_n - 1$ such that all the agents in $S_{m1}, S_{m1+1} \cdots, S_{m2-1}$ have potential not less than $\frac{1}{M_n}(\frac{na_n^2}{M_n})^{t_0-1}(1 + o(1))$. So under $P(t_0)$, the agents in S_{m2} have potential not less than

$$\frac{1}{M_n}\left(\frac{1}{M_n}\left(\frac{na_n^2}{M_n}\right)^{t_0-1}(1 + o(1)) \cdot |S_{m1+1}|\right) = \frac{1}{M_n}\left(\frac{na_n^2}{M_n}\right)^{t_0}(1 + o(1)) \ge \frac{c_n}{M_n}.$$

By the arbitrariness of j, we obtain

$$P(t_0)P(t_0-1)\cdots P(TK_n)e_i \geq \frac{c_n}{M_n}I,$$

hence

$$P((T+1)K_n-1)P((T+1)K_n-2)\cdots P(TK_n)e_i \geq \frac{c_n}{M_n}I, \tag{25}$$

then (23) holds.

By (23) and (10), we can calculate that

$$\tau(P((T+1)K_n-1)\cdots P(TK_n)) \leq 1-\frac{nc_n}{M_n},$$

from Lemma 3.1, we obtain

$$\begin{aligned}\Delta_{(T+1)\cdot K_n} &\leq \tau((P((T+1)K_n-1)\cdots P(TK_n))\Delta_{T\cdot K_n}\\ &\leq \Big(1-\frac{nc_n}{M_n}\Big)\Delta_{T\cdot K_n} = L_n\Delta_{T\cdot K_n} < \Delta_{T\cdot K_n}.\end{aligned} \tag{26}$$

The last inequality holds because $L_n<1$, the reason is that

$$\begin{aligned}\frac{nc_n}{M_n} &= \Big(\frac{na_n^2}{M_n}\Big)^{K_n}\frac{1}{a_n^2}(1+o(1))\\ &= \Big(\frac{n(\frac{1}{\sqrt{5}}(1-\eta_n)r_n)^2}{n\pi((1+\eta_n)r_n)^2}\Big)^{K_n}\frac{1}{a_n^2}(1+o(1))\\ &\leq \Big(\frac{1}{5\pi}\Big)^{K_n}\frac{1}{a_n^2}\\ &< \Big(\frac{1}{5\pi}\Big)^{\frac{2}{a_n}}\frac{1}{a_n^2}.\end{aligned} \tag{27}$$

Set $y=\frac{2}{a_n}$, then the right hand side of (27) is $f(y):=\frac{y^2}{4}(\frac{1}{5\pi})^y$, where $y\in(2,\infty)$ because of $a_n\in(0,1)$. Notice that $f(2)<1$ and $\lim_{y\to\infty}f(y)=0$, besides,

$$f'(y)=\Big(\frac{1}{5\pi}\Big)^y\log\Big(\frac{1}{5\pi}\Big)\frac{y^2}{4}+\Big(\frac{1}{5\pi}\Big)^y\frac{y}{2}<0 \tag{28}$$

on $(2,\infty)$, then $f(y)<1$ on $(2,\infty)$. Hence the last inequality of (26) holds .

Up to now, we have proved $\Delta_{k\cdot K_n}\leq L^k\Delta_1$ in the case of $k=T+1$, so the induction argument of $\Delta_{k\cdot K_n}\leq L_n^k\Delta_1$ is completed.

Hence, Δ_t will converge to 0 with exponential rate, that means the system will achieve consensus. ∎

Remark 3.4 Since the corresponding graphs are not symmetric due to the M-nearest-neighbor interaction, then estimating the spectral gap of the corresponding stochastic matrixes, which is an often-used method to measure the speed of the systems consensus, does not work in our situation. Hence, we use some new techniques. To avoid analyzing the spectral gap, we turn to estimating τ of the matrixes. Using the connectivity property under random framework, we can design an appropriate speed such that within a bounded time interval, τ of the products of corresponding stochastic matrixes is strictly smaller than 1. Then consensus analysis can be carried on.

Proof of Corollary 2.5 Now, we will estimate the order of Δ_1 under initial headings' distribution and then calculate the right-hand side of (6).

Throughout the sequel, for fixed η_n, we denote $h_n = \frac{na_n^2}{g_n \log n}$, which satisfies $\lim_{n\to\infty} h_n = \infty$ by the choice of a_n and g_n. Two useful lemmas are listed below.

Lemma 3.5 (see Lemma 8 in [7]) *Let* $S_j^{\sim} = \sum_{k\in S_j} \theta_k(0), j = 1, 2, \cdots, \left\lceil \frac{1}{a_n^2} \right\rceil$, *and denote*

$$D(a_n, g_n, h_n) = \left\{ \omega : \max_{1\le j\le \left\lceil \frac{1}{a_n^2} \right\rceil} \left| \widetilde{S_j} \right| \le na_n^2 \pi \sqrt{\frac{2}{h_n}}(1 + o(1)) \right\}.$$

Then $P\{D(a_n, g_n, h_n) \cap B_2^{\eta_n}\} = 1 - O(n^{-g_n})$.

Lemma 3.6 (Theorem 4 in [7]) *On the set* $D(a_n, g_n, h_n) \cap B_2^{\eta_n}$, *we have for* n *sufficiently large*

$$\Delta_1 \le 2\pi \sqrt{\frac{2}{h_n}}(1 + o(1)). \tag{29}$$

Hence, $P(D(a_n, g_n, h_n) \cap B_2^{\eta_n} \cap B_1^{\eta_n}) = 1 - O(n^{-g_n})$ from Lemma 3.5 and (14). And (6) and (29) both hold on the set $P(D(a_n, g_n, h_n) \cap B_2^{\eta_n} \cap B_1^{\eta_n})$. Substitute (29) into (6), we obtain that the system will achieve consensus under the condition

$$v_n < \frac{(\frac{n}{M_n})^{K_n} a_n^{2(K_n-1)} \eta_n r_n}{2\pi \sqrt{\frac{2}{h_n}}(1 + o(1))K_n} = \frac{(\frac{n}{M_n})^{K_n} a_n^{2(K_n-1)} \eta_n r_n \frac{na_n^2}{g_n \log n}}{2\pi\sqrt{2}(1 + o(1))K_n} \triangleq V_n.$$

When n ia large enough, by the definition in Theorem 2.2, we have

$$\begin{aligned} r_n &= \frac{1}{1+\eta_n}\sqrt{\frac{M_n}{n\pi}} \triangleq C_1 \sqrt{\frac{M_n}{n\pi}}, \\ a_n &= \frac{1-\eta_n}{\sqrt{5}(1+\eta_n)}\sqrt{\frac{M_n}{n\pi}} \triangleq C_2 \sqrt{\frac{M_n}{n\pi}}, \\ K_n &= 2\left\lceil \frac{1}{C_2}\sqrt{\frac{n\pi}{M_n}} + 1 \right\rceil = 2\Big(\frac{1}{C_2}\sqrt{\frac{n\pi}{M_n}}(1 + o(1)) + 1\Big). \end{aligned}$$

The last equality holds as $n \to \infty$.

Substitute all above into V_n, we have

$$\begin{aligned} V_n &= \frac{\eta_n}{2\pi\sqrt{2}(1 + o(1))} \cdot \frac{(\frac{n}{M_n})^{K_n} a_n^{2K_n} r_n n}{g_n \log n K_n} \\ &= \frac{\eta_n}{2\pi\sqrt{2}(1 + o(1))} \cdot \Big(\frac{C_2^2}{\pi}\Big)^{K_n} \cdot \frac{nr_n}{g_n \log n K_n} \\ &= \frac{\eta_n}{2\pi\sqrt{2}(1 + o(1))} \cdot C^{\sqrt{\frac{n}{M_n}}} \cdot \frac{M_n}{g_n \log n}, \end{aligned}$$

where $C = (\frac{C_2^2}{\pi})^{\frac{2\sqrt{\pi}}{C_2}} < (5\pi)^{-2\sqrt{5\pi}} < 1$. Notice that $M_n \gg \log n$ and $1 \ll g_n \ll \frac{M_n}{\log n}$, so $V_n = o\bigg((5\pi)^{-2\sqrt{5\pi n/M_n}} M_n/\log n\bigg)$. The proof is finished. ∎

Proof of Corollary 2.6 When $M_n = \alpha n(1 + o(1)$, the conclusion is obvious from Corollary 2.5. ∎

4 Simulation

In this section, we demonstrate some simulation examples. Here, the initial positions and headings of n agents are distributed uniformly and independently in $[0, 1]^2$ and $(-\pi, \pi)$ respectively, and the agents' speed is taken as $v = 0.01$. Figure 3 shows the change of the consensus frequency with the population size n when $M_n = 0.1n$, $M_n = 0.2n$ and $M_n = 0.3n$, respectively. From this simulation, we see that for the same speed $v = 0.01$, the systems with $\alpha = 0.2$ and $\alpha = 0.3$ will achieve consensus with the frequency very close to 1, when the number of agents is greater than 100 and 40 respectively. This is consistent with Theorem 2.6. At the same time, the system with $\alpha = 0.1$ can not achieve consensus with large probability when $n < 150$. It means that when n and v is fixed, the upper bound of the speed for consensus deceases as α decreases, which is consistent with Corollary 2.6.

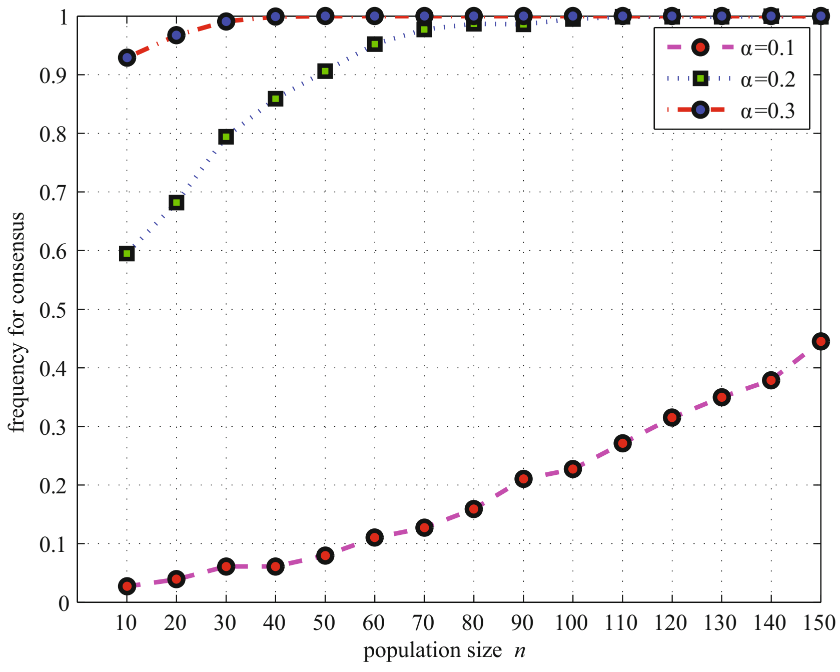

Figure 3 Simulation result for the system (1)–(3) with $v = 0.01$ and $M_n = 0.1n$, $M_n = 0.2n$ and $M_n = 0.3n$, respectively

5 Conclusion and Future Work

For a class of flocks motivated by starling birds whose interaction rules are modeled by the "M-nearest neighbor rule", this paper gives some easily verifiable conditions for consensus together with a complete theoretical analysis. In particular, we find that under some mild assumptions, the system will achieve consensus for any given speed almost surely, provided that the population size is large enough.

Of course, many interesting problems still remain open, for example, the case where noise effects should be considered, and the case where the number of agents are prescribed. These belong to further investigation.

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