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CONTROLLABILITY OF STOCHASTIC GAME-BASED CONTROL SYSTEMS*

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Abstract. It is well known that in classical control theory, the controller has a certain objective to achieve, and the plant to be controlled does not have its own objective. However, this is not the case in many practical situations in, for example, social, economic, and rapidly developing "intelligent" engineering systems. For these kinds of systems, the classical control theory cannot be applied directly. This motivates us to introduce a new control framework called game-based control systems (GBCSs), which has a hierarchical decision-making structure, i.e., a higher level regulator and lower level multiple agents. The regulator is regarded as the macrocontroller that makes decisions first, and then the agents try to optimize their respective objective functions, where a possible Nash equilibrium may be reached as a result of a noncooperative differential game. A fundamental issue in GBCSs is whether it is possible for the regulator to change the macrostates of the system by regulating the Nash equilibrium formed by the agents at the lower level. The investigation of this problem was initiated recently by the authors for deterministic systems. In this paper, we formulate this problem in the general stochastic nonlinear framework, and then focus on linear stochastic systems to give some explicit necessary and sufficient algebraic conditions on the controllability of the Nash equilibrium. In contrast to the classical controllability theory on forward differential equations, we now need to investigate the controllability of the associated forward-backward stochastic differential equations, which involves a more complicated investigation. Moreover, in the current stochastic case, which is more complicated than the deterministic case, we need some deep understanding of forward-backward stochastic differential equations.

Key words. noncooperative stochastic differential games, hierarchical structure, game-based control systems, Nash equilibrium, controllability, forward-backward stochastic differential equations

AMS subject classifications. 93B05, 93A13, 91A10, 91A23

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1. Introduction.

It is well known that, over the past half-century, considerable progress has been made in control theory, where the controller usually has a certain objective to achieve, whereas the plant to be controlled does not have its own objective, which is usually modeled by physical laws, such as the control of a car, an airplane, an industrial process, etc. However, the classical control framework is not applicable to many practical situations in, for example, social, economic, and rapidly developing "intelligent" engineering systems [1, 9, 11, 38, 40]. The common characteristic of these systems is that the plants to be controlled involve multiple rational agents who purse their own objectives, which may not be the same as the macrocontroller's objective, and hence such control systems should be described by differential games.

One concrete example comes from the problem of transboundary pollution [16, 22, 25, 6, 41]. It is well known that some pollutants, such as air pollution, can spread across an incredible distance. Consequently, pollution in one region can impact neighboring regions to cause transboundary pollution. Just as in the "public goods game," these negative externalities and the pursuit-of-self-interest maximization can result in an excess of pollution. Solving the problem calls for cooperation among

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different regions. Cooperative and noncooperative stochastic differential games have been formulated to model transboundary industrial pollution with emission permits trading [22, 6, 41]. In fact, full cooperation and complete noncooperation are two extreme cases. The more common situation is to introduce some mechanisms, such as contracts, associated agencies, and reputation systems, into the noncooperative game to promote cooperation. In the transboundary pollution problem, we can reasonably assume that different regions can reach an agreement or contract on the distribution of their joint net revenues through cooperation. For any given distribution contract, they will naturally form a noncooperative game. Hence, we can take the contract as a higher level regulator, which controls the pollution by regulating the Nash equilibrium formed by the different regions, and the existence of such a contract can be regarded as a controllability problem of transboundary pollution.

Another example comes from government regulation of macroeconomics. For various purposes, including remedying market failure, promoting cooperation, increasing social welfare, and solving asymmetric information problems, a government may regulate macroeconomics through legislation and policies. Using game theory and dynamic stochastic general equilibrium (DSGE) [23] instruments and methods, we can model the macroeconomic system as a noncooperative dynamic game with a government and multiple interest groups. The government can affect the system dynamics by enacting certain policies and, based on these policies the interest groups pursue their own interests. Hence, for any given government policy, the interest groups form a stochastic differential noncooperative game. By analyzing this model, the government may develop an appropriate policy to achieve some macroeconomic regulation purposes. If we take the policy as a higher level regulator, then it may achieve some satisfactory macroeconomic states by regulating the Nash equilibrium formed by the interest groups.

Many other examples exist in social, economic, and engineering systems. Moreover, many problems that were previously investigated by using the game theoretic framework may also be investigated by introducing a higher level regulator to induce the Nash equilibrium to a desired value. These include, among others, distributed game theoretic control [3, 14, 24], coverage optimization for mobile sensors [48], team formation control [12, 2], intelligent transportation systems [8], ecological systems [28, 47], and multiphase systems in chemical engineering [17].

In all of the above-mentioned examples, the systems to be regulated involve multiple active agents, whose behaviors are driven by their own interests. The individual rationality may result in strategic behaviors, but if these behaviors are ignored, the system dynamics may be seriously distorted and lead to misunderstanding; see, e.g., [11, 20]. Hence, it is necessary to model an individual's rational behavior by using game theory. This leads to the introduction and investigation of the game-based control system (GBCS), which has a hierarchical decision-making structure consisting of one regulator and multiple agents. The regulator is regarded as the macrocontroller and makes decisions first, and then the agents try to optimize their respective objectives to reach a possible Nash equilibrium as a result of a noncooperative dynamic game. We will delineate the details of the GBCS in subsection 2.1.

It goes without saying that different inputs of the higher level regulator may influence the Nash equilibrium of the lower level game and hence the performance of the system state of the GBCS. There may be a large number of state variables to represent the entire state of the GBCS, but the regulator may only care about a small subset of these state variables, which can reflect the macrostate of the GBCS. This phenomenon is ubiquitous in, for example, market regulation and corporate governance. Sometimes the state variables of the systems can be divided into macro- and microstate variables, and it is the macrostate that the higher level regulator needs to regulate. A fundamental issue in GBCS is the following: Is it possible for the regulator to change the macrostate by regulating the Nash equilibrium formed at the lower level? This leads to the investigation of controllability of the Nash equilibrium of GBCS, which can reflect the ability of the regulator to move the macrostate around in its entire configuration space. To the best of our knowledge, the first attempt devoted to the introduction and investigation of GBCS was given in [26, 27], followed by a preliminary investigation on controllability of GBCS for some special cases [44, 45, 46].

The main contributions of this paper are as follows. First, we formulate the general stochastic GBCSs as a two-level hierarchical structure, where the lower level is a noncooperative stochastic differential game among multiple agents, and the higher level is a macroregulator which can intervene in the lower level differential game to achieve a desired macrostate. Second, we introduce the concepts of exact V-controllability and total controllability (a special exact V-controllability). The exact V-controllability of general nonlinear stochastic GBCS is transformed into the controllability of a corresponding forward-backward stochastic differential equation (FBSDE) by using the maximum principle, which is different from the classical controllability problem, and may serve as a starting point for future related investigations. Third, we obtain some necessary and sufficient algebraic conditions on the exact V-controllability of the Nash equilibrium for general linear time-varying GBCS by solving the controllability problem of the associated FBSDEs, which is a key technical difficulty of the current paper that makes our analysis quite different from that in classical theory. For linear time-invariant GBCS, we give an explicit necessary and sufficient algebraic condition on the total controllability of the Nash equilibrium.

The remainder of this paper is organized as follows. In section 2, we introduce a general framework for GBCS and give some examples. In section 3, we will first study the controllability of general nonlinear stochastic GBCS, and then provide a complete solution for general linear stochastic systems. Proofs of the theorems are given in sections 4 and 5, and we conclude the paper with some remarks.

2. Problem formulation.

In this section, we will first give a general nonlinear framework for stochastic GBCS and introduce the concept of controllability, and then give two illustrative examples where the structure applies.

First, we introduce some notation to be used throughout this paper. All vectors are column vectors. The identity matrix of size $m \times m$ is denoted by I_m, and the null matrix of size $m \times n$ is denoted by $0_{m\times n}$ (or 0_m if $m = n$). For a matrix A, the operator $\text{rank}\{A\}$ means the rank of A, A^T denotes the transposition of A, $\text{Im}\{A\}$ represents the image space of A, and $\Lambda(A)$ denotes the set of all eigenvalues of A. For any subset $V \subset R^n$, $Span\{V\}$ denotes the linear subspace in R^n generated by V. We use the notation $\langle \cdot, \cdot \rangle$ for the inner product in R^n, and use f_x for the partial derivative of a function f with respect to x.

2.1. Game-based control systems.

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space endowed with a filtration $(\mathscr{F}_t, t \geqslant 0)$, and let $(w_t, t \geqslant 0)$ be a d-dimensional standard Brownian motion:

$$w_t = (w_t^1, w_t^2, \ldots, w_t^d)^T.$$

The filtration $\mathscr{F}_t$ is generated by this Brownian motion and satisfies the usual hypotheses (complete and right continuous). All processes mentioned here are assumed to be $\mathscr{F}_t$-adapted and square integrable. Let $L^2_{\mathscr{F}}(0,T;R^n)$ be the set of all valued square integrable processes, i.e.,

$$L^2_{\mathscr{F}}(0,T;R^n) = \left\{ X(\cdot) \text{ is } \{\mathscr{F}_t\}_{t\geq 0}\text{-adapted } : E\int_0^T \|X(t)\|^2 dt < \infty \right\}.$$

Consider the following hierarchical control system with one regulator and L agents:

$$\begin{cases} dx(t) = f(t, x(t), u_1(t), \ldots, u_L(t), u(t))dt \\ \qquad\qquad + \sigma(t, x(t), u_1(t), \ldots, u_L(t), u(t))dw(t), \\ x(0) = x_0(\in R^n), \end{cases} \tag{1}$$

where $u_i(t) \in AD_i \subset R^{m_i}$ stands for the strategy or control of the agent i, and $u(t) \in AD \subset R^m$ is the strategy or control of the regulator.

Assume that each agent wants to minimize a payoff function $J_i(u_i(\cdot), u_{-i}(\cdot), u(\cdot))$ by selecting $u_i(\cdot)$ from its admissible strategy set, where $u_{-i}(\cdot)$ represents the strategy profile of all agents except agent i.

The commonly used payoff function is as follows for each agent i:

$$\begin{aligned} &J_i(u_1(\cdot), u_2(\cdot), \ldots, u_L(\cdot), u(\cdot)) \\ &= EK_i(x(T)) + E\int_0^T L_i(x(t), u_1(t), u_2(t), \ldots, u_L(t), u(t))\, dt \end{aligned} \tag{2}$$

for any given finite time $T > 0$. Alternatively, one may also consider the following ergodic cost when $T \to \infty$:

$$\begin{aligned} &J_i(u_1(\cdot), u_2(\cdot), \ldots, u_L(\cdot), u(\cdot)) \\ &= \limsup_{T\to\infty} \frac{1}{T}\int_0^T L_i(x(t), u_1(t), u_2(t), \ldots, u_L(t), u(\cdot))\, dt. \end{aligned} \tag{3}$$

We only consider the finite time case in this paper.

It is well known that information structures play a crucial role in differential games [7, 13]. Here, we always assume that the information $\{f, \sigma, J_i, i = 1, 2, \ldots, L\}$ and the initial state $x(0)$ of the system are "common knowledge," which is the terminology used for information in game theory [10, Chapter 14]. In GBCS, the regulator will first make and announce its macrodecision, and then each agent makes a decision to minimize its own payoff function. The lower level agents have access to information about the regulator's input (i.e., they know the regulator's input) but do not know other agents' inputs when making their own decisions. Therefore, given the decision of the regulator, the agents in GBCS will form a noncooperative stochastic differential game at the lower level.

Depending on the information available to the lower level agents, there are two major cases in noncooperative differential games: open-loop information and feedback information. Under the open-loop information structure, all agents know just the initial state, and so the strategy $u_i(t)$ $(i = 1, 2, \ldots, L)$ of agent i is only a function of time and the initial state (open-loop strategy). In contrast, the feedback information structure allows all agents to observe the current system state, and then the strategy $u_i(t)$ $(i = 1, 2, \ldots, L)$ of agent i can depend on time and current state (feedback strategy). In this paper, we only consider the open-loop strategy, so the Nash equilibrium formed by the agents is also called an open-loop Nash equilibrium [4, Definition 5.6]. Let us now define $\mathcal{U}$ and $\mathcal{U}_i$ as the admissible control sets for the regulator and agent i, respectively. For any control $u(\cdot) \in \mathcal{U}$ of the regulator, the system will form a noncooperative differential game. If the Nash equilibrium strategy $(u_1^*(\cdot), u_2^*(\cdot), \dots, u_L^*(\cdot))$ of the agents exists, then we know that, for $i = 1, 2, \dots, L$,

$$J_i(u_i^*(\cdot), u_{-i}^*(\cdot), u(\cdot)) = \min_{u_i(\cdot)\in\mathcal{U}_i} J_i(u_i(\cdot), u_{-i}^*(\cdot), u(\cdot)). \tag{4}$$

As will be shown in the next section, under some mild regularity assumptions, given any decision of the regulator $u(\cdot) \in \mathcal{U}$, the corresponding agents' decisions $u_1(\cdot) \in \mathcal{U}_1, \dots, u_L(\cdot) \in \mathcal{U}_L$ will define a unique solution of the system dynamic (1) denoted by $t \mapsto x^{x_0,u,u_1,\dots,u_L}(t)$ on $[0,T]$ (in the sense of Definition 6.15 of [42, Chapter 1]).

If the Nash equilibrium of the noncooperative stochastic differential game (1) exists and is unique for some given input $u(\cdot) \in \mathcal{U}$ and $x(0) = x_0$, then, under some mild conditions, there is a unique state evolution process $x^*(t)(t \in [0,T])$ of the system, satisfying the following ordinary differential equation:

$$\begin{cases} dx^*(t) = f(t, x^*(t), u_1^*(t), u_2^*(t), \dots, u_L^*(t), u(t))dt \\ \qquad\qquad + \sigma(t, x^*(t), u_1^*(t), u_2^*(t), \dots, u_L^*(t), u(t))dw(t), \\ x^*(0) = x_0, \end{cases} \tag{5}$$

where $(u_1^*, u_2^*, \dots, u_L^*)$ is the Nash equilibrium corresponding to the regulator's strategy $u(t)(t \in [0,T])$. Hence, the system state dynamic is essentially determined by the regulator and thus can be regarded as a control system, where the regulator can change the state by regulating the Nash equilibrium formed at the lower level.

The above problem formulation can be directly extended to more general settings with hybrid dynamic and multilayers.

In many practical systems, there are macrostates and microstates. The dynamics of the macrostates can be affected by the lower level agents' states, which stand for the microstates but not their controls directly; i.e., the system is described by the following stochastic differential equation (SDE):

$$\begin{cases} dx(t) = f(t, x(t), x_1(t), x_2(t), \dots, x_L(t), u(t))dt \\ \qquad\qquad + \sigma(t, x(t), x_1(t), x_2(t), \dots, x_L(t), u(t))dw(t), \\ dx_i(t) = f_i(t, x(t), x_i(t), u_i(t), u(t))dt \\ \qquad\qquad + \sigma_i(t, x(t), x_i(t), u_i(t), u(t))dw(t), \\ x(0) = x_0,\ x_i(0) = x_{i,0},\ i = 1, 2, \dots, L, \end{cases} \tag{6}$$

and the payoff function to be minimized by $u_i(\cdot)$ of any agent i $(i = 1, 2, \dots, L)$ is

$$ES_i(x_i(T), T) + E\int_0^T L_i(x_i(\cdot), u_i(\cdot))\,dt$$

when T is finite. Here $x(t) \in R^n$ stands for the macrostate of the system, and $x_i(t) \in R^{n_i}$ is the state of the agent i $(i = 1, 2, \dots, L)$. In this case, the payoff function of agent i only depends on its own state x_i and input $u_i(\cdot)$.

It is easy to transform (6) into the form of system (1) as follows:

$$\begin{cases} dX(t) = \widetilde{f}(t, X(t), u_1(t), u_2(t), \dots, u_L(t), u(t))dt \\ \qquad\qquad + \widetilde{\sigma}(t, X(t), u_1(t), u_2(t), \dots, u_L(t), u(t))dw(t), \\ X(0) = X_0, \end{cases} \tag{7}$$

where

$$
\begin{aligned}
&X(t) = [x^T(t), x_1^T(t), \ldots, x_L^T(t)]^T, \\
&\widetilde{f}(t, X(t), u_1(t), u_2(t), \ldots, u_L(t), u(t)) \\
&\quad = \begin{bmatrix} f(t, x(t), x_1(t), x_2(t), \ldots, x_L(t), u(t)) \\ f_1(t, x(t), x_1(t), u_1(t), u(t)) \\ \vdots \\ f_L(t, x(t), x_L(t), u_L(t), u(t)) \end{bmatrix}, \\
&\widetilde{\sigma}(t, X(t), u_1(t), u_2(t), \ldots, u_L(t), u(t)) \\
&\quad = \begin{bmatrix} \sigma(t, x(t), x_1(t), x_2(t), \ldots, x_L(t), u(t)) \\ \sigma_1(t, x(t), x_1(t), u_1(t), u(t)) \\ \vdots \\ \sigma_L(t, x(t), x_L(t), u_L(t), u(t)) \end{bmatrix}.
\end{aligned} \tag{8}
$$

2.2. Controllability problem. As control systems, there are many interesting problems to be investigated. Here we are interested in whether or not the system macrostate can be driven from any initial state to any desired macrostate by the influence of the regulator, which can be captured by the following concept of exact V-controllability.

Let V be the space $L^2(\Omega, \mathscr{F}_T, P; R^h)$ $(h \leq n)$, and let P_h be the projection operator from R^n to R^h such that $P_h((x_1, \ldots, x_n)^T) = (x_1, \ldots, x_h)^T$.

DEFINITION 2.1. *The GBCS* (1) *is called exactly V-controllable if for any given initial state $x(0) = x_0 \in R^n$ and any terminal state $x^h(T) = x_T^h = \xi \in V$, there is a strategy $u(\cdot) \in \mathcal{U}$ of the regulator, under which the Nash equilibrium exists and is unique, and the solution $x^*(t)$ of the* (5) *satisfies $P_h x^*(T) = x_T^h$. Moreover, if $V = L^2(\Omega, \mathscr{F}_T, P; R^n)$, then we say that the GBCS is totally controllable.*

We remark that the initial states include all the components of x, but in the final states we are only interested in the partial state $x^h(T)$ in the above definition. Just as in the GBCS (7), $P_h x(T)$ are the first h components of state x and represents the macrostate.

2.3. Examples of GBCS. We will give two examples where the GBCS framework may be applied.

Example 2.2. Consider the problem of transboundary industrial pollution with emission permits trading. The dynamic of the system with two regions is described by [6]

$$
\begin{cases} dP(t) = (E_1(t) + E_2(t) - \theta_P P(t))dt + \sigma_P P(t)dw(t), \\ dS(t) = \mu_S S(t)dt + \sigma_S S(t)dw(t), \\ \ S(0) = S_0,\ P(0) = P_0, \end{cases} \tag{9}
$$

where $S(t)$ is the emission permits price, $P(t)$ denotes the stock of pollution in the environment, and $E_1(t)$ and $E_2(t)$ denote the emission levels of regions 1 and 2, respectively. Region i $(i = 1, 2)$ chooses its strategy $E_{Ci}(\cdot)$ to maximize its own objective functional,

$$
\begin{aligned}
&J_i(E_1,\, E_2, u(\cdot)) \\
&\quad = E\left\{ \int_{t_0}^{T} z_i(t)e^{-rt}\Big[L_1(t)+L_2(t)\Big]dt - z_i(T)\Big[K_1(T)+K_2(T)\Big]e^{rT} \right\},
\end{aligned} \tag{10}
$$

where

$$
\begin{aligned}
L_i(t) &= A_i - S(t)E_i(t) - \frac{1}{2}E_i^2(t) + S(t)E_{i0} - D_iP(t),\\
K_i(T) &= g_i(P(T) - \overline{P}_i)
\end{aligned}
$$

are the running cost and the salvage cost at time T, respectively, and $z = (z_1, z_2)^T$ represents the distribution of the joint net revenue; the specific meanings of other variables can be found in [6]. The two regions can make a collaborative contract on the revenues assignment $z(\cdot)$. This model can be taken as a cooperative game, which is a little different from that in the literature. When we let $z_1(t) = z_2(t) \equiv c$ (c is any nonzero constant), this model becomes a standard cooperative differential game or team optimal control.

An important issue in the model is the following: Given any terminal object of the pollution $P(T)$, is there an assignment contract $z(\cdot)$ such that the objective is achieved? This is obviously a typical problem on the controllability of the stochastic GBCS and has not yet been solved in the literature.

Example 2.3. Different stabilization policies have been deeply studied by economists for many years [15, 30, 31, 33, 39]. Here we consider the optimal economic stabilization policies problem under decentralized control and conflicting objectives. In many countries, macroeconomic policy is made and implemented by more than one authority who may have differing objectives [33].

The linear-quadratic difference games have been used to model these systems around a certain nominal state [15], and some stochastic elements have also been introduced into the model [39]. We assume that the diffusion term of the stochastic dynamic depends on the controls, which is reasonable and realistic in many practical systems. The continuous-time analogy of the discrete-time stochastic econometric dynamical model in the state variable is

$$
dx(t) = \left(Ax(t) + \sum_{i=1}^{k} B_iu_i(t) + Cz(t)\right)dt + \left(\overline{A}x(t) + \sum_{i=1}^{k} \overline{B}_iu_i(t) + \overline{C}z(t)\right)dw(t), \tag{11}
$$

where u_i is the control variable of authority i ($i = 1, \ldots, k$), who chooses u_i to minimize its own cost function

$$
J_i = E\int_0^T \Big(x^T(t)Q_ix(t) + u_i^T(t)R_iu_i(t)\Big)dt. \tag{12}
$$

Here, the input $z(t)$ can be taken as a higher level regulator, such as a law, a policy, or a contract. In different economic situations, the higher level regulator may need to regulate the equilibrium state $x(T)$ to some desired values by choosing some appropriate inputs $z(t)$. This is another typical problem on the controllability of GBCS, which can be solved directly by using Theorem 3.14 in subsection 3.3.

3. Main results. In this section, we first give an analysis of the controllability of the Nash equilibrium of GBCS with general nonlinear dynamics, and then focus on linear dynamic systems to give some explicit necessary and sufficient algebraic conditions on the controllability of the Nash equilibrium.

3.1. General nonlinear systems. Consider the general nonlinear system (1). For notational brevity, we only consider one-dimensional Brownian motion, $d = 1$. The case for multidimensional Brownian motion can be treated analogously.

Given any terminal time T and any input of the regulator, each agent i $(1, 2, \dots, L)$ wants to minimize its own payoff function, expressed by

$$\begin{aligned} &J_i(u_1(\cdot), u_2(\cdot), \dots, u_L(\cdot), u(\cdot)) \\ &= EK_i(x(T)) + E\int_0^T L_i(x(t), u_1(t), u_2(t), \dots, u_L(t), u(t))\, dt. \end{aligned} \tag{13}$$

Assumption 3.1. The input signals are admissible, i.e., they are taken from the following sets:

$$\begin{aligned} &u(\cdot) \in \mathcal{U} \triangleq \{u : [0, T] \to AD \mid u(\cdot) \in L^2_{\mathscr{F}}(0, T; R^m)\} \\ &u_i(\cdot) \in \mathcal{U}_i \triangleq \{u_i : [0, T] \to AD_i | u_i(\cdot) \in L^2_{\mathscr{F}}(0, T; R^{m_i})\}, \end{aligned} \tag{14}$$

where $AD \subseteq R^m$ and $AD_i \subseteq R^{m_i}$, $i = 1, 2, \dots, L$. $\{\mathscr{F}_t\}_{t\geq 0}$ is the natural filtration generated by $w(t)$, augmented by all of the P-null sets in $\mathscr{F}$.

Assumption 3.2. For any $u(\cdot) \in \mathcal{U}$, $u_i(\cdot) \in \mathcal{U}_i, i = 1, 2, \dots, L$, and initial point x_0, the state equation (1) admits a unique solution $x(\cdot) \equiv x(\cdot, u_1(\cdot), \dots, u_L(\cdot), u(\cdot))$ (in the sense of Definition 6.15 of [42, Chapter 1]) on $[0, T]$, and the cost function (13) is well defined for $i = 1, \dots, L$.

We remark that if equation (1) is linear and the cost function (13) is quadratic, then under certain conditions on the coefficients (see Assumption (L1) in [42, p. 301]), Assumption 3.2 holds. For general nonlinear systems, Assumption 3.2 holds if the following conditions hold [42, Chapter 1, section 6.4]: All of the functions f, σ, K_i, and L_i $(i = 1, \dots, L)$ are second-order continuous differentiable with respect to x, and there exist a constant $C > 0$ and a modulus of continuity $\varpi : [0, \infty) \to [0, \infty)$ such that for any $\varphi \in \{f,\ \sigma,\ K_i,\ L_i\}$, we have

$$\begin{cases} |\varphi(t, x, U) - \varphi(t, \overline{x}, \overline{U})| \leq C|x - \overline{x}| + \varpi(U - \overline{U}), \\ |\varphi_x(t, x, U) - \varphi_x(t, \overline{x}, \overline{U})| \leq C|x - \overline{x}| + \varpi(|U - \overline{U}|), \\ |\varphi_{xx}(t, x, U) - \varphi_{xx}(t, \overline{x}, \overline{U})| \leq \varpi(|U - \overline{U}| + |x - \overline{x}|), \\ |\varphi(t, 0, U)| \leq C \end{cases} \tag{15}$$

for any $t \in [0, T]$, x, $\overline{x} \in R^n$, and U, $\overline{U}$, where $U = (u_1^T, \dots, u_L^T, u^T)^T$ and $\overline{U} = (\overline{u}_1^T, \dots, \overline{u}_L^T, \overline{u}^T)^T$.

To solve the stochastic differential game defined by (1) and (13), we introduce the generalized Hamiltonian

$$\begin{aligned} G_i&(t, x, u_1, \dots, u_L, u, p, P) \\ &\triangleq \frac{1}{2}\sigma(t, x, u_1, \dots, u_L, u)^T P \sigma(t, x, u_1, \dots, u_L, u) \\ &\quad + \langle p, f(t, x, u_1, \dots, u_L, u)\rangle - L_i(t, x, u_1, \dots, u_L, u) \\ \forall\ &(t, x, u_1, \dots, u_L, u, P) \in [0, T] \times R^n \times R^{m_1} \times \cdots \times R^{m_L} \times R^m \times S^n, \end{aligned} \tag{16}$$

where $S^n = \{A \in R^{n\times n} : A^T = A\}$, $i = 1, \dots, L$.

Next, we define an $\boldsymbol{H}$-function,

$$
\begin{aligned}
&\boldsymbol{H}_i(t, x, u_1, \dots, u_{i-1}, u_i, \overline{u}_i, u_{i+1}, \dots, u_L, u, p, P)\\
&\quad\triangleq G_i(t, x, u_1, \dots, u_{i-1}, u_i, u_{i+1}, \dots, u_L, u, p, P)\\
&\qquad + \sigma(t, x, u_1, \dots, u_{i-1}, u_i, u_{i+1}, \dots, u_L, u)^T[q - P\overline{\sigma}],
\end{aligned}
\tag{17}
$$

where $\overline{\sigma} = \sigma(t, x, u_1, \dots, u_{i-1}, \overline{u}_i, u_{i+1}, \dots, u_L, u)$.

Assumption 3.3. For any $(t, x) \in [0, T] \times R^n$, $u(\cdot) \in \mathcal{U}$, any vectors $p_1, \dots, p_L \in R^n$, and any matrices $P_1, \dots, P_L \in S^n$, there exists a unique point $(u_1^*, \dots, u_L^*) \in AD_1 \times \cdots \times AD_L$ such that for any $i = 1, 2, \dots, L$,

$$
u_i^* = \arg\min_{u_i \in AD_i} \boldsymbol{H}_i(t, x, u_1^*, \dots, u_{i-1}^*, u_i, u_i^*, u_{i+1}^*, \dots, u_L^*, u, p_i, P_i).
$$

The corresponding map is denoted by

$$
\begin{aligned}
&(t, x, p_1, \dots, p_L, P_1, \dots, P_L, u(\cdot))\\
&\to (u_1^*(t, x, p_1, \dots, p_L, P_1, \dots, P_L, u(\cdot)), \dots, u_L^*(t, x, p_1, \dots, p_L, P_1, \dots, P_L, u(\cdot))).
\end{aligned}
$$

We remark that the above assumption can be interpreted as the existence and uniqueness condition of the Nash equilibrium of a "one-shot" game, which has been used in [5]. If the $\boldsymbol{H}_i$-function has some properties, such as concavity and coerciveness, this assumption will be satisfied. Examples in the nonlinear case that can verify Assumption 3.3 include affine-nonlinear systems [5].

If Assumption 3.3 holds and the Nash equilibrium of the stochastic differential game formed by agents at the lower level exists for some $u(t), t \in [0, T]$, $x(0) = x_0$, then we can get the following equations by applying the stochastic maximum principle [42, Chapter 3, Theorem 3.3]:

$$
\begin{cases}
dx(t) = f(t, x(t), U^*(t), u(t))dt + \sigma(t, x(t), U^*(t), u(t))dw(t),\\
dp_i(t) = -\Big\{f_x^T(t, x(t), U^*(t), u(t))p_i(t) + \sigma_x^T(t, x(t), U^*(t), u(t))q_i(t)\\
\qquad\qquad - (L_i)_x(t, x(t), U^*(t), u(t))\Big\}dt + q_i(t)dw(t),\\
x(0) = x_0, p_i(T) = (K_i)_x(x(T)),\ i = 1, \dots, L,
\end{cases}
\tag{18}
$$

where $U^*(\cdot) = (u_1^*(\cdot), \dots, u_L^*(\cdot))$ are the open-loop Nash equilibrium controls of the L agents and they satisfy

$$
\begin{aligned}
&u_i^*(t)\\
&\arg\min_{u_i \in AD_i} \boldsymbol{H}_i(t, x(t), u_1^*(t), \dots, u_{i-1}^*(t), u_i, u_i^*(t), u_{i+1}^*(t), \dots, u_L^*(t), u(t), p_i(t), P_i(t)),\\
&i = 1, 2, \dots, L.
\end{aligned}
$$

Here, $p_i(t)$ is the solution of (18), and $P_i(t)$ is the solution of the following equations:

$$
\begin{cases}
dP_i(t) = -\Big\{f_x^T(t, x(t), U^*(t), u(t))P_i(t) + P_i(t)f_x(t, x(t), U^*(t), u(t))\\
\qquad + \sigma_x^T(t, x(t), U^*(t), u(t))P_i(t)\sigma_x(t, x(t), U^*(t), u(t))\\
\qquad + \sigma_x^T(t, x(t), U^*(t), u(t))Q_i(t) + Q_i(t)\sigma_x(t, x(t), U^*(t), u(t))\\
\qquad + (H_i)_{xx}(t, x(t), U^*(t), u(t), p_i(t), q_i(t))\Big\}dt + Q_i(t)dw(t),\\
P_i(T) = -(K_i)_{xx}(x(T)),\ i = 1, \dots, L,
\end{cases}
\tag{19}
$$

where the Hamiltonian H_i is defined as follows:

$$(20)\qquad \begin{aligned} H_i(t,x,U,u,\,p,q) &= \langle p, f(t,x,U,u)\rangle + q^T\sigma(t,x,U,u) - L_i(t,x,U),\\ &\quad (t,x,U,u,p,q)\in[0,T]\times R^n\times R^{m_1+\cdots+m_L}\times R^m\times R^n\times R^n. \end{aligned}$$

Equations (18) and (19) are backward stochastic differential equations (BSDEs). Under Assumptions 3.1–3.3, there exists a unique adapted solution $(P_i(\cdot), Q_i(\cdot))$ of (19) for any given $p_i(t),\ q_i(t),\ t\in[0,T]$ [21, Theorem 4.2].

From the above analysis, we know that $u_i^*(t), i = 1,\ldots,L$, is a function of $t, x(t), p_i, P_i$, so we can rewrite the functions in (18) and (19) as $F, \Sigma, \Lambda_i, \Gamma_i$, which do not have the variables $u_i^*(t), i = 1,\ldots,L$.

Assumption 3.4. For any input $u(\cdot)\in\mathcal{U}$ of the regulator and any initial value x_0, the Nash equilibrium of the stochastic differential game defined by (1) and (13) exists.

We remark that the input $u(\cdot)\in\mathcal{U}$ in our problem formulation is an open-loop control, i.e., it is just a function of time t. Hence, following an argument similar to that for the existence of optimal control of time-varying nonlinear systems in, e.g., [34, 35], one may investigate the existence of the Nash equilibrium. In particular, if the payoff functions of the agents are all convex [42, Chapter 3, Theorem 5.2], then it may be shown that the existence of the solution of the forward-backward equation implies the existence of the Nash equilibrium in Assumption 3.4. In this case, Assumption 3.4 is equivalent to the existence of the solution of (18), and we can remove it from Theorem 3.6, which will be presented shortly. The well-known dynamic programming can also be used to study the existence problem of the Nash equilibrium by applying the method of viscosity solution to the corresponding coupled Hamilton–Jacobi–Isaacs equation [42, Chapter 4].

When we apply the maximum principle to solve the Nash equilibrium formed by the lower level agents, we will arrive at a forward-backward differential equation with forward state $x(t)$ and backward adjoint states $\{p_i(t), P_i(t) : i = 1,\ldots,L\}$. This motivates us to introduce the concept of exact V-controllability of an FBSDE as follows.

Definition 3.5. *The coupled FBSDE*

$$(21)\qquad \begin{cases} dx(t) = f(t,x(t),y(t),u(t))dt + \sigma(t,x(t),y(t),u(t))dw(t),\\ dy(t) = g(t,x(t),y(t),u(t))dt + q(t)dw(t),\\ \ x(0) = x_0,\ y(T) = G(x(T)) \end{cases}$$

is called exactly V-controllable if for any given initial state $x(0) = x_0 \in R^n$ and any terminal state $x^h(T) = x_T^h = \xi\in V$, there is an admissible strategy $u(\cdot)\in\mathcal{U}$ of the regulator, under which the trajectory of (21) *exists on $[0,T]$, is unique, and satisfies $P_h x(T) = x_T^h$. Moreover, if $V = L^2(\Omega,\mathscr{F}_T,P;R^n)$, then we say that FBSDE* (21) *is totally controllable.*

Theorem 3.6. *Suppose that Assumptions* 3.1–3.4 *hold; then the stochastic GBCS defined by* (1) *and* (13) *is exactly V-controllable if the following FBSDE is exactly*

V-controllable:

$$
\left\{
\begin{aligned}
dx(t) &= F(t, x(t), p_1(t), \dots, p_L(t), P_1(t), \dots, P_L(t), u(t))dt \\
&\quad + \Sigma(t, x(t), p_1(t), \dots, p_L(t), P_1(t), \dots, P_L(t), u(t))dw(t), \\
dp_i(t) &= \Lambda_i(t, x(t), p_1(t), \dots, p_L(t), P_1(t), \dots, P_L(t), q_i(t), u(t))dt \\
&\quad + q_i(t)dw(t), \\
dP_i(t) &= \Gamma_i(t, x(t), p_1(t), \dots, p_L(t), P_1(t), \dots, P_L(t), Q_i(t), u(t))dt \\
&\quad + Q_i(t)dw(t), \\
x(0) &= x_0,\ p_i(T) = (K_i)_x(x(T)),\ P_i(T) = -(K_i)_{xx}(x(T)),\ i = 1, \dots, L.
\end{aligned}
\right. \tag{22}
$$

Moreover, if FBSDE (22) admits a unique solution for any $x_0 \in R^n$ and $u(\cdot) \in \mathcal{U}$, the reverse also holds.

Although it is not easy to get an explicit condition for the controllability of general nonlinear GBCS in terms of the system structure, Theorem 3.6 may provide a necessary basis for further study of nonlinear GBCS, reminiscent of the roles played by the general maximum principle and dynamic programming in optimal nonlinear control. As in optimal control theory, explicit conditions for controllability may be obtained for linear GBCS with quadratic payoff functions of the agents. This is the content of the next section.

3.2. General linear-quadratic systems. Consider the following general noncooperative linear-quadratic differential game with one regulator and L agents:

$$
\left\{
\begin{aligned}
dx(t) &= \left(A(t)x(t) + \sum_{i=1}^{L} B_i(t)u_i(t) + B(t)u(t) \right) dt \\
&\qquad + \left(C(t)x(t) + \sum_{i=1}^{L} D_i(t)u_i(t) + D(t)u(t) \right) dw(t), \\
x(0) &= x_0.
\end{aligned}
\right. \tag{23}
$$

Let $U_i = L^2_{\mathscr{F}}(0, T; R^{m_i})$ and $U = L^2_{\mathscr{F}}(0, T; R^m)$. The payoff function to be minimized by $u_i(\cdot)$ of any agent i $(i = 1, 2, \dots, L)$ is

$$
\begin{aligned}
J_i(u_1(\cdot), u_2(\cdot), \dots, u_L(\cdot)) &= \frac{1}{2} E x^T(T) Q_{iT} x(T) \\
&\quad + \frac{1}{2} E \int_0^T \left[x^T(t) Q_i(t) x(t) + u_i^T(t) R_i(t) u_i(t) \right] dt,
\end{aligned} \tag{24}
$$

where for any $t \in [0, T]$, $R_i^{-1}(t)$ exists, $R_i(t), Q_i(t)$, and Q_{iT} are symmetric, and all entries of the matrices $A(t)$, $B(t)$, $C(t)$, $D(t)$, $B_i(t)$, $C_i(t)$, $D_i(t)$, $Q_i(t)$, $R_i(t)$, $R_i^{-1}(t)$, $i = 1, 2, \dots, L$, are piecewise smooth functions of time. Here, we only consider the deterministic matrices, but many of the results of this paper can be easily extended to random matrices.

From Theorem 3.6, we know that the exact V-controllability of GBCS is equivalent to the exact V-controllability of the corresponding FBSDE (22). In order to rewrite (22) of the linear-quadratic GBCS in a compact form, we introduce the following notation:

$$X(t) = \begin{bmatrix} x(t) \\ p_1(t) \\ \vdots \\ p_L(t) \end{bmatrix}, \ p(t) = \begin{bmatrix} p_1(t) \\ p_2(t) \\ \vdots \\ p_L(t) \end{bmatrix}, \ q(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_L(t) \end{bmatrix} \tag{25}$$

and

$$\begin{aligned} S(t) &= \left[B_1(t)R_1^{-1}(t)B_1^T(t),\ \ldots,\ B_L(t)R_L^{-1}(t)B_L^T(t)\right], \\ T(t) &= \left[D_1(t)R_1^{-1}(t)D_1^T(t),\ \ldots,\ D_L(t)R_L^{-1}(t)D_L^T(t)\right], \\ M(t) &= \left[B_1(t)R_1^{-1}(t)D_1^T(t),\ \ldots,\ B_L(t)R_L^{-1}(t)D_L^T(t)\right], \\ N(t) &= \left[D_1(t)R_1^{-1}(t)B_1^T(t),\ \ldots,\ D_L(t)R_L^{-1}(t)B_L^T(t)\right], \end{aligned} \tag{26}$$

$$\begin{aligned} Q(t) &= \left[Q_1^T(t),\ \ldots,\ Q_L^T(t)\right]^T, \\ Q_T &= \left[-Q_{1T}^T,\ \ldots,\ -Q_{LT}^T\right]^T, \\ \overline{A}(t) &= \begin{bmatrix} A(t) & S(t) \\ -Q(t) & -I_L \otimes A^T(t) \end{bmatrix}, \ \overline{A}_1(t) = \begin{bmatrix} C(t) & T(t) \\ 0 & 0 \end{bmatrix}, \\ \overline{B}(t) &= \left[B^T(t), 0, \ldots, 0\right]^T, \qquad \overline{B}_1(t) = \left[D^T(t), 0, \ldots, 0\right]^T, \\ \overline{C}(t) &= \begin{bmatrix} M(t) \\ -I_L \otimes C^T(t) \end{bmatrix}, \qquad \overline{C}_1(t) = \begin{bmatrix} N(t) \\ I_{nL} \end{bmatrix}, \\ \widehat{C}(t) &= \left[0, I_{nL}\right] \in R^{Ln\times(L+1)n}. \end{aligned}$$

The corresponding stochastic equation (22) of the linear-quadratic stochastic GBCS is a linear forward-backward equation, which has the following form:

$$\begin{cases} dX(t) = \Big(\overline{A}(t)X(t) + \overline{B}(t)u(t) + \overline{C}(t)q(t)\Big)dt \\ \qquad\qquad + \Big(\overline{A}_1(t)X(t) + \overline{B}_1(t)u(t) + \overline{C}_1(t)q(t)\Big)dw(t), \\ x(0) = x_0,\ p(T) = Q_T x(T). \end{cases} \tag{27}$$

Hence, the matrices $\overline{A}(t)$ and $\overline{A}_1(t)$ defined above are just the system matrices of (22), and the matrices $\overline{B}(t)$ and $\overline{B}_1(t)$ are the control matrices.

Assumption 3.7. The following FBSDE admits a unique solution:

$$\begin{cases} dX(t) = \Big(\overline{A}(t)X(t) + \overline{C}(t)q(t)\Big)dt \\ \qquad\qquad + \Big(\overline{A}_1(t)X(t) + \overline{C}_1(t)q(t)\Big)dw(t), \\ x(0) = 0,\ p(T) = Q_T x(T), \end{cases} \tag{28}$$

where $X(t) = (x^T(t), p^T(t))^T$.

Assumption 3.8. The following Riccati differential equations have a set of strongly regular solutions K_j on $[0,T]$ for $j = 1, 2, \ldots, L$:

$$\begin{cases} \dot{K}_j(t) = -A^TK_j - K_jA - Q_j - C^TK_jC \\ \qquad\qquad + (B_j^TK_j + D_j^TK_jC)^T(R_j + D_j^TK_jD_j)^{-1}(B_j^TK_j + D_j^TK_jC), \\ K_j(T) = Q_{jT}, \end{cases} \tag{29}$$

where the strong regularity of K_j means that there is a positive number $\lambda > 0$, which satisfies the following condition [36]:

$$R_j(t) + D_j^T(t)K_j(t)D_j(t) \geq \lambda I \quad \text{a.e. } t \in [0, T]. \tag{30}$$

We remark that if the matrices $Q_i(t)$, Q_{iT} $(i = 1, 2, \ldots, L)$ are positive semidefinite and $R_i(t) > 0$, which is the standard condition of stochastic linear quadratic optimal control, then Assumption 3.8 holds automatically [42].

We are now in a position to state our main results on controllability of general linear-quadratic stochastic GBCS; proofs are deferred to section 4. We first present a necessity theorem.

THEOREM 3.9. *If the stochastic GBCS* (23)–(24) *is exactly V-controllable, then the following two conditions hold:*

1. *For any set N of zero-measure, we have*

$$\overline{V} \subseteq Span\left\{ \bigcup_{t\in[0,T]-N} \operatorname{Im}(D(t)) \right\}, \tag{31}$$

where $\overline{V} = \{v \in R^n : v_i = 0,\ h < i \leq n\}$.

2. *The matrix*

$$\left[\begin{bmatrix} 0_{n\times nL} \\ I_{nL} \end{bmatrix}, \overline{\Phi}_T \overline{Q}_T \begin{bmatrix} 0_{h\times(n-h)} \\ I_{n-h} \end{bmatrix}, M(T) \right] \tag{32}$$

is of full rank, where

$$\begin{aligned} M(T) &= E\int_0^T \overline{\Phi}(t)\overline{B}(t)\overline{B}(t)^T\overline{\Phi}T(t)\,dt, \\ \overline{\Phi}_T &= E\overline{\Phi}(T),\ \overline{Q}_T = \begin{bmatrix} I_n \\ Q_T \end{bmatrix} \in R^{(L+1)n\times n}, \end{aligned} \tag{33}$$

and the matrix $\overline{\Phi}(t)$ is defined by

$$\begin{cases} d\overline{\Phi}(t) &= -\overline{\Phi}(t)\overline{A}(t)dt - \overline{\Phi}(t)\overline{C}(t)\widehat{C}(t)dw(t), \\ \overline{\Phi}(0) &= I_{(L+1)n\times(L+1)n}. \end{cases} \tag{34}$$

We remark that the first necessary condition (31) is about the diffusion term, and the second condition (32) is, in some sense, about the controllability of the "expectation system." It is worth mentioning that for deterministic GBCS, the second condition is not only necessary but also sufficient for total controllability [44].

THEOREM 3.10. *Assume that Assumptions* 3.7–3.8 *hold. If the GBCS defined by* (23)–(24) *is exactly V-controllable, then for any $x^h \in L^2(\Omega, \mathscr{F}_T, P; R^h)$, there exist $x_F \in L^2(\Omega, \mathscr{F}_T, P; R^{n-h})$ and $u(\cdot) \in U$ such that the following BSDE has a solution:*

$$\begin{cases} dX(t) = \Big(\overline{A}(t)X(t) + \overline{B}(t)u(t) + \overline{C}(t)q(t)\Big)dt \\ \qquad\qquad + \Big(\overline{A}_1(t)X(t) + \overline{B}_1(t)u(t) + \overline{C}_1(t)q(t)\Big)dw(t), \\ x(T) = x_T,\ p(T) = Q_T x(T), \end{cases} \tag{35}$$

where $x_T = \begin{bmatrix} x^h \\ x_F \end{bmatrix} \in L^2(\Omega, \mathscr{F}_T, P; R^n)$*. Moreover, if the BSDE* (35) *has a solution for any* $x_T \in L^2(\Omega, \mathscr{F}_T, P; R^n)$ *and any* $u(\cdot) \in U$*, then the stochastic GBCS* (23)–(24) *is exactly* V*-controllable if and only if the matrix*

$$\left[\begin{bmatrix} 0 \\ I_{nL} \end{bmatrix}, \overline{\Phi}_T \overline{Q}_T \begin{bmatrix} 0_{h\times(n-h)} \\ I_{n-h} \end{bmatrix}, M(T) \right], \tag{36}$$

which is the same as the matrix in Theorem 3.9*, is of full rank.*

Note that by the definition of $\overline{B}_1(t)$ and $\overline{C}_1(t)$, if $\operatorname{rank}(D(t)) = n$ for any $t \in [0, T]$, then we can deduce that $m \geq n$, and the matrix $\widetilde{B}_1(t) = [\overline{B}_1(t), \overline{C}_1(t)]$ is of full rank for any $t \in [0, T]$. In this case, we can find an invertible $(Ln+m) \times (Ln+m)$-matrix $H(t)$ such that $\widetilde{B}_1(t)H(t) = [I_{(L+1)n}, 0]$, and a matrix $K(t)$ such that $\widetilde{B}_1(t)K(t) = -\overline{A}_1(t)$.

We now introduce the transition matrix $\Phi(t)$ defined by

$$\begin{cases} d\Phi(t) &= -\Phi(t)\widehat{A}(t)dt - \Phi(t)\widehat{A}_1(t)dw(t), \\ \Phi(0) &= I_{(L+1)n\times(L+1)n}, \end{cases} \tag{37}$$

together with the controllability matrix $M(T)$ defined by

$$M(T) = E\int_0^T \Phi(t)\widehat{B}(t)\widehat{B}(t)^T\Phi^T(t)\,dt \in R^{(L+1)n\times(L+1)n}, \tag{38}$$

where

$$\begin{aligned} \widehat{A}(t) &= \overline{A}(t) + \widetilde{B}(t)K(t) \in R^{(L+1)n\times(L+1)n}, \\ \widehat{A}_1(t) &= \widetilde{B}(t)H(t)\begin{bmatrix} I_{(L+1)n} \\ 0 \end{bmatrix} \in R^{(L+1)n\times(L+1)n}, \\ \widehat{B}(t) &= \widetilde{B}(t)H(t)\begin{bmatrix} 0 \\ I_{m-n} \end{bmatrix} \in R^{(L+1)n\times(m-n)}, \\ \widetilde{B}(t) &= [\overline{B}(t), \overline{C}(t)], \quad \widetilde{B}_1(t) = [\overline{B}_1(t),\ \overline{C}_1(t)]. \end{aligned} \tag{39}$$

THEOREM 3.11. *Under Assumptions* 3.7–3.8 *with* $\operatorname{rank}(D(t)) = n$ *for any* $t \in [0, T]$*, the stochastic GBCS defined by* (23)–(24) *is totally controllable if and only if the following matrix is of full rank:*

$$\left[\begin{bmatrix} 0 \\ I_{nL} \end{bmatrix},\ M(T) \right]. \tag{40}$$

The condition $\operatorname{rank}(D(t)) = n$ for any $t \in [0, T]$ in Theorem 3.11 seems to be restricted. It means that the dimension of the input of the regulator is greater than or equal to the dimension of the system state. But for some special case, we can show that this condition is necessary for the controllability of the GBCS.

Assumption 3.12. The matrix $D(t)$ is time-invariant, i.e., $D(t) \equiv D$.

THEOREM 3.13. *Suppose that Assumptions* 3.7, 3.8*, and* 3.12 *hold. Then, the stochastic GBCS defined by* (23)–(24) *is totally controllable if and only if the following two conditions hold:*

1. $\text{rank}(D) = n$.
2. *The following matrix is of full rank:*

$$\left[\begin{bmatrix} 0 \\ I_{nL} \end{bmatrix},\ M(T)\right].$$

We remark that we only consider the energy-finite control of the regulator; i.e., the admissible control of the regulator belongs to L^2. In this case, the condition $\text{rank}(D) = n$ is necessary, but it does not hold if we take the regulator's control in L^1 space [19].

3.3. Time-invariant linear-quadratic system. When the GBCS (23)–(24) is time-invariant, i.e., the matrices $A(t)$, $B(t)$, $C(t)$, $D(t)$, $B_i(t)$, $D_i(t)$, $Q_i(t)$, $R_i(t)$ $(i = 1, 2, \dots, L)$ are independent of time t, we can denote them by $A, B, C, D, B_i, D_i, Q_i, R_i$ $(i = 1, 2, \dots, L)$, respectively, and get a simpler and more explicit criterion for controllability.

THEOREM 3.14. *Assume that the stochastic GBCS defined by* (23)–(24) *is time-invariant and Assumptions* 3.7–3.8 *hold. Then, the GBCS is totally controllable if and only if the following rank conditions hold:*

1. $\text{rank}(D) = n$,
2. $\text{rank}(Q_C) = n$,

where

$$Q_C = [I_n,\ 0]\left[\widehat{B},\ \widehat{A}\widehat{B},\ \widehat{A}_1\widehat{B},\ \widehat{A}\widehat{A}_1\widehat{B},\ \widehat{A}_1\widehat{A}\widehat{B},\ \dots\right],$$

in which the matrices $\widehat{A}$, $\widehat{A}_1$, and $\widehat{B}$ are time-invariant matrices corresponding to the matrices $\widehat{A}(t)$, $\widehat{A}_1(t)$, and $\widehat{B}(t)$ defined in (39).

The proof of this theorem is deferred to section 4.

We remark that the computation of the rank of the seemingly infinite many-column matrix Q_C in the above theorem can be completed within finite steps, because the matrices involved are finite dimensional.

Let us now illustrate the main results of Theorem 3.14 by a numerical example. Consider Example 2.3 in section 2.3. For simplicity, we assume that there are two authorities, $x(t) \in R^2$, and the matrices in (11) and (12) have the following forms:

$$\begin{aligned}
A &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \overline{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\\
B_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \overline{B}_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix},\ \overline{B}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix},\\
Q_1 &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ Q_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ R_1 = 1,\ R_2 = 1,\\
C &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},\ \overline{C} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.
\end{aligned}$$

We can calculate the corresponding matrices defined in (39) as

$$\widehat{A} = \begin{bmatrix} 2 & 0 & 5 & 0 & 0 & 0 \\ 1 & 2 & 4 & 0 & 0 & 5 \\ -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 \end{bmatrix}, \ \widehat{A}_1 = \begin{bmatrix} 1 & 1 & & & & \\ 1 & 1 & & & & \\ & & -1 & 0 & & \\ & & 0 & -1 & & \\ & & & & -1 & 0 \\ & & & & 0 & -1 \end{bmatrix},$$
$$\widehat{B} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$

where the values of the blank positions in the matrices are 0. Because

$$\widehat{A}_1 \widehat{B} = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$

we can get $\mathrm{rank}(Q_C) = 2$. By Theorem 3.14, we know that the GBCS is totally controllable.

We remark that here we have just used some fictitious datum to illustrate the verification of the controllability of GBCS. It would be interesting to use the concept of structure controllability [18] to simplify the verification when the values of the system matrices are not exactly known, which is a topic for future investigation.

4. Proofs of the theorems. In this section, we present the main proofs of the theorems; auxiliary material is given in the appendix.

To start, let us temporarily assume that for any $u(t) (t \in [0, T])$ and any initial states $x_0, x_{i,0}$ $(i = 1, 2, \ldots, L)$, the open-loop Nash equilibrium $(u_1^*(\cdot), u_2^*(\cdot), \ldots, u_L^*(\cdot))$ exists and is unique, so $u_i^*(\cdot)$ is the unique solution of the following stochastic linear-quadratic optimal control problem for any $i \in \{1, \ldots, L\}$:

$$\text{(41)} \quad \begin{cases} dx(t) = \left(A(t)x(t) + \sum_{j \neq i} B_j(t)u_j^*(t) + B_i(t)u_i(t) + B(t)u(t) \right) dt \\ \qquad\qquad + \left(C(t)x(t) + \sum_{j \neq i} D_j(t)u_j^*(t) + D_i(t)u_i(t) + D(t)u(t) \right) dw(t), \\ x(0) = x_0, \end{cases}$$

and the payoff function to be minimized by $u_i(\cdot)$ is

$$\text{(42)} \quad \begin{aligned} &J_i(u_1(\cdot), u_2(\cdot), \ldots, u_L(\cdot)) \\ &\qquad = \frac{1}{2} E \int_0^T \left[x^T(t)Q_i(t)x(t) + u_i^T(t)R_i(t)u_i(t) \right] dt + Ex^T(T)Q_{iT}x(T). \end{aligned}$$

According to the maximum principle, the control $u_i^*(\cdot)$ satisfies

$$\text{(43)} \quad \begin{cases} u_i^*(t) = R_i^{-1}(t)\left(B_i^T(t)p_i(t) + D_i^T(t)q_i(t) \right), \\ dx(t) = \left(A(t)x(t) + \sum_{j=1}^L B_j(t)u_j^*(t) + B(t)u(t) \right) dt \\ \qquad\qquad + \left(C(t)x(t) + \sum_{j=1}^L D_j(t)u_j^*(t) + D(t)u(t) \right) dw(t), \\ dp(t) = -\left(Q_i(t)x(t) - A^T(t)p_i(t) - C^T(t)q_i(t) \right) dt + q_i(t)dw(t), \\ x(0) = x_0, p_i(T) = -Q_{iT}x(T), \ i = 1, 2, \ldots, L. \end{cases}$$

Using the notation defined by (25) and (26), we can rewrite (43) as follows:

$$\begin{cases} dx(t) = \Big(A(t)x(t) + S(t)p(t) + M(t)q(t) + B(t)u(t)\Big)dt \\ \qquad\qquad + \Big(C(t)x(t) + T(t)p(t) + N(t)q(t) + D(t)u(t)\Big)dw(t), \\ dp(t) = \Big(Q(t)x(t) - I_L \otimes A^T(t)p(t) - I_L \otimes C^T(t)q(t)\Big)dt + q(t)dw(t), \\ x(0) = x_0,\ p(T) = Q_T x(T). \end{cases} \tag{44}$$

We note that in (44), the terminal condition of process $p(t)$ rather than the initial condition is given (this is a BSDE), and (44) is a coupled FBSDE. For more details about BSDE, we refer the reader to the original paper by Pardoux and Peng [29]. Such FBSDEs have been studied by many researchers; see, e.g., [42, 43]. Equation (44) can also be rewritten more compactly as (27).

4.1. Proof of Theorem 3.9. To prove Theorem 3.9, we first give some lemmas and claims.

LEMMA 4.1. *If Assumption* 3.8 *holds, then for any given input* $u(t)$ $(t \in [0,T])$ *of the regulator and any initial state* x_0*, the noncooperative differential game defined by* (23)–(24) *admits an open-loop Nash equilibrium if and only if the FBSDE* (44) *has a solution. Moreover, if both Assumption* 3.7 *and Assumption* 3.8 *hold, then the solution is unique.*

Proof. See Appendix A. ∎

The following lemma can be found in [32].

LEMMA 4.2. *It is impossible to find* $(a, b) \in L^2_{\mathscr{F}}(0,T;R) \times L^2_{\mathscr{F}}(0,T;R)$ *and* $x \in R$ *with*

$$\lim_{t\to T} E\|b(t) - b(T)\|^2 = 0$$

such that

$$\zeta = x + \int_0^T a(s)ds + \int_0^T b(s)dw(s),$$

where

$$\zeta = \int_0^T \varphi(s)dw(s),$$

$$\varphi(t) = \begin{cases} +1 & \textit{when } t \in [(1-2^{-2i})T, (1-2^{-2i-1})T),\ i = 0, 1, \ldots, \\ -1 & \textit{otherwise.} \end{cases}$$

Consider the BSDE

$$\begin{cases} dx(t) = b(x(t), v(t), t)dt + \sigma(x(t), v(t), t)dw(t), \\ x(T) = x_T. \end{cases} \tag{45}$$

We assume that (45) satisfies the following conditions: For any $(x, v) \in R^n \times R^m$,

$$\begin{aligned} &b(x, v, \cdot) \in L^2_{\mathscr{F}}(0,T;R^n), \\ &\sigma(x, v, \cdot) \in L^2_{\mathscr{F}}(0,T;R^n), \\ &\lim_{t\to T} E\|\sigma(x, v, t) - \sigma(x, v, T)\|^2 = 0, \end{aligned}$$

the functions $b(x, v, t)$, $\sigma(x, v, t)$ satisfy the linear growth with respect to (x, v) uniformly in $t \in [0, T]$, and $\sigma(x, v, t)$ is Lipschitzian with respect to x uniformly in (v, t).

Similarly to [32], we can define the V-E-well-posedness of (45), where V is a linear subspace of space $L^2(\Omega, \mathscr{F}_T, P; R^n)$.

Definition 4.3. *The system* (45) *is called V-E-well-posed if for any terminal state $x^h(T) = x^h_T = \xi \in V$, there exists at least one $v(t) \in L^2_{\mathscr{F}}(0, T; R^m)$, under which the solution $x(t)$ of* (45) *satisfies the condition $P_h x(T) = x^h_T$.*

In Definition 4.3, we only require that the terminal state satisfies some conditions, but if we want to know which initial states can get to the desired terminal state, then we should consider a related definition of exactly controllable.

Definition 4.4. *The system* (45) *is called exactly W-V-controllable if for any initial state $x(0) = x_0 \in W \subseteq R^n$ and any terminal state $x^h(T) = x^h_T = \xi \in V$, there exists at least one $v(t) \in L^2_{\mathscr{F}}(0, T; R^m)$, under which the solution $x(t)$ of* (45) *satisfies the condition $x(0) = x_0$ and $P_h x(T) = x^h_T$.*

If we let $v(t) = [u^T(t), q^T(t)]^T$, then (46) can be transformed into the form of (45) as follows:

$$\begin{cases} dX(t) = \Big(\overline{A}(t)X(t) + \widetilde{B}(t)v(t)\Big)dt \\ \qquad\qquad + \Big(\overline{A}_1(t)X(t) + \widetilde{B}_1(t)v(t)\Big)dw(t), \\ x(0) = x_0, \ p(T) = Q_T x(T), \end{cases} \tag{46}$$

where

$$\widetilde{B}(t) = \left[\overline{B}(t), \overline{C}(t)\right], \ \widetilde{B}_1(t) = \left[\overline{B}_1(t), \overline{C}_1(t)\right].$$

From the definitions, we see that the GBCS (23)–(24) is exactly V-controllable if and only if there exists a subspace W of $R^{(L+1)n}$ such that $P_n(W) = R^n$, (46) is exactly W-V-controllable, and the terminal condition $p(T) = Q_T x(T)$ is satisfied. It means that (46) is V-E-well-posed.

Claim 4.5. *Let V be the space $L^2(\Omega, \mathscr{F}_T, P; R^h)$ $(0 < h \leq n)$. A necessary condition for the V-E-well-posedness of* (45) *is that for any $a \in R^m$, $b \in \overline{V} = \{v \in R^n : \|v\| = 1, v_i = 0, h < i \leq n\}$, there exists $(x, v) \in R^n \times R^m$ such that*

$$b^T(\sigma(x, v, t) - \sigma(x, a, t)) \not\equiv 0 \quad \textit{for almost all } t \in [0, T]. \tag{47}$$

Proof. We know that any solution $(x(t), v(t))$ of (46) satisfies

$$\begin{aligned} &\lim_{t \to T} E\|\sigma(x(t), a, t) - \sigma(x(T), a, T)\|^2 = 0, \\ &b(x(\cdot), v(\cdot), \cdot) \in L^2_{\mathscr{F}}(0, T; R^n), \\ &\sigma(x(\cdot), v(\cdot), \cdot) \in L^2_{\mathscr{F}}(0, T; R^n). \end{aligned}$$

If condition (47) is false, then we can find $a \in R^m$, $b \in \overline{V}$ such that for any (x, v),

$$b^T(\sigma(x, v, t) - \sigma(x, a, t)) = 0.$$

Let $\hat{x}_T = \xi = \zeta b$ (ζ is defined in Lemma 4.2). Because (46) is V-E-well-posed, there exists a vector $x_F \in L^2(\Omega, \mathscr{F}_T, P; R^{n-h})$ such that for the terminal state

$$x_T = \hat{x}_T + \hat{x}_F = \hat{x}_T + \begin{bmatrix} 0 \\ x_F \end{bmatrix} = \zeta \begin{bmatrix} b_1 \\ \vdots \\ b_h \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_F \end{bmatrix} = \begin{bmatrix} \zeta b_1 \\ \vdots \\ \zeta b_h \\ x_F \end{bmatrix} \in L^2(\Omega, \mathscr{F}_T, P; R^n),$$

(46) admits a solution $(x(t), v(t))$, i.e.,

$$x_T = \hat{x}_T + \hat{x}_F = x_0 + \int_0^T b(x(s), v(s), s)ds + \int_0^T \sigma(x(s), v(s), s)dw(s),$$

which yields

$$\begin{aligned} \zeta &= \zeta b^T b = \zeta b^T b + b^T \hat{x}_F = b^T(\zeta b + \hat{x}_F) = b^T x_T \\ &= b^T x_0 + \int_0^T b^T b(x(s), v(s), s)ds + \int_0^T b^T \sigma(x(s), v(s), s)dw(s) \\ &= b^T x_0 + \int_0^T b^T b(x(s), v(s), s)ds + \int_0^T b^T \sigma(x(s), a, s)dw(s) \\ &= b^T x_0 + \int_0^T \overline{a}(s)ds + \int_0^T \overline{b}(s)dw(s). \end{aligned}$$

This contradicts Lemma 4.2, so the proof is complete. ∎

Similarly to [32], we consider the special case

$$\sigma(x, v, t) = \sigma_1(x, t) + G_1(t)v,$$

where G_1 is an $n \times m$ time-variant matrix and $\sigma_1(x, t)$ is uniformly Lipschitzian with respect to x. We can give a simple necessary condition for the V-E-well-posedness of (46) for this special case.

CLAIM 4.6. *A necessary condition for the V-E-well-posedness of* (46) *is that for any zero-measure set $I \subseteq [0, T]$, we have*

$$\overline{V} \subseteq Span\left\{ \bigcup_{t \in [0,T]-I} \mathrm{Im}(G_1(t)) \right\}, \tag{48}$$

where $\overline{V} = \{v \in R^n : \|v\| = 1, v_i = 0, h < i \leq n\}$.

Proof. We know that any solution $(x(t), v(t))$ of (46) satisfies

$$\begin{aligned} &\lim_{t \to T} E\|\sigma(x(t), a, t) - \sigma(x(T), a, T)\|^2 = 0, \\ &b(x(\cdot), v(\cdot), \cdot) \in L^2_{\mathscr{F}}(0, T; R^n), \\ &\sigma(x(\cdot), v(\cdot), \cdot) \in L^2_{\mathscr{F}}(0, T; R^n). \end{aligned}$$

If condition (48) is false, then there exists a nonzero vector $b \in \overline{V}$ such that $b^T G_1(t) = 0$ for almost all $t \in [0, T]$, so for any pair (x, v), we have

$$b^T(\sigma(x, v, t) - \sigma(x, a, t)) = b^T(G_1(t)(v - a)) = 0.$$

This contradicts Claim 4.5. The proof is complete. ∎

CLAIM 4.7. *If we define the linear space*

$$W = \left\{ E\int_0^T \Phi(t)B(t)u(t)\,dt \ : \ u(t)(t\in[0,T]) \ \in L^2_{\mathscr{F}}(0,T;R^m) \right\} \tag{49}$$

and matrix

$$M = E\int_0^T \Phi(s)B(s)B^T(s)\Phi^T(s)\,ds, \tag{50}$$

where $\Phi(\cdot)\in L^2_{\mathscr{F}}(0,T;R^{n\times n})$ *and* $B(\cdot)\in L^2_{\mathscr{F}}(0,T;R^{n\times m})$, *then we have*

$$\mathrm{Im}(M) \ = \ W.$$

Proof. The proof is divided into two steps.

(1) First, we show that $\mathrm{Im}(M)\subseteq W$.

Let $\{e_i, i=1,2,\ldots,n\}$ be a basis of the space R^n. Then

$$\begin{aligned} Me_i &= E\int_0^T \Phi(t)B(t)B^T(t)\Phi^T(t)\,dte_i \ = \ E\int_0^T \Phi(t)B(t)B^T(t)\Phi^T(t)e_i\,dt \\ &= E\int_0^T \Phi(t)B(t)u(t)\,dt, \end{aligned}$$

where we have taken $u(t)=B^T(t)\Phi^T(t)e_i\in L^2_{\mathscr{F}}(0,T;R^m)$. Thus, we have $Me_i\in W$ for any $i=1,2,\ldots,n$, and $\mathrm{Im}(M)\subseteq W$ holds.

(2) Second, we prove that $\mathrm{Im}(M)=W$.

We use a contradiction argument and assume that this does not hold, i.e., $\mathrm{Im}(M)\subsetneq W$. Then, there is a nonzero vector $0\neq z\in W$ and $z^TMz=0$, so we have

$$0 = z^TE\int_0^T \Phi(t)B(t)B^T(t)\Phi^T(t)\,dtz \ = \ E\int_0^T \|z^T\Phi(t)B(t)\|^2\,dt,$$

which means that

$$z^T\Phi(t)B(t)=0 \quad \text{a.e. } 0\leqslant t\leqslant T.$$

Because of $z\in W$, there is $u(t)$ such that $z=E\int_0^T\Phi(t)B(t)u(t)\,dt$, so we have

$$\|z\|^2 = \ z^Tz = \ z^TE\int_0^T \Phi(t)B(t)u(t)\,dt = E\int_0^T z^T\Phi(t)B(t)u(t)\,dt = 0,$$

which contradicts $z\neq 0$, and so the proof is complete. ∎

Now, we give the proof of Theorem 3.9.

Proof. (1) *Proof of the first condition.*We know that any solution $(x(t),v(t))$ of (46) satisfies

$$\begin{aligned} &\lim_{t\to T} E\|\sigma(x(t),a,t)-\sigma(x(T),a,T)\| = 0, \\ &b(x(\cdot),v(\cdot),\cdot)\in L^2_{\mathscr{F}}(0,T;R^n), \\ &\sigma(x(\cdot),v(\cdot),\cdot)\in L^2_{\mathscr{F}}(0,T;R^n). \end{aligned}$$

Because the V-E-well-posed condition of (46) is necessary for the exactly V-controllability of GBCS (23)–(24), from Claim 4.6, we have for any zero-measure set $N \subseteq [0,T]$,

$$\overline{V}_{(L+1)n} \subseteq Span\left\{ \bigcup_{t\in[0,T]-N} \mathrm{Im}\left(\begin{bmatrix} D(t) & N(t) \\ 0 & I_{Ln} \end{bmatrix} \right) \right\},$$

where $\overline{V}_{(L+1)n} = \{v \in R^{(L+1)n} : \|v\| = 1, v_i = 0, h < i \le (L+1)n\}$. The last $(L+1)n$ components of any vector in $\overline{V}_{(L+1)n}$ are zero, so

$$\overline{V} \subseteq Span\left\{ \bigcup_{t\in[0,T]-N} \mathrm{Im}(D(t)) \right\},$$

where $\overline{V} = \{v \in R^n : \|v\| = 1, v_i = 0, h < i \le n\}$. The proof is complete.

(2) *Proof of the second condition.* Applying the Ito formula to $\overline{\Phi}(t)X(t)$, where $\overline{\Phi}(t)$ and $X(t)$ are defined by (34) and (27), respectively, we have

$$\begin{aligned}
&\overline{\Phi}(T)X(T)\\
&\quad = X(0) + \int_0^T \overline{\Phi}(t)dX(t) + \int_0^T (d\overline{\Phi}(t))X(t)\\
&\qquad + \int_0^T -\overline{\Phi}(t)\overline{C}(t)\widehat{C}(t)\Big(\overline{A}_1(t)X(t) + \overline{B}_1(t)u(t) + \overline{C}_1(t)q(t)\Big)dt\\
&\quad = X(0) + \int_0^T \overline{\Phi}(t)\Big(\overline{A}(t)X(t) + \overline{B}(t)u(t) + \overline{C}(t)q(t)\Big)dt\\
&\qquad + \int_0^T \overline{\Phi}(t)\Big(\overline{A}_1(t)X(t) + \overline{B}_1(t)u(t) + \overline{C}_1(t)q(t)\Big)dw(t)\\
&\quad - \int_0^T \overline{\Phi}(t)\overline{A}(t)X(t)dt \; - \int_0^T \overline{\Phi}(t)\overline{C}(t)\widehat{C}(t)X(t)dw(t)\\
&\quad - \int_0^T \overline{\Phi}(t)\overline{C}(t)\widehat{C}(t)\Big(\overline{A}_1(t)X(t) + \overline{B}_1(t)u(t) + \overline{C}_1(t)q(t)\Big)dt.
\end{aligned}$$

By simple algebraic manipulations and using the relations $\widehat{C}(t)\overline{C}_1(t) = I_{Ln}$, $\overline{C}(t)\ \widehat{C}(t)\ \overline{B}_1(t) = 0$, and $\overline{C}(t)\widehat{C}(t)\overline{A}_1(t) = 0$, we can get

$$\begin{aligned}
\Phi(T)X(T) = X(0) &+ \int_0^T \overline{\Phi}(t)\overline{B}(t)u(t)dt\\
&+ \int_0^T \overline{\Phi}(t)\Big(\overline{A}_1(t)X(t) + \overline{B}_1(t)u(t) + \overline{C}_1(t)q(t)\Big)dw(t)\\
&- \int_0^T \overline{\Phi}(t)\overline{C}(t)\widehat{C}(t)X(t)dw(t).
\end{aligned}$$

Taking expectation on both sides of the above relation yields

$$E\overline{\Phi}(T)X(T) = EX(0) + E\int_0^T \overline{\Phi}(t)\overline{B}(t)u(t)dt. \tag{51}$$

If the GBCS (23)–(24) is exactly V-controllable, then for any initial state x_0, there is an input $u(t)$, under which the solution of (27) exists and $P_h(x(T)) = 0$; i.e., for any

x_0, there are $p(0) \in R^{Ln}$, $u(t)$ such that

$$E\overline{\Phi}(T)\overline{Q}_T \begin{bmatrix} 0_{h\times 1} \\ x_F \end{bmatrix} = E \begin{bmatrix} x_0 \\ p_0 \end{bmatrix} + E\int_0^T \overline{\Phi}(t)\overline{B}(t)u(t)dt,$$

where $x_F \in L^2(\Omega, \mathscr{F}_T, P; R^{n-h})$ is some vector. By simple algebraic manipulations, we get that for any $x_0 \in R^n$, there are a vector p_0 and a control $u(\cdot)$ such that

$$\begin{bmatrix} x_0 \\ 0 \end{bmatrix} = E \begin{bmatrix} 0 \\ p_0 \end{bmatrix} + E\int_0^T \overline{\Phi}(t)\overline{B}(t)u(t)dt + E\overline{\Phi}(T)\overline{Q}_T \begin{bmatrix} 0_{h\times 1} \\ x_F \end{bmatrix}.$$

This leads to

$$\mathrm{Im}\left(\begin{bmatrix} I_n \\ 0 \end{bmatrix} \right) \subseteq \mathrm{Im}\left(\left[\begin{bmatrix} 0 \\ I_{Ln} \end{bmatrix},\ \overline{W},\ \overline{\Phi}_T\overline{Q}_T \begin{bmatrix} 0 \\ I_{n-h} \end{bmatrix} \right] \right),$$

where $\overline{W}$ is a basis matrix of linear subspace W. Thus, the matrix

$$\left[\begin{bmatrix} 0 \\ I_{Ln} \end{bmatrix},\ \overline{W},\ \overline{\Phi}_T\overline{Q}_T \begin{bmatrix} 0 \\ I_{n-h} \end{bmatrix} \right]$$

is of full rank. By Claim 4.7, we know that the following matrix is also full rank:

$$\left[\begin{bmatrix} 0 \\ I_{Ln} \end{bmatrix},\ M(T),\ \overline{\Phi}_T\overline{Q}_T \begin{bmatrix} 0 \\ I_{n-h} \end{bmatrix} \right],$$

so the proof is complete. ∎

4.2. Proof of Theorem 3.10.

Proof. The first part of this theorem can be derived directly from the definition of V-controllability. Now we only need to prove the second part.

By Lemma 4.1, we know that for any input $u(\cdot)$, the existence of a unique Nash equilibrium is equivalent to the existence of a unique solution of the corresponding FBSDE (44). Then, by Theorem 3.6, we only need to study the exact V-controllability of FBSDE (44).

The necessity is just the result of Theorem 3.9, so we only need to prove the sufficiency.

For any initial state $x_0 \in R^n$ and terminal state $x^h \in L^2_{\mathscr{F}_T}(R^h)$, by the existence of the solution of BSDE (35), we know that there exist $x_F^{x^h} \in L^2(\Omega, \mathscr{F}_T, P; R^{n-h})$ and $u^{x^h}(\cdot) \in U$ such that BSDE (35) admits a solution $(X^{x^h}(\cdot), q^{x^h}(\cdot))$, and because of Assumption 3.7 and Assumption 3.8, the solution is unique.

If the matrix (36) is of full rank and BSDE (35) admits a solution for any terminal state, then for any given initial state $x \in R^n$, there exists an input $u^{(x)}(\cdot) \in U$ such that FBSDE (44) admits a unique solution for $x(0) = x$ and $x(T) = \begin{bmatrix} 0 \\ x^0 \end{bmatrix}$ for some $x^0 \in L^2(\Omega, \mathscr{F}_T, P; R^{n-h})$.

If we let $u(\cdot) = u^{(x_0 - x^{x^h}(0))} + u^{x^h}$, where $x^{x^h}(0)$ are the first n components of $X^{x^h}(0)$, then we can easily verify that, under this input, the FBSDE (44) admits a unique solution for $x(0) = x_0$ and $x(T) = \begin{bmatrix} x^h \\ \overline{x}_F \end{bmatrix}$, so the GBCS is V-controllable. ∎

4.3. Proof of Theorem 3.11.

Proof. Under the assumptions of Theorem 3.11, we can use the linear transformation

$$v(t) = H(t)\begin{bmatrix} Z(t) \\ U(t) \end{bmatrix} + K(t)X(t) \tag{52}$$

to transform (46) into the following equivalent form:

$$\begin{cases} dX(t) = \Big(\widehat{A}(t)X(t) + \widehat{A}_1(t)Z(t) + \widehat{B}(t)U(t)\Big)dt + Z(t)dw(t), \\ \quad x(0) = x_0,\ p(T) = Q_T x(T). \end{cases} \tag{53}$$

The equivalence between (46) and (53) is that if there is a pair $(Z(\cdot), U(\cdot))$ such that (53) has a solution $X(\cdot)$, then we can construct a $v(t)$ such that (46) has the same solution $X(\cdot)$ and vice versa.

The sufficiency and necessity will be proved separately.

(1) *Necessity.* Applying the Ito formula to $\Phi(t)X(t)$, where $\Phi(t)$ and $X(t)$ are defined by (37) and (53), respectively, we have

$$\begin{aligned} &\Phi(T)X(T) \\ &\quad = X(0) + \int_0^T \Phi(t)dX(t) + \int_0^T (d\Phi(t))X(t) \ + \int_0^T -\Phi(t)\widehat{A}_1(t)Z(t)dt \\ &\quad = X(0) + \int_0^T \Phi(t)\Big(\widehat{A}(t)X(t) + \widehat{A}_1(t)Z(t) + \widehat{B}(t)U(t)\Big)dt \ + \int_0^T \Phi(t)Z(t)dw(t) \\ &\qquad - \int_0^T \Phi(t)\widehat{A}(t)X(t)dt \ - \int_0^T \Phi(t)\widehat{A}_1(t)X(t)dw(t) \ - \int_0^T \Phi(t)\widehat{A}_1(t)Z(t)dt \\ &\quad = X(0) + \int_0^T \Phi(t)\widehat{B}(t)U(t)dt + \int_0^T \Phi(t)\Big(Z(t) - \widehat{A}_1(t)X(t)\Big)dw(t). \end{aligned}$$

Taking expectation on both sides of the above relation yields

$$E\Phi(T)X(T) = EX(0) + E\int_0^T \Phi(t)\widehat{B}(t)U(t)dt. \tag{54}$$

Consider the following BSDE:

$$\begin{cases} dX(t) = (\widehat{A}(t)X(t) + \widehat{A}_1(t)Z(t) + \widehat{B}(t)U(t))dt + Z(t)dw(t), \\ X(T) = 0, \end{cases} \tag{55}$$

where $x \in L^2_{\mathscr{F}_T}(R^n)$. According to [32], for each $U(t) \in L^2_{\mathscr{F}}(0,T;R^m)$, there exists a unique pair $(X(\cdot), Z(\cdot)) \in L^2_{\mathscr{F}}(0,T;R^{(2L+1)n})$ satisfying the above BSDE. We denote this pair by $(X^{U,x}(\cdot), Z^{U,x}(\cdot))$.

Define the spaces

$$\begin{aligned} S_0 &= \{X^{U,0}(0) :\ U(t) \in L^2_{\mathscr{F}}(0,T;R^m)\}, \\ S_{x_0} &= \{x^{U,0}(0) :\ X^{U,0}(0) \in S_0\}, \end{aligned} \tag{56}$$

where $x^{U,0}(0)$ are the first n components of $X^{U,0}(0)$. The space S_{x_0} consists of the initial states which can be controlled to zero; i.e., if the initial state of the GBCS

(23)–(24) is $x_0 \in S_{x_0}$, then there is an input of the regulator, under which the state of the system will reach 0 at time T.

If the GBCS (23)–(24) is totally controllable, then for any initial state x_0, there is an input $U(t)$, under which the solution of (55) exists and $x(0) = x_0$, where $x(0)$ are the first n components of $X(0)$, which means that $S_{x_0} = R^n$.

If we can prove that $S_0 = \mathrm{Im}(M(T))$, then $S_{x_0} = R^n$ can imply the full rank of the matrix in Theorem 3.11, and so the proof of necessity is completed.

Now we prove the equality $S_0 = \mathrm{Im}(M(T))$.

Upon substituting the terminal state $X(T) = 0$ into (54), we get

$$X(0) = -E\int_0^T \Phi(t)\widehat{B}(t)U(t)dt. \tag{57}$$

By Claim 4.7, we obtain $S_0 = \mathrm{Im}(M(T))$.

(2) *Sufficiency.* If the matrix in Theorem 3.11 is of full rank, then $S_{x_0} = R^n$; i.e., for any initial state x_0, there is an input $U(t)$, under which the solution of (55) exists and $x(0) = x_0$. Denote the corresponding regulator input $U^{x_0}(\cdot)$

For the given terminal state $x_T \in L^2_{\mathscr{F}_T}(R^n)$, we know that there is a unique pair $(X^{X_T}(\cdot), Z^{X_T}(\cdot))$, which solves the following BSDE:

$$\begin{cases} dX(t) = (\widehat{A}(t)X(t) + \widehat{A}_1(t)Z(t))dt + Z(t)dw(t), \\ \ x(T) = x_T,\ p(T) = Q_T x(T). \end{cases} \tag{58}$$

For the given initial state $x_0 \in R^n$, let the input of the regulator be $U^{x_0 - x^{x_T}(0)}(\cdot)$; then (53) is solvable and $x(T) = x_T$.

The above analysis shows that under the conditions of Theorem 3.11, for any given initial state $x_0 \in R^n$ and terminal state $x_T \in L^2_{\mathscr{F}_T}(R^n)$, we can construct an input of the regulator, under which the solution of (53) is solvable and $x(T) = x_T$. This is just the definition of total controllability of GBCS, and so the proof of sufficiency is complete. ∎

4.4. Proof of Theorem 3.13. From Theorem 3.11, we know that the condition of Theorem 3.13 is sufficient. We only need to prove that the condition $\mathrm{rank}(D) = n$ is necessary, which can be obtained from Theorem 3.9, and so the proof is complete. ∎

4.5. Proof of Theorem 3.14. To prove Theorem 3.14, we need the following claim, which can be found in [32].

CLAIM 4.8 ([32]). *Consider the following BSDE:*

$$\begin{cases} -dx(t) = (Ax(t) + A_1 z(t) + Bu(t))dt - z(t)dw(t), \\ \ x(T) = 0. \end{cases} \tag{59}$$

For each $u(\cdot) \in L^2_{\mathscr{F}}(0,T;R^m)$, there exists a unique pair $(x(\cdot), z(\cdot)) \in L^2_{\mathscr{F}}(0,T;R^{2n})$ satisfying the above BSDE. We denote this pair by $(x^u(\cdot), z^u(\cdot))$. Then we have the following relation:

$$\{x_0^u : u(\cdot) \in L^2_{\mathscr{F}}(0,T;R^m)\} = \mathrm{Im}([B, AB, A_1B, AA_1B, A_1AB, \ldots]). \tag{60}$$

Proof. From Theorem 3.13, we know that if we have the relation $\mathrm{Im}(M(T)) = \mathrm{Im}(Q_C)$, then the theorem holds. In fact, this relation can be deduced from (55)–(57), the relation $S_0 = \mathrm{Im}(M(T))$ (see the proof of Theorem 3.11), and Claim 4.8. Hence, the proof is complete. ∎

5. Conclusions. In this paper, we have investigated the controllability of the Nash equilibrium of a class of stochastic game-based control systems (GBCSs). The motivation for studying GBCSs comes from rich situations in the real world, where the global-regulator can change the macrostates of the system by regulating the Nash equilibrium formed by the agents at the lower level via noncooperative differential games. This framework extends the classical framework of control theory and can be applied in the regulation of a wider class of practical complex systems. We have first introduced a general framework for the controllability of stochastic GBCSs and transformed the controllability problem of stochastic GBCSs into the controllability problem of corresponding forward-backward stochastic differential equations, and then presented some necessary and sufficient conditions for the controllability of linear-quadratic stochastic GBCSs. Compared with the controllability of classical control systems, the key difficulty in this work is that we have to analyze the controllability of the associated forward-backward stochastic differential equations, which has rarely been explored in the literature. For future investigation, it would be interesting to generalize the results of this paper to more complicated situations or to investigate other important control problems, such as robust and adaptive control.

Appendix A. Proof of Lemma 4.1. We believe that the result of the lemma is not new. Since we cannot find an exact reference, for convenience we sketch a proof here.

The necessity of the lemma is obvious. Now, we give the proof of sufficiency.

For any fixed input $u(t), t \in [0, T]$, suppose that the FBSDE (44) admits a solution $(x(\cdot), p(\cdot), q(\cdot))$; we can prove that the strategy profile

$$\left\{ u_i^*(t) = R_i^{-1}(t)\Big(B_i^T(t)p_i(t) + D_i^T(t)q_i(t)\Big),\ i = 1, \ldots, L \right\} \tag{61}$$

is a Nash equilibrium.

The fact that $u_i^*(\cdot)(i = 1, \ldots, L)$ is an open-loop admissible strategy for agent i is the consequence of the boundedness of $R_i^{-1}(t)$, $B_i(t)$, $D_i(t)$ and the fact that $p_i(\cdot) \in L^2_{\mathscr{F}}(0, T; R^n)$, $q_i(\cdot) \in L^2_{\mathscr{F}}(0, T; R^n)$.

Now, we choose any agent i, without loss of generality, assume $i = 1$, and fix the strategies of the other $L - 1$ agents as the control in (61). If we can prove that $u_1^*(\cdot)$ is the optimal control of agent 1, then (61) is a Nash equilibrium. We do this next.

Substituting $u_i^*(\cdot)$ $(i = 2, \ldots, L)$ for $u_i(\cdot)$ $(i = 2, \ldots, L)$ in systems (23) and (24), we solve the following minimization problem for agent 1:

$$\min_{u_1(\cdot) \in U_1} J_1(u_1(\cdot), u_2^*(\cdot), \ldots, u_L^*(\cdot)), \tag{62}$$

where

$$\begin{cases} dx(t) = \left(A(t)x(t) + B_1(t)u_1(t) + \displaystyle\sum_{i=2}^{L} B_i(t)u_i^*(t) + B(t)u(t)\right)dt \\ \qquad\qquad + \left(C(t)x(t) + D_1(t)u_1(t) + \displaystyle\sum_{i=2}^{L} D_i(t)u_i^*(t) + D(t)u(t)\right)dw(t), \\ x(0) = x_0. \end{cases} \tag{63}$$

If Assumption 3.8 holds, i.e., equations (29) have a set of strongly regular solutions, then the minimization problem has a unique open-loop optional control, and

the map $u_1(\cdot) \to J^0(u_1(\cdot)) = J_1^0(u_1(\cdot), u_2^*, \dots, u_L^*)$ is uniformly convex [36], where
(64)

$$J^0(u_1(\cdot)) = \frac{1}{2} E\left\{ \int_0^T \Big(x_1^T(t) Q_1(t) x_1(t) + u_1^T(t) R_1(t) u_1(t) \Big) dt + x_1^T(T) Q_{1T} x_1(T) \right\},$$

and $x_1(\cdot)$ is the solution to the following SDE:

$$\text{(65)} \qquad \begin{cases} dx_1(t) = \Big(A(t)x_1(t) + B_1(t)u_1(t) \Big) dt + \Big(C(t)x_1(t) + D_1(t)u_1(t) \Big) dw(t), \\ x_1(0) = 0, \ t \in [0,T]. \end{cases}$$

The uniform convexity implies that for any $u_1(\cdot) \in U_1$, $J^0(u_1(\cdot)) \geq 0$.

For any $\lambda \in R$ and $v(\cdot) \in U_1$, using a method similar to that in [37], we have

$$\begin{aligned} &J_1(u_1^*(\cdot) + \lambda v(\cdot), u_2^*, \dots, u_L^*) - J_1(u_1^*(\cdot), u_2^*, \dots, u_L^*) \\ &\quad = \lambda^2 J^0(v(\cdot)) \geq 0, \end{aligned}$$

so $u_1^*(\cdot)$ is the unique open-loop optimal control.

From the proof above, we know that for any solution $(x(\cdot), p(\cdot), q(\cdot))$ of FBSDE (44), we can construct a Nash equilibrium and vice versa, and so there is a 1-1 relation between the solution of (44) and the Nash equilibrium of the stochastic differential game (23)–(24) formed by the L lower level agents.

If both Assumption 3.7 and Assumption 3.8 hold, then we will prove that for any input $u(\cdot)$ of the regulator, the Nash equilibrium is unique. We use a contradiction argument and assume that this does not hold, i.e., for some input $u(\cdot)$, there are at least two different Nash equilibria; then we know that (44) admits at least two different solutions $(x^1(\cdot), p^1(\cdot), q^1(\cdot))$ and $(x^2(\cdot), p^2(\cdot), q^2(\cdot))$, and we can conclude that the tuple $(x(\cdot), p(\cdot), q(\cdot)) = (x^1(\cdot) - x^2(\cdot), p^1(\cdot) - p^2(\cdot), q^1(\cdot) - q^2(\cdot)) \neq 0$ is a solution of the following FBSDE:

$$\text{(66)} \qquad \begin{cases} dx(t) = \Big(A(t)x(t) + S(t)p(t) + M(t)q(t) \Big) dt \\ \qquad\qquad + \Big(C(t)x(t) + T(t)p(t) + N(t)q(t) \Big) dw(t), \\ dp(t) = \Big(Q(t)x(t) - I_L \otimes A^T(t)p(t) - I_L \otimes C^T(t)q(t) \Big) dt + q(t)dw(t), \\ x(0) = 0, \ p(T) = Q_T x(T). \end{cases}$$

It is obvious that $(0,0,0)$ is also a solution of FBSDE (66). We obtain that there are at least two different solutions of (66). This contradicts Assumption 3.7, and so the proof of Lemma 4.1 is completed. ∎

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