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STOCHASTIC ADAPTIVE SWITCHING CONTROL BASED ON MULTIPLE MODELS*

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Abstract. It is well known that the transient behaviors of the traditional adaptive control may be very poor in general, and that the adaptive control designed based on switching between multiple models is an intuitively appealing and practically feasible approach to improve the transient performances. In this paper, we shall prove that for a typical class of linear systems disturbed by random noises, the multiple model based least-squares (LS) adaptive switching control is stable and convergent, and has the same convergence rate as that established for the standard least-squares-based self-tunning regulators. Moreover, the mixed case combining adaptive models with fixed models is also considered.

Key words. Multiple model, switching control, least-squares, adaptive control, convergence rate.

1 Introduction

In an uncertain and complex environment, the approach of "optimal" switching is often used for making decisions through predicting and comparing the effects of multiple schemes. In the area of control, the multiple model approach, which has been used to improve estimations and control accuracies, can be traced back at least to 1960s–1970s (see, e.g., [1]–[3]). Some practical applications have also been reported[4,5]. In adaptive control, switching controller based on multiple models has also been used to reduce the dependence of the prior knowledge about the systems[6]–[8]. In [9] and [10], the use of multiple fixed models was studied. By comparing the prediction errors of the fixed models and switching based on the "certainty equivalence principle", the author defined a supervisory controller, and proved the tracking performance and robustness of the control system; however, the multiple fixed models need to be chosen with care. Recently, [11] introduced and studied the adaptive control problem of the mixed case, where adaptive models are combined with fixed models.

All the above mentioned papers deal with continuous-time systems only. Recently, [12] tried to extend the results of [11] to the discreet-time case. However, when analyzing the RLS based controller, the authors either assume the persistent excitation condition as in [13], or use essentially a stochastic gradient algorithm which has poor convergent rate in general. Furthermore, the proof of stability appears to be incomplete for the stochastic adaptive control based on mixed multiple models.

In this paper, we will consider a typical class of linear systems, and give a rigorous proof of stability and optimality for multiple-models-based minimum variance adaptive control.

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2 Problem Formulation

Consider the following linear discrte-time stochastic system:

$$A(z)y_t = B(z)u_{t-1} + C(z)w_t, \ \ t \geq 0, \tag{1}$$

where $\{y_t\}, \{u_t\}$, and $\{w_t\}$ are the system output, input and noise processes respectively. We assume that $y_t = u_t = w_t = 0$, $\forall t < 0$, $A(z), B(z)$ and $C(z)$ are polynomials in the backward-shift operator z:

$$\begin{aligned} A(z) &= 1 + a_1 z + \cdots + a_p z^p, & p \geq 1, \\ B(z) &= b_1 + b_2 z + \cdots + b_q z^{q-1}, & q \geq 1, \\ C(z) &= 1 + c_1 z + \cdots + c_r z^r, & r \geq 0, \end{aligned}$$

where a_i, $1 \leq i \leq p$; b_j, $1 \leq j \leq q$, c_k, $1 \leq k \leq r$ are unknown coefficients; p, q and r are the upper bounds for the true orders. Now, introduce the unknown parameter vector:

$$\theta = [-a_1, -a_2, \cdots, -a_p, b_1, b_2, \cdots, b_q, c_1, c_2, \cdots, c_r]^{\mathrm{T}}, \tag{2}$$

and the corresponding regressor:

$$\varphi_t^0 = [y_t, y_{t-1}, \cdots, y_{t-p+1}, u_t, u_{t-1}, \cdots, u_{t-q+1}, w_t, w_{t-1}, \cdots, w_{t-r+1}]^{\mathrm{T}}, \tag{3}$$

$$\varphi_t = [y_t, y_{t-1}, \cdots, y_{t-p+1}, u_t, u_{t-1}, \cdots, u_{t-q+1}, \hat{w}_t, \hat{w}_{t-1}, \cdots, \hat{w}_{t-r+1}]^{\mathrm{T}}, \tag{4}$$

where $\hat{w}_t$ is the estimation of w_t. Then system (1) can be rewritten as

$$y_{t+1} = \theta^{\mathrm{T}} \varphi_t^0 + w_{t+1}, \quad t \geq 0. \tag{5}$$

Our control objective is, at any instant t, to construct a feedback control u_t based on the past measurements $\{y_0, y_1, \cdots, y_t, u_0, u_1, \cdots, u_{t-1}\}$ so that the following averaged tracking error is asymptotically minimized

$$J_t \triangleq \frac{1}{t} \sum_{i=1}^{t} (y_i - y_i^*)^2, \tag{6}$$

where $\{y_i^*\}$ is a known reference signal.

We need the following standard conditions:

(A.1) The noise sequence $\{w_t, \mathcal{F}_t\}$ is a martingale difference sequence, i.e., $E[w_{t+1}|\mathcal{F}_t] = 0$, and satisfies

$$\lim_{t \to \infty} \frac{1}{t} \sum_{i=1}^{t} w_i^2 = \sigma^2 > 0, \quad \text{a.s.} \tag{7}$$

$$\sup_t E\left[|w_{t+1}|^{\beta} \middle| \mathcal{F}_t\right] < \infty, \ \text{a.s.} \quad \text{for some } \beta > 2. \tag{8}$$

(A.2) $\max\limits_{|z|=1} |C(z) - 1| < 1$.

(A.3) $B(z) \neq 0$, $|z| \leq 1$.

(A.4) $\{y_t^*\}$ is a bounded reference sequence independent of $\{w_t\}$.

We remark that if $\{d_t\}$ is a nondecreasing positive deterministic sequence such that

$${w_t}^2 = O(d_t), \ \text{a.s.} \tag{9}$$

then under Condition (A.1), d_n can be taken as

$$d_t = t^{\delta}, \quad \forall \delta \in \left(\frac{2}{\beta}, 1\right) \tag{10}$$

where β is given by (A.1)[14].

Conventional adaptive control is based on a single identification model (e.g. LS one), which usually leads to large transient errors if the initial values of the algorithm are not properly chosen. In order to improve the transient behaviors of the control algorithms, a natural idea is to use parallel algorithms with multiple different initial values[8],[11]. Compared with the previous results, the main contribution of this paper is to design and to rigorously prove the stability, optimality and convergence rate of a multiple-models-based stochastic adaptive switching control.

3 Multiple Models Based on LS Algorithm

Let $I_1, I_2, \cdots, I_M$ be M predictive models described by

$$I_i: \quad \hat{y}_i(t+1) = \varphi_t^{\mathrm{T}} \hat{\theta}_i(t), \quad i = 1, 2, \cdots, M, \ t = 1, 2, \cdots.$$

where φ_t is defined by (4), and $\hat{\theta}_i(t)$ is the estimate of θ given by the ith estimation algorithm at time t, corresponding to an initial value $\hat{\theta}_i(0)$. At any instant, one of the models is chosen according to a performance index, and the corresponding controller is used to control the system. We consider the problem in the following two cases.

3.1 Multiple adaptive models

First of all, we state some properties of the standard single LS algorithm for the estimation of the unknown parameter θ :

$$\theta_{t+1} = \theta_t + a_t P_t \varphi_t (y_{t+1} - \varphi_t^{\mathrm{T}}, \theta_t), \tag{11}$$

$$P_{t+1} = P_t - a_t P_t \varphi_t \varphi_t^{\mathrm{T}} P_t, \quad a_t = (1 + \varphi_t^{\mathrm{T}} P_t \varphi_t)^{-1}, \tag{12}$$

$$\varphi_t = [y_t, \cdots, y_{t-p+1}, u_t, \cdots, u_{t-q+1}, \hat{w}_t, \cdots, \hat{w}_{t-r+1}]^{\mathrm{T}}, \tag{13}$$

$$\hat{w}_t = y_t - \varphi_{t-1}^{\mathrm{T}} \theta_t \tag{14}$$

where the initial values θ_0, $\varphi_0 \neq 0$, $P_0 > 0$ can be chosen arbitrarily.

Let $\{j_t\}$ be a sequence of integers taking values in $\{0, 1, \cdots, d\}$, $d = p + q + r$, defined by

$$j_t = \operatorname*{argmax}_{0 \le j \le d} |b_{1t} + e_{p+1}^{\mathrm{T}} P_t^{\frac{1}{2}} e_j| \tag{15}$$

where $e_0 = 0$, e_j, $1 \le j \le d$ is the jth column of the $d \times d$ identity matrix, and b_{1t} is the estimate for b_1 given by θ_t.

To guarantee that the estimated "high frequency" gain b_1 is not too small in the minimum variance adaptive control, we slightly modify the LS algorithm as follows[15]:

$$\hat{\theta}_t = \begin{cases} \theta_t & \text{if } |b_{1t}| \ge \dfrac{\beta_0}{\sqrt{\log \rho_{t-1} + t}}, \\ \theta_t + P_t^{\frac{1}{2}} e_{j_t} & \text{otherwise} \end{cases} \tag{16}$$

where $\rho_t \overset{\triangle}{=} 1 + \sum_{i=0}^{t} \|\varphi_i\|^2$, and β_0 is an arbitrary positive constant. In practice, β_0 may be taken as a lower bound to $|b_1|$ if it is available.

The following lemma states that the LS-based algorithm (11)–(16) has the same convergence rate as the standard LS. The proof of it is almost the same as Theorem 6.3 in [15].

Lemma 1 *Under Conditions* (A.1) *and* (A.2), *for any initial values* $(\theta_0, \varphi_0, P_0)$, *if* $\{u_t\}$ *is adapted to* $\mathcal{G}_t \triangleq \sigma(y_j, y^*_{j+1}, j \le t)$, *then the estimation* $\{\hat{\theta}_t\}$ *given by the LS-based algorithm* (11)–(16) *satisfies*

$$\text{(H.1)} \qquad \|\hat{\theta}_t\|^2 = O(\log \rho_{t-1}), \quad a.s.$$

$$\text{(H.2)} \qquad \sum_{i=1}^{t+1} \|\hat{w}_i - w_i\|^2 = O(\log \rho_t), \quad a.s.$$

$$\text{(H.3)} \qquad \sum_{i=1}^{t} \frac{(\varphi_i^{\mathrm{T}} \tilde{\theta}_i)^2}{1 + \varphi_i^{\mathrm{T}} P_i \varphi_i} = O(\log \rho_t), \quad a.s.$$

$$\text{(H.4)} \qquad |\hat{b}_{1t}| \ge C_1 \frac{1}{\sqrt{\log(\rho_{t-1} + t)}}, \quad a.s.$$

where $C_1 > 0$ *is a random variable,* $\hat{b}_{1t}$ *is the estimate for* b_1 *given by* $\hat{\theta}_t$, *and* $\tilde{\theta}_t \triangleq \theta - \hat{\theta}_t$.

In the study of switching control using multiple adaptive models, the estimates of the unknown parameter $\hat{\theta}_i(t)$, $i = 1, 2, \cdots, M$ are all given by the LS-based algorithm (11)–(16). However, their initial values $(\theta_i(0), \varphi_i(0), P_i(0))$ are different, resulting in different prediction models I_i. Notice that

$$\varphi_i(t) = [y_t, \cdots, y_{t-p+1}, u_t, \cdots, u_{t-q+1}, \hat{w}_i(t), \cdots, \hat{w}_i(t-r+1)]^{\mathrm{T}}, \tag{17}$$

$$\hat{w}_i(t) = y_t - \theta_i^{\mathrm{T}}(t) \varphi_i(t-1). \tag{18}$$

It is obvious that for each model I_i, the corresponding values of $\varphi_i(t)$, $P_i(t)$, $\rho_i(t)$ are also different.

Denote

$$e_i(t) \triangleq y_t - \varphi_{t-1}^{\mathrm{T}} \hat{\theta}_i(t-1),$$

$$J_i(t) \triangleq \frac{1}{t} \sum_{j=1}^{t} e_i^2(j), \quad i = 1, 2, \cdots, M,$$

$$i_t \triangleq \operatorname*{argmin}_{1 \le i \le M} J_i(t). \tag{19}$$

At any instant t, the prediction model corresponding to the minimum of $J_i(t)$, $i = 1, 2, \cdots, M$ is chosen to determine the input $u(t)$, i.e.

$$\hat{y}_{i_t}(t+1) = \varphi_{i_t}^{\mathrm{T}}(t) \hat{\theta}_{i_t}(t) = y^*_{t+1} \tag{20}$$

or

$$u_t = \frac{1}{\hat{b}_{1_{i_t}}(t)} (\hat{a}_{1_{i_t}}(t) y_t + \cdots + \hat{a}_{p_{i_t}}(t) y_{t-p+1} - \hat{b}_{2_{i_t}}(t) u_{t-1} - \cdots - \hat{b}_{q_{i_t}}(t) u_{t-q+1}$$
$$- \hat{c}_{1_{i_t}}(t) \hat{w}_{i_t}(t) - \cdots - \hat{c}_{r_{i_t}}(t) \hat{w}_{i_t}(t-r+1) + y^*_{t+1}) \tag{21}$$

where $\hat{a}_{j_{i_t}}(t)$, $\hat{b}_{l_{i_t}}(t)$ and $\hat{c}_{k_{i_t}}(t)$ are the components of $\hat{\theta}_{i_t}(t)$.

Define

$$\delta_i(t) \triangleq \operatorname{tr}(P_i(t) - P_i(t+1)), \quad \delta_t \triangleq \max_{1 \le i \le M} \delta_i(t). \tag{22}$$

Theorem 1 *For the system* (1), *let the conditions* (A.1)–(A.4) *be satisfied, and let the control law be defined by* (21). *Then the closed-loop system is globally stable, optimal and has the following rate of convergence:*

$$R_t = O(\log t + \varepsilon_t) \tag{23}$$

where

$$R_t \triangleq \sum_{i=1}^{t}(y_i - y_i^* - w_i)^2, \tag{24}$$

$$\varepsilon_t = (\log t) \max_{1\le j\le t}\{\delta_j j^{\varepsilon} d_j\},\ \forall \varepsilon > 0. \tag{25}$$

3.2 Multiple models consisting of fixed and adaptive models

We now suppose, without loss of generality, that $\hat{\theta}_1(t)$ is given by adaptive algorithm (11)–(16), and $\hat{\theta}_i(t) = \theta_i,\ i = 2,3,\cdots,M$ are fixed estimates for the unknown parameter. For the fixed models, we still use (12) to construct $\{P_i(t)\}$, $i = 2,3,\cdots,M$; moreover, for each fixed parameter estimate : $\theta_i = [-a_{1_i},\cdots,-a_{p_i},b_{1_i},\cdots,b_{q_i},c_{1_i},\cdots,c_{r_i}]^{\mathrm{T}}$, we can define the following polynomials in the backward-shift operator z:

$$\begin{aligned} A_i(z) &= 1 + a_{1_i}z + \cdots + a_{p_i}z^p, \\ B_i(z) &= b_{1_i} + b_{2_i}z + \cdots + b_{q_i}z^{q-1}, \\ C_i(z) &= 1 + c_{1_i}z + \cdots + c_{r_i}z^r. \end{aligned} \tag{26}$$

Since $C(z)$ satisfies the strictly positive real condition in the system (1), it is natural to require the fixed noise models $C_i(z)$ be stable, i.e., $C_i(z) \neq 0$, $|z| \le 1$, $i = 2,3,\cdots,M$.

Furthermore, we need the following strengthened noise condition:

(A.1)$'$ $\{w_t,\mathcal{F}_t\}$ is a martingale difference sequence, and satisfies:

$$E\left[w_{t+1}^2 \middle| \mathcal{F}_t\right] = \sigma^2, \quad \sup_t E\left[w_{t+1}^4 \middle| \mathcal{F}_t\right] < \infty, \text{ a.s.} \tag{27}$$

Define

$$\bar{r}_t \triangleq \sum_{j=0}^{t}(y_j^2 + u_j^2), \tag{28}$$

$$S_i(t) \triangleq \frac{1}{\log \bar{r}_{t-1}}\sum_{j=0}^{t-1}\frac{(y_{j+1} - \hat{\theta}_i^{\mathrm{T}}(j)\varphi_i(j))^2}{1 + \varphi_i^{\mathrm{T}}(j)P_i(j)\varphi_i(j)}, \quad i = 1,2,\cdots,M, \tag{29}$$

$$I_i(t) \triangleq \max_{1\le j\le t}(S_i(j) - S_1(j)), \quad i = 1,2,\cdots,M, \tag{30}$$

$$\wedge_t \triangleq \{i : 1 \le i \le M | I_i(t) \le K\} \tag{31}$$

where $K > 0$ is a constant. Since $I_1(t) \equiv 0$, it is obvious that $1 \in \wedge_t$, hence $\wedge_t \neq \emptyset$. Let

$$i_t^* \triangleq \operatorname*{argmin}_{i\in\wedge_t} S_i(t), \tag{32}$$

then at time t, $u(t)$ is determined by the following equation:

$$\hat{\theta}_{i_t^*}^{\mathrm{T}}(t)\varphi_{i_t^*}(t) = y_{t+1}^*. \tag{33}$$

Theorem 2 *For the system* (1), *let Conditions* (A.1)$'$, (A.2)–(A.4) *hold and let the control law be defined by* (32) *and* (33). *Then the closed-loop system is globally stable, optimal and has the following rate of convergence*

$$R_t = O(\log t) + (\varepsilon_t) \tag{34}$$

where R_t and ε_t are defined by (24), (25) *respectively.*

4 The Proofs of the Main Theorems

In this section, We first present two lemmas which will be used in the proof of Theorem 1. Now we introduce the following notations:

$$\tilde{\theta}_i(t) \triangleq \theta - \hat{\theta}_i(t), \quad \alpha_i(t) \triangleq \frac{(\varphi_i^{\mathrm{T}}(t)\tilde{\theta}_i(t))^2}{1+\varphi_i^{\mathrm{T}}(t)P_i(t)\varphi_i(t)}, \quad \alpha_t \triangleq \max_{1\le i\le M} \alpha_i(t), \tag{35}$$

$$r_i(t) \triangleq 1 + \sum_{j=0}^{t} \|\varphi_i(j)\|^2, \quad r_t \triangleq \max_{1\le i\le M} r_i(t). \tag{36}$$

Lemma 2 *Consider the closed-loop system* (1) *with the control given by* (19) *and* (20). *If Conditions* (A.1)–(A.4) *are satisfied, then there exists a positive random process* $\{L_t\}$ *such that* $y_t^2 \le L_t, \quad L_{t+1} \le (\lambda + C_2 f_t)L_t + \xi_t$, *where the constants* $\lambda \in (0,1)$, $C_2 > 0$. *And*

$$f_t = [\alpha_t\delta_t \log(t+r_t)]^2 + \alpha_t\delta_t, \tag{37}$$

$$\xi_t = O(d_t \log^5(t+r_t)). \tag{38}$$

Proof By (5) and (20), we have

$$\begin{aligned} y_{t+1} &= \varphi_{i_t}^{\mathrm{T}}(t)\theta - \varphi_{i_t}^{\mathrm{T}}(t)\hat{\theta}_{i_t}(t) + \theta^{\mathrm{T}}(\varphi_t^0 - \varphi_{i_t}(t)) + y_{t+1}^* + w_{t+1} \\ &= \varphi_{i_t}^{\mathrm{T}}(t)\tilde{\theta}_{i_t}(t) + \theta^{\mathrm{T}}(\varphi_t^0 - \varphi_{i_t}(t)) + y_{t+1}^* + w_{t+1}. \end{aligned} \tag{39}$$

From the definition of α_t and (H.3) of Lemma 1, it follows that

$$\begin{aligned} \sum_{j=0}^{t}\alpha_j \le \sum_{j=0}^{t}\sum_{i=1}^{M}\alpha_i(j) &= \sum_{j=0}^{t}\frac{(\varphi_1^{\mathrm{T}}(j)\tilde{\theta}_1(j))^2}{1+\varphi_1^{\mathrm{T}}(j)P_1(j)\varphi_1(j)} + \cdots + \sum_{j=0}^{t}\frac{(\varphi_M^{\mathrm{T}}(j)\tilde{\theta}_M(j))^2}{1+\varphi_M^{\mathrm{T}}(j)P_M(j)\varphi_M(j)} \\ &= O(\log r_1(t) + \cdots + \log r_M(t)) = O(\log r_t). \end{aligned} \tag{40}$$

Particularly, $\alpha_t = O(\log r_t)$, $\alpha_{i_t}(t) \le \alpha_t = O(\log r_t)$. Hence

$$\begin{aligned} (\varphi_{i_t}^{\mathrm{T}}(t)\hat{\theta}_{i_t}(t))^2 &= \alpha_{i_t}(t)(1+\varphi_{i_t}^{\mathrm{T}}(t)P_{i_t}(t)\varphi_{i_t}(t)) \\ &= \alpha_{i_t}(t)(1+\varphi_{i_t}^{\mathrm{T}}(t)(P_{i_t}(t)-P_{i_t}(t+1))\varphi_{i_t}(t) + \varphi_{i_t}^{\mathrm{T}}(t)P_{i_t}(t+1)\varphi_{i_t}(t)) \\ &\le \alpha_{i_t}(t)\delta_{i_t}(t)\|\varphi_{i_t}(t)\|^2 + 2\alpha_{i_t}(t) \\ &\le \alpha_t\delta_t\|\varphi_{i_t}(t)\|^2 + 2\alpha_t = \alpha_t\delta_t\|\varphi_{i_t}(t)\|^2 + O(\log r_t). \end{aligned} \tag{41}$$

By (H.2), we have

$$(\theta^{\mathrm{T}}(\varphi_t^0 - \varphi_{i_t}(t)))^2 \le \|\theta\|^2 \sum_{j=0}^{r-1}(w_{t-j} - \hat{w}_{i_t}(t-j))^2 = O(\log r_{i_t}(t-1)) = O(\log r_{t-1}). \tag{42}$$

From Condition (A.4) and (9), it is seen that

$$\begin{aligned} y_{t+1}^2 &\le 3(\varphi_{i_t}^{\mathrm{T}}(t)\hat{\theta}_{i_t}(t))^2 + 3(\theta^{\mathrm{T}}(\varphi_t^0 - \varphi_{i_t}(t)))^2 + 3(y_{t+1}^* + w_{t+1})^2 \\ &\le 3\alpha_t\delta_t\|\varphi_{i_t}(t)\|^2 + O(d_t) + C\ \log r_t). \end{aligned} \tag{43}$$

By the stability of $B(z)$, it is known from (1) that there is a constant $\lambda \in (0,1)$ such that

$$u_{t-1}^2 = O\Big(\sum_{j=0}^{t} \lambda^{t-j} y_j^2\Big) + O\Big(\sum_{j=0}^{t} \lambda^{t-j} w_j^2\Big). \tag{44}$$

Hence

$$\begin{aligned}
\|\varphi_{i_t}(t)\|^2 - u_t^2 &= O\Big(\sum_{j=0}^{t} \lambda^{t-j} y_j^2\Big) + O\Big(\sum_{j=0}^{t} \lambda^{t-j} w_j^2\Big) + O\Big(\sum_{j=0}^{r-1} \hat{w}_{i_t}^2(t-j)\Big)\\
&= O\Big(\sum_{j=0}^{t} \lambda^{t-j} y_j^2\Big) + O\Big(\sum_{j=0}^{r-1} (\hat{w}_{i_t}(t-j) - w_{t-j})^2\Big) + O\Big(\sum_{j=0}^{r-1} w_{t-j}^2\Big) + O(d_t)\\
&= O\Big(\sum_{j=0}^{t} \lambda^{t-j} y_j^2\Big) + O(\log r_{t-1}) + O(d_t).
\end{aligned} \tag{45}$$

By (H.1) and (H.4) of Lemma 1, it is seen that

$$\|\hat{\theta}_{i_t}(t)\|^2 \le \sum_{i=1}^{M} \|\hat{\theta}_i(t)\|^2 = O\Big(\sum_{i=1}^{M} \log r_i(t-1)\Big) = O(\log r_{t-1}), \tag{46}$$

$$|\hat{b}_{1_{i_t}}(t)| \ge \min_{1\le i\le M} |\hat{b}_{1_i}(t)| \ge \min_{1\le i\le M} \frac{\bar{C}_i}{\sqrt{\log(t + r_{t-1})}} \ge \frac{\bar{C}}{\sqrt{\log(t + r_{t-1})}}, \text{ a.s.} \tag{47}$$

where $\bar{C}_i \ge \bar{C} > 0$ is a random variable. From this and (21), we have

$$\begin{aligned}
u_t^2 &= O\Big(\log^2(t + r_{t-1})\Big(\sum_{j=0}^{p-1} y_{t-j}^2 + \sum_{j=1}^{q-1} u_{t-j}^2 + \sum_{j=0}^{r-1} \hat{w}_{i_t}^2(t-j)\Big) + \log(t + r_{t-1})\Big)\\
&= O\Big(\log^2(t + r_{t-1})\Big(\sum_{j=0}^{t} \lambda^{t-j} y_j^2 + \sum_{j=0}^{t} \lambda^{t-j} w_j^2 + \sum_{j=0}^{r-1} \hat{w}_{i_t}^2(t-j)\Big) + \log(t + r_{t-1})\Big)\\
&= O(\log^2(t + r_{t-1})(L_t + d_t + \log(t + r_{t-1}))\\
&= O(L_t \log^2(t + r_{t-1})) + O(d_t \log^3(t + r_{t-1})),
\end{aligned} \tag{48}$$

where $L_t \triangleq \sum_{j=0}^{t} \lambda^{t-j} y_j^2$. Substituting this into (45), we have

$$\|\varphi_{i_t}(t)\|^2 = O(L_t \log^2(t + r_{t-1})) + O(d_t \log^3(t + r_{t-1})). \tag{49}$$

Note that $b_1 u_t = \varphi_{i_t}^{\rm T}(t)\tilde{\theta}_{i_t}(t) + y_{t+1}^* + (b_1 u_t - \theta^{\rm T}\varphi_{i_t}(t))$ and by (41), (44), it follows that

$$\begin{aligned}
b_1^2 u_t^2 &\le 3(\varphi_{i_t}^{\rm T}(t)\tilde{\theta}_{i_t}(t))^2 + O(1 + |b_1 u_t - \theta^{\rm T}\varphi_{i_t}(t)|^2)\\
&\le 3\alpha_t \delta_t \|\varphi_{i_t}(t)\|^2 + O(\log r_t) + O(L_t + d_t + \log r_{t-1}).
\end{aligned} \tag{50}$$

Therefore,

$$u_t^2 = O(\alpha_t \delta_t \|\varphi_{i_t}(t)\|^2) + O(L_t + d_t + \log r_t). \tag{51}$$

From (45), we have

$$\|\varphi_{i_t}\|^2 = O(\alpha_t \delta_t \|\varphi_{i_t}(t)\|^2) + O(L_t + d_t + \log r_t). \tag{52}$$

Substituting (49) into this, we get

$$\|\varphi_{i_t}(t)\|^2 = O(\alpha_t\delta_t L_t \log^2(t+r_{t-1})) + O(L_t + d_t\log^4(t+r_{t-1})). \tag{53}$$

Finally, substituting the above into (43), we find that there exists a constant $C_2 > 0$ such that

$$y_{t+1}^2 \le C_2 f_t L_t + \xi_t, \tag{54}$$

where f_t and ξ_t are defined by (37) and (38). Moreover by the definition of L_t, we have

$$L_{t+1} \le \lambda L_t + y_{t+1}^2 \le (\lambda + C_2 f_t)L_t + \xi_t. \tag{55}$$

Hence the proof of Lemma 2 is completed.

Lemma 3 *Under the conditions of Lemma* 2, *we have*

$$\|\varphi_{i_t}(t)\|^2 = O((t+r_t)^\varepsilon d_t), \quad a.s. \quad \forall \varepsilon > 0.$$

Proof By the definition of δ_t, it is seen that

$$\delta_t = \max_{1\le i\le M} \delta_i(t) \le \sum_{i=1}^M \mathrm{tr}(P_i(t) - P_i(t+1)). \tag{56}$$

Hence

$$\sum_{t=0}^\infty \delta_t \le \sum_{i=1}^M \sum_{t=0}^\infty \mathrm{tr}(P_i(t) - P_i(t+1)) < \infty. \tag{57}$$

From this and (40), we can prove the lemma by proceeding along the same lines as those for Lemma 6.2 in [15]. The details will not be repeated here.

Proof of Theorem 1 By (39), we have

$$\begin{aligned} R_{t+1} &= \sum_{j=0}^t (y_{j+1} - y_{j+1}^* - w_{j+1})^2 = \sum_{j=0}^t \Big(\varphi_{i_j}^{\mathrm{T}}(j)\tilde{\theta}_{i_j}(j) + \theta^{\mathrm{T}}(\varphi_j^0 - \varphi_{i_j}(j))\Big)^2 \\ &\le 2\sum_{j=0}^t (\varphi_{i_j}^{\mathrm{T}}(j)\tilde{\theta}_{i_j}(j))^2 + 2\sum_{j=0}^t \|\theta\|^2 \|\varphi_j^0 - \varphi_{i_j}(j)\|^2 \\ &= O\Big(\sum_{j=0}^t (\alpha_j\delta_j\|\varphi_{i_j}(j)\|^2 + 2\alpha_j)\Big) + O\Big(\sum_{i=1}^M\sum_{j=0}^t (w_j - \hat{w}_i(j))^2\Big). \end{aligned} \tag{58}$$

From (40) and (H.2) of Lemma 1, it follows that

$$R_{t+1} = O\Big(\sum_{j=0}^t \alpha_j\delta_j\|\varphi_{i_j}(j)\|^2\Big) + O(\log r_t).$$

From Lemma 3, it is obvious that for any $\varepsilon > 0$, we have

$$R_{t+1} = O(\log r_t) + O\Big(\max_{1\le j\le t}\{\delta_j(j+r_j)^\varepsilon d_j\}\log r_t\Big). \tag{59}$$

Therefore, for (23), it suffices to prove that $r_t = O(t)$. From above, we have

$$R_{t+1} = O((t+r_t)^\varepsilon d_t).$$

By Conditions (A.1) and (A.4), it follows that

$$\sum_{j=0}^{t+1} y_j^2 = O(t) + R_{t+1} = O(t) + O((t+r_t)^\varepsilon d_t). \tag{60}$$

By this and Condition (A.3), it follows from (1) that

$$\sum_{j=0}^{t} u_j^2 = O\Big(\sum_{j=0}^{t+1} y_j^2\Big) + O\Big(\sum_{j=0}^{t+1} w_j^2\Big) = O(t) + O((t+r_t)^\varepsilon d_t). \tag{61}$$

$$\begin{aligned}\sum_{j=0}^{t+1} \hat{w}_i^2(j) &= O\Big(\sum_{j=0}^{t+1}(w_j - \hat{w}_i(j))^2\Big) + O\Big(\sum_{j=0}^{t+1} w_j^2\Big)\\ &= O(\log r_i(t)) + O(t) = O(\log r_t) + O(t).\end{aligned} \tag{62}$$

From the definition of $r_i(t)$, it is seen that for each $i = 1, 2, \cdots, M$, we have $r_i(t) = O((t + r_t)^\varepsilon d_t) + O(t)$. Hence,

$$r_t = \max_{1\le i\le M} r_i(t) = O((t+r_t)^\varepsilon t^\delta) + O(t), \quad \forall \delta \in \Big(\frac{2}{\beta}, 1\Big) \tag{63}$$

By taking ε small enough such that $\varepsilon + \delta < 1$, we get

$$\frac{r_t}{t} = O(1) + O\left(\left(\frac{r_t}{t}\right)^\varepsilon \frac{1}{t^{1-\varepsilon-\delta}}\right) = O(1) + o\left(\left(\frac{r_t}{t}\right)^\varepsilon\right). \tag{64}$$

From this, we see that $r_t = O(t)$ holds, hence

$$R_{t+1} = O(\log t) + O(\varepsilon_t), \quad \text{a.s.} \tag{65}$$

where ε_t is defined by (25). Obviously, $R_t = o(t)$. Moreover, by the definition of J_t and Condition (A.1) (see [16]), we get

$$\lim_{t\to\infty} J_t = \sigma^2, \text{ a.s.} \tag{66}$$

Hence the optimality of the control is also true.

In the following, we present three auxiliary lemmas which will be needed in the proof of Theorem 2.

Lemma 4 *Consider the closed-loop system* (1) *with the control based on mixed models, Let Conditions* (A.1)$'$, (A.2) *and* (A.4) *be satisfied, and the adaptive model use the LS algorithm* (11)–(16). *If the fixed noise models,* $C_i(z)$, $i = 2, 3, \cdots, M$ *are stable, then* $\bar{r}_t$ *is equivalent to* $r_i(t)$, $i = 1, 2, \cdots, M$ *in the sense that, there exist some positive random variables* m_i, M_i *such that*

$$m_i r_i(t) \le \bar{r}_t \le M_i r_i(t), \quad i = 1, 2, \cdots, M. \tag{67}$$

Proof For the adaptive model, we know from (H.2) of Lemma 1 that,

$$\sum_{j=0}^{t}(w_j - \hat{w}_1(j))^2 = O(\log r_1(t-1)). \tag{68}$$

By Condition (A.2), we know that $C(z)$ is stable, hence

$$\sum_{j=0}^{t} w_j^2 = O\Big(\sum_{j=0}^{t} y_j^2\Big) + O\Big(\sum_{j=0}^{t-1} u_j^2\Big). \tag{69}$$

Therefore,

$$\sum_{j=0}^{t} \hat{w}_1^2(j) = O(\bar{r}_t) + O(\log r_1(t)). \tag{70}$$

From this and the definition of $r_1(t)$, we have

$$r_1(t) = O\Big(\sum_{j=0}^{t} y_j^2\Big) + O\Big(\sum_{j=0}^{t} u_j^2\Big) + O\Big(\sum_{j=0}^{t} \hat{w}_1^2(j)\Big) = O(\bar{r}_t) + o(r_1(t)). \tag{71}$$

Hence $r_1(t) = O(\bar{r}_t)$.

For the fixed models I_i, $i = 2, 3, \cdots, M$, we have

$$\hat{w}_i(t) = y_t - \theta_i^{\mathrm{T}} \varphi_i(t-1). \tag{72}$$

From this and (26), it follows that

$$A_i(z) y_t = B_i(z) u_{t-1} + C_i(z) \hat{w}_i(t). \tag{73}$$

Since $C_i(z)$ is stable, we have

$$\sum_{j=0}^{t} \hat{w}_i^2(j) = O\Big(\sum_{j=0}^{t} y_j^2\Big) + O\Big(\sum_{j=0}^{t-1} u_j^2\Big). \tag{74}$$

Therefore $r_i(t) = O(\bar{r}_t)$, $i = 2, 3, \cdots, M$. Moreover, from the definitions of $r_i(t)$ and $\bar{r}_t$, it is obvious that $\bar{r}_t \leq r_i(t)$, $i = 1, 2, \cdots .M$. Thus the proof of this lemma is completed.

Lemma 5 *If Condition* (A.3) *holds in additional to those of Lemma* 4*, then for the fixed models* I_i, $i = 2, 3, \cdots, M$, *there exists a constant* $0 < \rho < 1$ *such that*

$$\hat{w}_i^2(t) = O\Big(\sum_{j=0}^{t} \rho^{t-j} y_j^2\Big) + O(d_t). \tag{75}$$

Proof By (73) and the stability of $C_i(z)$, it is known that for each fixed model I_i, $i = 2, 3. \cdots, M$, there exists a constant $0 < \lambda_i < 1$ such that

$$\hat{w}_i^2(t) = O\Big(\sum_{j=0}^{t} \lambda_i^{t-j} y_j^2\Big) + O\Big(\sum_{j=0}^{t} \lambda_i^{t-j} u_{j-1}^2\Big). \tag{76}$$

From Condition (A.3), it follows that

$$u_{t-1}^2 = O\Big(\sum_{j=0}^{t} \lambda^{t-j} y_j^2\Big) + O\Big(\sum_{j=0}^{t} \lambda^{t-j} w_j^2\Big). \tag{77}$$

Substituting this into (76), we get

$$\hat{w}_i^2(t) = O\Big(\sum_{j=0}^{t} \lambda_i^{t-j} y_j^2\Big) + O\Big(\sum_{k=0}^{t} (t-k+1) \bar{\lambda}^{t-k} y_k^2\Big) + O\Big(\sum_{k=0}^{t} (t-k+1) \bar{\lambda}^{t-k} w_k^2\Big). \tag{78}$$

where $\bar{\lambda} \triangleq \max\{\lambda, \lambda_2, \cdots, \lambda_M\}$. Let $\rho \triangleq \sqrt{\bar{\lambda}}$, it is obvious that $0 < \rho < 1$, hence there exists a constant $C_3 > 0$ such that for any $x > 0$, we have

$$\frac{x\bar{\lambda}^x}{\rho^x} = x\rho^x \le C_3. \tag{79}$$

Therefore,

$$(t-k+1)\bar{\lambda}^{t-k} \le \frac{1}{\bar{\lambda}} C_3 \rho^{t-k+1} = \frac{C_3}{\rho}\rho^{t-k}. \tag{80}$$

Substituting this into (78), we have

$$\hat{w}_i^2(t) = O\Big(\sum_{j=0}^{t} \rho^{t-j} y_j^2\Big) + O(d_t). \tag{81}$$

Hence the lemma is true.

Lemma 6 *For the system* (1), *let Conditions* $(A.1)'$, (A.2)–(A.4) *be satisfied. If the control law is determined by* (32) *and* (33), *then there exists a random time* $t_1 > 0$ *such that for* $t > t_1$, *the value of* i_t^* *will belong to the set* $N \triangleq \{i : 1 \le i \le M \,|\, I_i(t) \le K, \forall\, t > 0\}$, *and*

$$\sum_{j=0}^{t-1} \frac{(\tilde{\theta}_i^{\mathrm{T}}(j)\varphi_i(j) + \theta^{\mathrm{T}}(\varphi_j^0 - \varphi_i(j)))^2}{1 + \varphi_i^{\mathrm{T}}(j)P_i(j)\varphi_i(j)} = O(\log \bar{r}_{t-1}), \ a.s.\ \forall\, i \in N \tag{82}$$

Proof By the definition of $I_i(t)$, we see that for each $i = 1, 2, \cdots, M$, $I_i(t)$ is nondecreasing. If for some i, $\lim_{t\to\infty} I_i(t) > K$, then after a period of time, i will no longer belong to the set $\wedge_t$. Moreover, from the definition for N, we know that there exists some $t_1 > 0$ such that $\wedge_t = N$, $t > t_1$. Obviously, $1 \in N$.

By the definition of $S_i(t)$, we know that

$$S_i(t) = \frac{1}{\log \bar{r}_{t-1}} \sum_{j=0}^{t-1} \frac{(\tilde{\theta}_i^{\mathrm{T}}(j)\varphi_i(j) + \theta^{\mathrm{T}}(\varphi_j^0 - \varphi_i(j)) + w_{j+1})^2}{1 + \varphi_i^{\mathrm{T}}(j)P_i(j)\varphi_i(j)}. \tag{83}$$

Denote $g_i(t) \triangleq \theta^{\mathrm{T}}(\varphi_t^0 - \varphi_i(t)) + \tilde{\theta}_i^{\mathrm{T}}(t)\varphi_i(t)$, then $S_i(t)$ can be rewritten as follows

$$S_i(t) = \frac{1}{\log \bar{r}_{t-1}}(S_i^1(t) + S_i^2(t) + S_i^3(t)) \tag{84}$$

where

$$S_i^1(t) \triangleq \sum_{j=0}^{t-1} \frac{g_i^2(j)}{1 + \varphi_i^{\mathrm{T}}(j)P_i(j)\varphi_i(j)}, \qquad S_i^2(t) \triangleq \sum_{j=0}^{t-1} \frac{2g_i(j)w_{j+1}}{1 + \varphi_i^{\mathrm{T}}(j)P_i(j)\varphi_i(j)}, \tag{85}$$

$$S_i^3(t) \triangleq \sum_{j=0}^{t-1} \frac{w_{j+1}^2}{1 + \varphi_i^{\mathrm{T}}(j)P_i(j)\varphi_i(j)}. \tag{86}$$

By the martingale convergence theorem, we have

$$S_i^2(t) = O\Big((S_i^1(t))^{\frac{1}{2}+\eta}\Big), \ \forall\, \eta > 0. \tag{87}$$

From (H.2), (H.3), it follows that $S_1^1(t) = O(\log r_1(t-1))$. Moreover, from Lemma 4, we see that $r_1(t) = O(\bar{r}_t)$. Hence $S_1^1(t) = O(\log \bar{r}_{t-1})$, and $S_1^2(t) = O(\log \bar{r}_{t-1})$.

By the definition of $S_i^3(t)$, it is known that

$$\begin{aligned}S_1^3(t)-S_i^3(t)&=\sum_{j=0}^{t-1}\Big(\frac{1}{1+\varphi_1^{\mathrm T}(j)P_1(j)\varphi_1(j)}-\frac{1}{1+\varphi_i^{\mathrm T}(j)P_i(j)\varphi_i(j)}\Big)w_{j+1}^2\\&\le\sum_{j=0}^{t-1}\frac{\varphi_i^{\mathrm T}(j)P_i(j)\varphi_i(j)}{(1+\varphi_1^{\mathrm T}(j)P_1(j)\varphi_1(j))(1+\varphi_i^{\mathrm T}(j)P_i(j)\varphi_i(j))}w_{j+1}^2\\&\le\sum_{j=0}^{t-1}\frac{\varphi_i^{\mathrm T}(j)P_i(j)\varphi_i(j)}{1+\varphi_i^{\mathrm T}(j)P_i(j)\varphi_i(j)}(w_{j+1}^2-\sigma^2)+\sum_{j=0}^{t-1}\frac{\varphi_i^{\mathrm T}(j)P_i(j)\varphi_i(j)}{1+\varphi_i^{\mathrm T}(j)P_i(j)\varphi_i(j)}\sigma^2.\qquad(88)\end{aligned}$$

From (12), it is seen that $P_i^{-1}(j+1)=P_i^{-1}(j)+\varphi_i(j)\varphi_i^{\mathrm T}(j)$. Then taking determinants on both sides, we have

$$|P_i^{-1}(j+1)|=|P_i^{-1}(j)|\Big(1+\varphi_i^{\mathrm T}(j)P_i(j)\varphi_i(j)\Big).\qquad(89)$$

Therefore,

$$\begin{aligned}\sum_{j=0}^{t-1}\frac{\varphi_i^{\mathrm T}(j)P_i(j)\varphi_i(j)}{1+\varphi_i^{\mathrm T}(j)P_i(j)\varphi_i(j)}&=\sum_{j=0}^{t-1}\Big(1-\frac{|P_i^{-1}(j)|}{|P_i^{-1}(j+1)|}\Big)\\&\le\sum_{j=0}^{t-1}\Big(-\log\frac{|P_i^{-1}(j)|}{|P_i^{-1}(j+1)|}\Big)=\log|P_i^{-1}(t)|+\log|P_i(0)|.\qquad(90)\end{aligned}$$

Since

$$P_i^{-1}(t+1)=P_i^{-1}(0)+\sum_{j=0}^{t}\varphi_i(j)\varphi_i^{\mathrm T}(j).\qquad(91)$$

It follows that

$$\log|P_i^{-1}(t)|\le d\log\lambda_{\max}(P_i^{-1}(t))\le d\log r_i(t-1)+O(1)\qquad(92)$$

where $d=p+q+r$. Substituting the above into (90), it follows from Lemma 4 that

$$\sum_{j=0}^{t-1}\frac{\varphi_i^{\mathrm T}(j)P_i(j)\varphi_i(j)}{1+\varphi_i^{\mathrm T}(j)P_i(j)\varphi_i(j)}=O(\log\bar r_{t-1}).\qquad(93)$$

Note that $\Big\{\frac{\varphi_i^{\mathrm T}(j)P_i(j)\varphi_i(j)}{1+\varphi_i^{\mathrm T}(j)P_i(j)\varphi_i(j)}(w_{j+1}^2-\sigma^2),\ \mathcal F_{j+1}\Big\}$ is a martingale difference sequence, by Condition (A.1)$'$ and (93), it follows from the the martingale convergence theorem that

$$\sum_{j=0}^{t-1}\frac{\varphi_i^{\mathrm T}(j)P_i(j)\varphi_i(j)}{1+\varphi_i^{\mathrm T}(j)P_i(j)\varphi_i(j)}(w_{j+1}^2-\sigma^2)=O\big(\log^{\frac12+\eta}\bar r_{t-1}\big),\quad\forall\,\eta>0.\qquad(94)$$

Therefore, for each $i\in N$, we have

$$\begin{aligned}\frac{1}{\log\bar r_{t-1}}(S_i^1(t)+S_i^2(t))&=S_i(t)-S_1(t)+\frac{1}{\log\bar r_{t-1}}(S_1^1(t)+S_1^2(t)+S_1^3(t)-S_i^3(t))\\&\le I_i(t)+O(1)\le K+O(1)=O(1).\qquad(95)\end{aligned}$$

Moreover, from (87), it is known that $S_i^1(t) = O(\log \bar{r}_{t-1}), \forall i \in N$. Hence the proof of the lemma is completed.

Proof of Theorem 2 By (39) and the definition of $g_i(j)$, we know that

$$y_{t+1} = g_{i_t^*}(t) + y_{t+1}^* + w_{t+1}. \tag{96}$$

From Lemma 6, it follows that for $t > t_1$, $i_t^* \in N$. Define $\beta_t = \frac{g_{i_t^*}^2(t)}{1+\varphi_{i_t^*}^{\mathrm{T}}(t)P_{i_t^*}(t)\varphi_{i_t^*}(t)}$, then for $t \geq t_1 + 1$, we have

$$\sum_{j=t_1+1}^{t} \beta_j \leq \sum_{i \in N} \sum_{j=t_1+1}^{t} \frac{g_i^2(j)}{1 + \varphi_i^{\mathrm{T}}(j)P_i(j)\varphi_i(j)}. \tag{97}$$

Using the result of Lemma 6 again, we get $\sum_{j=0}^{t} \beta_j = O(1) + \sum_{j=t_1+1}^{t} \beta_j = O(\log \bar{r}_t)$, particularly, $\beta_t = O(\log \bar{r}_t)$. Hence, we have

$$\begin{aligned} y_{t+1}^2 &\leq 2g_{i_t^*}^2(t) + 2(y_{t+1}^* + w_{t+1})^2 = 2\beta_t(1 + \varphi_{i_t^*}^{\mathrm{T}}(t)P_{i_t^*}(t)\varphi_{i_t^*}(j)) + O(d_t) \\ &\leq 2\beta_t\delta_t\|\varphi_{i_t^*}(t)\|^2 + O(\log \bar{r}_t) + O(d_t) \end{aligned} \tag{98}$$

where δ_t is still defined by (22).

By Condition (A.3), (H.2) of Lemma 1 and Lemma 5, it follows that

$$\begin{aligned} \|\varphi_{i_t^*}(t)\|^2 - u_t^2 &= O\Big(\sum_{j=0}^{t} \lambda^{t-j} y_j^2\Big) + O\Big(\sum_{j=0}^{t} \lambda^{t-j} w_j^2\Big) + O\Big(\sum_{j=0}^{r-1} \hat{w}_{i_t^*}^2(t-j)\Big) \\ &= O\Big(\sum_{j=0}^{t} \rho^{t-j} y_j^2\Big) + O(\log \bar{r}_{t-1}) + O(d_t). \end{aligned} \tag{99}$$

From (H.1),(H.4) and the fact that $\hat{\theta}_i(t) = \theta_i$, $i = 2, 3, \cdots, M$, we have

$$\|\hat{\theta}_{i_t^*}(t)\|^2 \leq \sum_{i=1}^{M} \|\hat{\theta}_i(t)\|^2 = O(\log r_1(t-1)) + O(1) = O(\log \bar{r}_{t-1}), \tag{100}$$

$$|\hat{b}_{1_{i_t^*}}(t)| \geq \frac{C_4}{\sqrt{\log(t + \bar{r}_{t-1})}}, \text{ a.s.} \tag{101}$$

where $C_4 > 0$ is a random variable. Similar to (21), we know that

$$\begin{aligned} u_t^2 &= O\Big(\log^2(t + \bar{r}_{t-1})\Big(\sum_{i=0}^{t} \lambda^{t-j} y_j^2 + \sum_{i=0}^{t} \lambda^{t-i} w_j^2 + \sum_{j=0}^{r-1} \hat{w}_{i_t^*}^2(t-j)\Big) + \log(t + \bar{r}_{t-1})\Big) \\ &= O(\log^2(t + \bar{r}_{t-1})(L_t' + d_t + \log(t + \bar{r}_{t-1}))) \\ &= O(L_t' \log^2(t + \bar{r}_{t-1})) + O(d_t \log^3(t + \bar{r}_{t-1})) \end{aligned} \tag{102}$$

where $L_t' \triangleq \sum_{j=0}^{t} \rho^{t-j} y_j^2$. Moreover, by (99), it follows that

$$\|\varphi_{i_t^*}(t)\|^2 = O(L_t' \log^2(t + r_{t-1})) + O(d_t \log^3(t + r_{t-1})). \tag{103}$$

Note that $b_1 u_t = g_{i_t^*}(t) + y_{t+1}^* + (b_1 u_t - \theta^{\mathrm{T}} \varphi_{i_t^*}(t)) - \theta^{\mathrm{T}}(\varphi_t^0 - \varphi_{i_t^*}(t))$. Hence

$$\begin{aligned} b_1^2 u_t^2 &\leq 4g_{i_t^*}^2(t) + O(1 + |b_1 u_t - \theta^{\mathrm{T}} \varphi_{i_t^*}(t)|^2) + O\Big(\sum_{j=0}^{r-1} (w_{t-j} - \hat{w}_{i_t^*}(t-j))^2\Big) \\ &= 4\beta_t\delta_t\|\varphi_{i_t^*}(t)\|^2 + O(\log \bar{r}_t) + O(L_t' + d_t + \log \bar{r}_{t-1}). \end{aligned} \tag{104}$$

Using (99) again, it follows that

$$\|\varphi_{i_t^*}\|^2 = O(\beta_t\delta_t\|\varphi_{i_t^*}(t)\|^2) + O(L_t' + d_t + \log \bar{r}_t). \tag{105}$$

Substituting (103) into this, we get

$$\|\varphi_{i_t^*}(t)\|^2 = O(\beta_t\delta_t L_t' \log^2(t+\bar{r}_{t-1})) + O(L_t' + d_t\log^4(t+\bar{r}_{t-1})). \tag{106}$$

Finally, substituting this into (98), we know that there exists a constant $C_5 > 0$ such that

$$y_{t+1}^2 \le C_5 f_t' L_t' + \xi_t', \tag{107}$$

where

$$f_t' \triangleq [\beta_t\delta_t \log(t+\bar{r}_t)]^2 + \beta_t\delta_t, \tag{108}$$

$$\xi_t' \triangleq O(d_t\log^5(t+\bar{r}_t)). \tag{109}$$

Similar to the proof of Lemma 6.2 in [15], we can prove that for $t \ge t_1 + 1$,

$$\|\varphi_{i_t^*}(t)\|^2 = O((t+\bar{r}_t)^\varepsilon),\ \forall\, \varepsilon > 0. \tag{110}$$

Therefore, for $t \ge t_1 + 1$, we have

$$\begin{aligned}
\sum_{j=0}^{t}(y_{j+1} - y_{j+1}^* - w_{j+1})^2 &= \sum_{j=0}^{t_1}(y_{j+1} - y_{j+1}^* - w_{j+1})^2 + \sum_{j=t_1+1}^{t} g_{i_j^*}^2(j) \\
&\le O(1) + \sum_{j=t_1+1}^{t} \beta_j(1 + \varphi_{i_j^*}^{\mathrm{T}}(j)P_{i_j^*}(j)\varphi_{i_j^*}(j)) \\
&= O(1) + O(\log \bar{r}_t) + \sum_{j=t_1+1}^{t} \beta_j\delta_j\|\varphi_{i_j^*}(j)\|^2 \\
&= O(\log \bar{r}_t) + O\Big(\max_{0\le j\le t}\{\delta_j(j+\bar{r}_j)^\varepsilon d_j\}\log \bar{r}_t\Big).
\end{aligned} \tag{111}$$

Notice the definition of $\bar{r}_t$, we can prove that $\bar{r}_t = O(t)$ in a similar way as that for Theorem 1. Hence the proof of Theorem 2 is completed.

5 Simulation Results

In Sections 3 and 4, the convergence and optimality properties of stochastic adaptive control using multiple models are discussed. In this section, we use an example to demonstrate the performances of the switching adaptive control.

5.1 The Problem

Consider the following linear time-invariant discrete-time plant described by

$$\begin{aligned}
y(t+1) = 3y(t) - 2y(t-1) - y(t-2) + u(t) + 0.8u(t-1) \\
+0.4u(t-2) + w(t+1) + 0.5w(t)
\end{aligned} \tag{112}$$

where $\{w(t)\}$ is a white noise sequence which is normally distributed with zero mean and variance $\sigma^2 = 0.04$.

The above plant can also be written as

$$y(t+1) = \varphi_0^{\mathrm{T}}(t)\theta + w(t+1),$$

where

$$\varphi_0^{\mathrm{T}}(t) = [y(t),\ y(t-1),\ y(t-2),\ u(t),\ u(t-1),\ u(t-2),\ w(t)],$$
$$\theta^{\mathrm{T}} = [3,\ -2,\ -1,\ 1,\ 0.8,\ 0.4,\ 0.5].$$

Let us assume that θ is an unknown parameter vector of the plant that has to be estimated. The objective of the control is to track the following reference signal $y^*(t)$ ([12]).

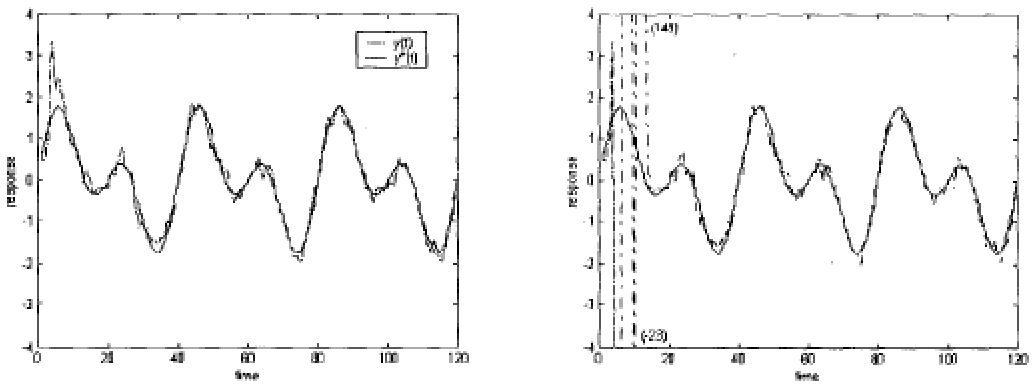
$$y^*(t) = \sin\left(\frac{\pi t}{20}\right) + \sin\left(\frac{\pi t}{10}\right).$$

5.2 The Simulations

Simulation 1 The comparison of the transient responses between multiple-adaptive-model-based switching controller and controller based on single adaptive model is shown in Fig.1(a) and (b). For the switching controller, five adaptive models are used, which have the following initial values respectively:

$$\theta_1 = [4.9,\ 3,\ -2.4,\ 2.1,\ 1.6,\ 0.8,\ 0.7],$$
$$\theta_2 = [2.85,\ -2,\ -0.93,\ 1.05,\ 0.87,\ 0.45,\ 0.52],$$
$$\theta_3 = [2.7,\ -1.8,\ -1.34,\ 1.12,\ 0.91,\ 0.51,\ 0.62],$$
$$\theta_4 = [4.7,\ 2.1,\ -1.4,\ 2,\ 1.65,\ -0.25,\ 0.17],$$
$$\theta_5 = [1.31,\ -2.9,\ 1.21,\ -1.6,\ 1.5,\ 0.24,\ 0.78].$$

At each instant, the performance index $J_i(t) = (1/t)\sum_{j=1}^{t} e_i^2(j)$ is computed for all the models, and the model corresponding to the minimum of $J_i(t)$ is chosen to determine the control input. The response of the switching controller based on multiple models is found to be satisfactory (see Fig.1(a)). However, the single adaptive model based controller with initial value $\theta_0 = \theta_1$, will result in large transient errors as shown in Fig.1(b).

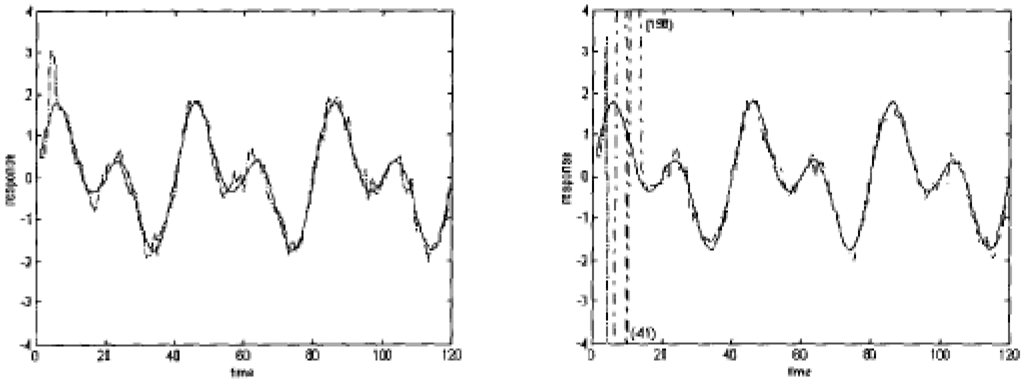


(a) (b)

Fig.1 Comparison between multiple-adaptive-model-based control and tranditional adaptive control

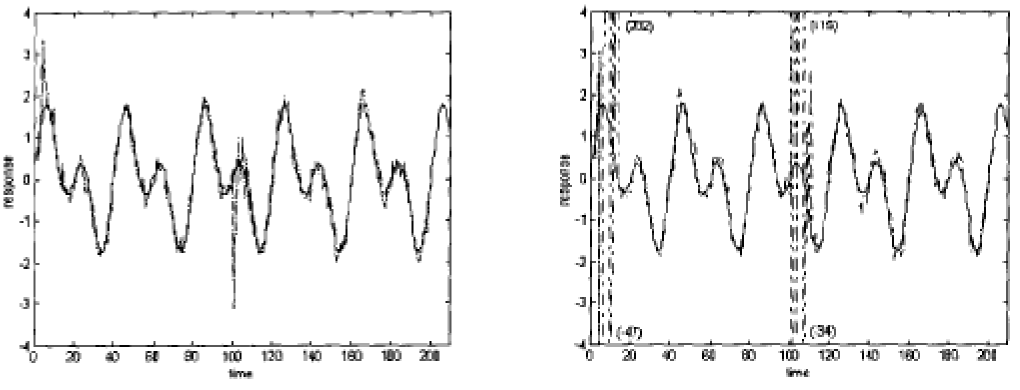
Simulation 2 This simulation compares the transient response of mixed-model-based switching controller with that of tranditional adaptive control. Fig.2(a) corresponds to the switching controller, where the initial value of the adaptive model is θ_1, and the fixed models are specified by the parameter vectors $\theta_i\,(i=2,3,\cdots,5)$ given in Simulation 1.

Fig.2(b) shows the response of the controller based on a single adaptive model, where the initial estimation of θ is $\theta_0=\theta_1$. It is obvious that the response of switching controller is much more satisfactory. In fact $(1+\varphi_i^{\mathrm{T}}(j)P_i(j)\varphi_i(j))^{-1}$ acts as a time-varing "forgetting" factor in the performance index $S_i(t)$ for each model. This implies that past errors are less weighted than present ones, and thus the adptive model will have the chance to be chosen if its estimation for the unknown parameter is good enough.



(a) (b)

Fig.2 Comparison between mixed-model-based control and tranditional control



(a) (b)

Fig.3 Comparison of mixed-model-based control and tranditional adaptive control for a switched system

Simulation 3 In this experiment, the plant parameters are assumed to vary periodically, i.e., starting from time 0, the plant parameter vector is a constant in every 100 units of time. For simplicity, here the plant parameter is assumed to switch between

$$\theta=[3,\ -2,\ -1,\ 1,\ 0.8,\ 0.4,\ 0.5]^{\mathrm{T}}$$

and

$$\bar{\theta}=[1.13,\ 3.9,\ 2.27,\ 3.14,\ 3.02,\ 0.5,\ 0.89]^{\mathrm{T}}.$$

Fig.3(a) shows the response of controller using multiple mixed models, where the initial value of the adaptive model and the fixed models are the same as those in Simulation 2. Obviously, the performance of switching controller is significantly improved, compared to the response of the controller using a single adaptive model [Fig.3(b)].

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